THE VORONOVSKAYA THEOREM
FOR SOME LINEAR POSITIVE OPERATORS
IN EXPONENTIAL WEIGHT SPACES

L. REMPUŠKA AND M. SKORUPKA

Abstract
In this note we give the Voronovskaya theorem for some linear positive
operators of the Szasz-Mirakjan type defined in the space of functions continuous
on \([0, +\infty)\) and having the exponential growth at infinity.
Some approximation properties of these operators are given in [3], [4].

1. Preliminaries

1.1. Let \(R_0 := [0, +\infty), N := \{1, 2, \ldots\}, N_0 := N \cup \{0\}\) and let \(w_r(\cdot), r > 0,\)
be the weight function defined on \(R_0\) by the formula
\[
w_r(x) := e^{-rx}.
\]
Similarly as in [1] we denote by \(C_r, r > 0,\) the space of real-valued
functions \(f\) defined on \(R_0\) and such that \(w_r f\) is a uniformly continuous
and bounded function on \(R_0.\) The norm in \(C_r\) is defined by
\[
\|f\|_r := \sup_{x \in R_0} w_r(x)|f(x)|.
\]
For a fixed \(r > 0\) let
\(C_r^2 := \{f \in C_r : f', f'' \in C_r\}.\)

1.2. In [3] were introduced the following operators of the Szasz-Mirak-
jan type for functions \(f \in C_r, r > 0,\)
\[
I_n^{(1)}(f; x) := \sum_{k=0}^{\infty} p_{n,k}(x) f \left( \frac{2k}{n} \right),
\]
\[
I_n^{(2)}(f; x) := \sum_{k=0}^{\infty} p_{n,k}(x) \frac{n}{2} \int_{I_{n,k}} f(t) dt,
\]

Keywords. Voronowskaya theorem, linear positive operator.
$x \in R_0$, $n \in N$, where

\begin{equation}
\label{4}
p_{n,k}(x) := \frac{1}{\cosh nx} \frac{(nx)^{2k}}{(2k)!}, \quad k \in N_0,
\end{equation}

\sinh x$, \cosh x$, \tanh x$ are the elementary hyperbolic functions and $I_{n,k} := \left[ \frac{2k}{n}, \frac{2k+2}{n} \right]$, $k \in N_0$.

In [4] were introduced the operators

\begin{equation}
\label{5}
L_n^{(3)}(f; x) := \frac{f(0)}{1 + \sinh nx} + \sum_{k=0}^{\infty} q_{n,k}(x) f \left( \frac{2k + 1}{n} \right),
\end{equation}

\begin{equation}
\label{6}
L_n^{(4)}(f; x) := \frac{f(0)}{1 + \sinh nx} + \sum_{k=0}^{\infty} q_{n,k}(x) \frac{n}{2} \int_{I_{n,k}^*} f(t) \, dt,
\end{equation}

$x \in R_0$, $n \in N$, where

\begin{equation}
\label{7}
q_{n,k}(x) := \frac{1}{1 + \sinh nx} \frac{(nx)^{2k+1}}{(2k+1)!},
\end{equation}

and $I_{n,k}^* := \left[ \frac{2k+1}{n}, \frac{2k+3}{n} \right]$ for $k \in N_0$.

We observe that the above operators are linear positive operators well-defined on every space $C_r$, $r > 0$, and

\begin{equation}
\label{8}
L_n^{(i)}(1; x) = 1, \quad 1 \leq i \leq 4,
\end{equation}

for all $x \in R_0$ and $n \in N$.

In [3] and [4] it was proved that $L_n^{(i)}$, $1 \leq i \leq 4$, are operators from $C_r$ into $C_{s}$ for every fixed $s > r > 0$ provided $n$ is large enough. Moreover in [3], [4] some approximation properties of these operators were given. In particular in [3], [4] we proved the following

**Theorem A.** Suppose that $r, s, n_0$ are fixed numbers such that $s > r > 0$, $n_0 \in N$ and $n_0 > r (\ln \frac{3}{r})^{-1}$. If $f \in C_r$, then there exists a positive constant $M_1 \equiv M_1(n_0, r, s)$ depending only on $n_0, r, s$ such that for all $x \in R_0$, $n_0 < n \in N$ and $1 \leq i \leq 4$

\[
\omega_{s}(x) \left| L_n^{(i)}(f; x) - f(x) \right| \leq M_1 \omega \left( f, C_r; \sqrt{\frac{x + 1}{n}} \right),
\]

where $\omega(f; C_r; t)$ is the modulus of continuity of $f$, i.e.,

\[
\omega(f; C_r; t) := \sup_{0 < h \leq t} \| f(\cdot + h) - f(\cdot) \|_r.
\]
2. Auxiliary results

In this part we shall give some properties of the operators $L_n^{(i)}$. Let

$$S(nx) := \frac{\sinh nx}{1 + \sinh nx},$$

$$T(nx) := \frac{\cosh nx}{1 + \sinh nx},$$

$$V(nx) := 1 - \tanh nx,$$

for $n \in \mathbb{N}$ and $x \in R_0$. By elementary calculations from (2)-(8) and (9) we obtain

**Lemma 1.** For all $x \in R_0$ and $n \in \mathbb{N}$ we have

$$L_n^{(1)}(t - x; x) = -x V(nx),$$

$$L_n^{(1)}((t - x)^2; x) = \left( 2x^2 - \frac{x}{n} \right) V(nx) + \frac{x}{n},$$

$$L_n^{(1)}((t - x)^4; x) = \left( 8x^4 - \frac{12x^3}{n} + \frac{4x^2}{n^2} - \frac{x}{n^3} \right) V(nx) + \frac{3x^2}{n^2} + \frac{x}{n^3},$$

$$L_n^{(2)}(t - x; x) = -x V(nx) + \frac{1}{n},$$

$$L_n^{(2)}((t - x)^2; x) = \left( 2x^2 - \frac{3x}{n} \right) V(nx) + \frac{x}{n} + \frac{4}{3n^2},$$

$$L_n^{(2)}((t - x)^4; x) = \left( 8x^4 - \frac{28x^3}{n} + \frac{32x^2}{n^2} - \frac{21x}{n^3} \right) V(nx) + \frac{12x}{n^2} + \frac{16}{5n^4},$$

$$L_n^{(3)}(t - x; x) = x (T(nx) - 1),$$

$$L_n^{(3)}((t - x)^2; x) = x^2 (S(nx) - 2T(nx) + 1) + \frac{x}{n} V(nx),$$

$$L_n^{(3)}((t - x)^4; x) = x^4 (7S(nx) - 8T(nx) + 1) + \frac{12x^3}{n} (T(nx) - S(nx))$$

$$+ \frac{x^2}{n^2} (7S(nx) - 4T(nx)) + \frac{x}{n^2} T(nx),$$

$$L_n^{(4)}(t - x; x) = x (T(nx) - 1) + \frac{1}{n} S(nx),$$

$$L_n^{(4)}((t - x)^2; x) = x^2 (S(nx) - 2T(nx) + 1)$$

$$+ \frac{2x}{n} (T(nx) - S(nx)) + \frac{4}{3n^2} S(nx),$$

$$L_n^{(4)}((t - x)^4; x) = x^4 (7S(nx) - 8T(nx) + 1) + \frac{28x^3}{n} (T(nx) - S(nx))$$

$$+ \frac{x^2}{n^2} (35S(nx) - 32T(nx)) + \frac{17x}{n^3} T(nx).$$
Using Lemma 1, we shall prove two lemmas.

**Lemma 2.** For every fixed $x_0 \in R_0$ one has

\[
\lim_{n \to \infty} nL_n^{(i)}(t - x_0; x_0) = \begin{cases} 
0 & \text{if } i = 1, 3, \\
1 & \text{if } i = 2, 4,
\end{cases}
\]

and

\[
\lim_{n \to \infty} nL_n^{(i)}((t - x_0)^2; x_0) = x_0 \quad \text{for } 1 \leq i \leq 4.
\]

**Proof:** We shall prove only (10) and (11) for $i = 3$, because the proof for $i = 1, 2, 4$ is analogous.

By Lemma 1 and (9) we have

\[
nL_n^{(3)}(t - x; x) = \frac{nx}{e^{2nx}(1 + \sinh nx)} - \frac{nx}{1 + \sinh nx},
\]

\[
nL_n^{(3)}((t - x)^2; x) = \frac{nx^2}{1 + \sinh nx} - \frac{2nx^2}{e^{nx}(1 + \sinh nx)} + \frac{x \cosh nx}{1 + \sinh nx},
\]

for every $x \in R_0$ and $n \in N$, which immediately yield (10) and (11).

**Lemma 3.** For every fixed $x_0 \in R_0$ there exists a positive constant $M_2(x_0)$, depending only on $x_0$, such that for all $n \in N$

\[
L_n^{(i)}((t - x_0)^4; x_0) \leq M_2(x_0)n^{-2}, \quad 1 \leq i \leq 4.
\]

**Proof:** For example we shall prove (12) for $L_n^{(1)}$. By (9) we have for $n \in N, p \in N$ and $x \in R_0$

\[
0 \leq x^p V(nx) = \frac{2x^p}{e^{2nx} + 1} \leq 2^{1-p} p! n^{-p}.
\]

Applying the above inequality to the formula given in Lemma 1, we obtain

\[
L_n^{(1)}((t - x_0)^4; x_0) \leq \frac{47}{n^4} + \frac{3x^2_0}{n^2} + \frac{x_0}{n^4} \leq M_2(x_0)n^{-2},
\]

for every fixed $x_0 \geq 0$ and for all $n \in N$.

The proof of (12) for $i = 2, 3, 4$ is similar.

In the papers [3] (for $L_n^{(i)}, i = 1, 2$) and [4] (for $L_n^{(i)}, i = 3, 4$) we proved the following two lemmas.
Lemma 4. Let $s > r > 0$ and let $n_0$ be a natural number such that
\begin{equation}
n_0 > r \left( \ln \frac{s}{r} \right)^{-1}.
\end{equation}
Then there exists a positive constant $M_3 \equiv M_3(r, s, n_0)$ depending only on $r$, $s$, $n_0$ such that for all $n > n_0$ and $i = 1, 2, 3, 4$
\begin{equation}
\left\| L_n^{(i)} \left( \frac{1}{w_r(t)} ; \cdot \right) \right\|_s \leq M_3.
\end{equation}

Lemma 5. Suppose that $r, s$ and $n_0$ are a numbers as in Lemma 4. Then there exists a positive constant $M_4 \equiv M_4(r, s, n_0)$ depending only on $r$, $s$, $n_0$ such that for all $x \geq 0$, $n > n_0$ and $i = 1, 2, 3, 4$
\begin{equation}
w_s(x) L_n^{(i)} \left( \frac{(t - x)^2}{w_r(t)} ; x \right) \leq M_3 \frac{x + 1}{n}.
\end{equation}

Applying the above lemmas, we shall prove

Lemma 6. Suppose that $x_0$ is a fixed point on $R_0$ and $\varphi(\cdot ; x_0)$ is a function belonging to a give space $C_r$, $r > 0$, such that $\lim_{t \to \infty} \varphi(t; x_0) = 0$, $\lim_{t \to 0^+} \varphi(t; 0) = 0$. Then
\begin{equation}
\lim_{n \to \infty} L_n^{(i)} \left( \varphi(t; x_0) ; x_0 \right) = 0 \quad \text{for} \quad 1 \leq i \leq 4.
\end{equation}

Proof: We shall prove (15) for $i = 1$, because the proof of (15) for $i = 2, 3, 4$ is analogous.

Choose $\varepsilon > 0$ and $M_3$ as in Lemma 4. Then by the properties of $\varphi(\cdot ; x_0)$ there exist positive constants $\delta \equiv \delta (\varepsilon, M_3)$ and $M_5$ such that
\begin{align*}
w_r(t) |\varphi(t; x_0)| &< \frac{\varepsilon}{2 M_3} \quad \text{for} \quad |t - x_0| < \delta, \\
w_r(t) |\varphi(t; x_0)| &< M_5 \quad \text{for} \quad t \geq 0.
\end{align*}

Denoting by $Q_{n,1} := \{ k \in N_0 : |\frac{2k}{n} - x_0| < \delta \}$ and $Q_{n,2} := \{ k \in N_0 : |\frac{2k}{n} - x_0| \geq \delta \}$, we get for $s > r$ and $n > n_0$ by (1)-(4) and Lemma 4
\begin{align*}
w_s(x_0) L_n^{(1)} \left( \varphi(t; x_0) ; x_0 \right) &\leq w_s(x_0) \sum_{k=0}^{\infty} p_{n,k}(x_0) \left| \varphi \left( \frac{2k}{n} ; x_0 \right) \right| \\
&= w_s(x_0) \sum_{k \in Q_{n,1}} p_{n,k}(x_0) \left| \varphi \left( \frac{2k}{n} ; x_0 \right) \right| \\
&\quad + w_s(x_0) \sum_{k \in Q_{n,2}} p_{n,k}(x_0) \left| \varphi \left( \frac{2k}{n} ; x_0 \right) \right| \\
&:= \sum_{1} + \sum_{2}
\end{align*}
and 
\[
\sum_{1} < \frac{\varepsilon}{2M_3} w_s(x_0) \sum_{k=0}^{\infty} p_{n,k}(x_0) \left( w_r \left( \frac{2k}{n} \right) \right)^{-1} < \frac{\varepsilon}{2},
\]
\[
\sum_{2} \leq M_5 w_s(x_0) \sum_{k \in Q_{n,2}} p_{n,k}(x_0) \left( w_r \left( \frac{2k}{n} \right) \right)^{-1}.
\]

Since \(1 \leq \delta^{-2} \left( \frac{2k}{n} - x_0 \right)^2\) if \(|\frac{2k}{n} - x_0| \geq \delta\), we have
\[
\sum_{2} \leq M_5 \delta^{-2} w_s(x_0) \sum_{k \in Q_{n,2}} p_{n,k}(x_0) \left( w_r \left( \frac{2k}{n} \right) \right)^{-1} \left( \frac{2k}{n} - x_0 \right)^2
\]
\[
\leq M_5 \delta^{-2} w_s(x_0) L_n^{(1)} \left( \frac{(t-x_0)^2}{w_r(t)} ; x_0 \right),
\]
which by (14) and (13) yields
\[
\sum_{2} \leq M_5 M_4 \frac{x_0 + 1}{n\delta^2} \text{ for all } n > n_0.
\]

It is obvious that for fixed numbers \(\varepsilon > 0, \delta > 0, M_3 > 0, M_4 > 0,\)

\(n_0 \in N\) and \(x_0 \geq 0\) there exist a natural number \(n_1 > n_0\) depending on

the above parameters such that for all \(n_1 < n \in N\)
\[
M_4 M_5 \frac{x_0 + 1}{n\delta^2} < \frac{\varepsilon}{2}.
\]

Hence we have
\[
\sum_{2} < \frac{\varepsilon}{2} \text{ for all } n > n_1.
\]

Consequently,
\[
w_s(x_0) | L_n^{(1)} (\varphi(t;x_0); x_0) | < \varepsilon \text{ for } n > n_1,
\]
which proves that
\[
\lim_{n \to \infty} w_s(x_0) L_n^{(1)} (\varphi(t;x_0); x_0) = 0.
\]
From this and (1) assertion (15) follows for \(x_0\) and \(i = 1\). Thus the proof

is completed.
3. Theorems of the Voronovskaya type

The Voronovskaya theorem for the Bernstein operators is given in [2]. We shall prove a similar theorem for the operators \( L_n^{(i)} \).

**Theorem 1.** Let \( f \in C^r_2 \) with some \( r > 0 \). Then

\[
\lim_{n \to \infty} n \left\{ L_n^{(i)}(f; x) - f(x) \right\} = \frac{x}{2} f''(x)
\]

for every \( x \in R_0 \) and \( i = 1, 3 \).

**Proof:** Let \( x_0 \geq 0 \) be an arbitrary fixed point and \( i = 1 \). By the Taylor formula we have for \( t \geq 0 \)

\[
f(t) = f(x_0) + f'(x_0)(t-x_0) + \frac{1}{2} f''(x_0)(t-x_0)^2 + \psi(t; x_0)(t-x_0)^2,
\]

where \( \psi(\cdot; x_0) \) is a function belonging to the space \( C_r \) and \( \lim_{t \to x_0} \psi(t; x_0) = 0 \). By (2), (8) and (17) we get

\[
L_n^{(1)}(f(t); x_0) = f(x_0) + f'(x_0) L_n^{(1)}(t - x_0; x_0) + \frac{1}{2} f''(x_0) L_n^{(1)}((t - x_0)^2; x_0) + L_n^{(1)}(\psi(t; x_0)(t-x_0)^2; x_0)
\]

for every \( n \in N \). Using Lemma 2, we have

\[
\lim_{n \to \infty} n L_n^{(1)}(t - x_0; x_0) = 0,
\]

\[
\lim_{n \to \infty} n L_n^{(1)}((t - x_0)^2; x_0) = x_0.
\]

By (2) and the Hölder inequality we have for every \( n \in N \)

\[
\left| L_n^{(1)}(\psi(t; x_0)(t-x_0)^2; x_0) \right| \leq \left\{ L_n^{(1)}(\psi^2(t; x_0); x_0) \right\}^{1/2} \left\{ L_n^{(1)}((t-x_0)^4; x_0) \right\}^{1/2}.
\]

Since for the function \( \varphi(t; x_0) := \psi^2(t; x_0), t \geq 0 \), we have \( \varphi(\cdot; x_0) \in C_{2r} \) and \( \lim_{t \to x_0} \varphi(t; x_0) = 0 \), we get by Lemma 6

\[
\lim_{n \to \infty} L_n^{(1)}(\psi^2(t; x_0); x_0) \equiv \lim_{n \to \infty} L_n^{(1)}(\varphi(t; x_0); x_0) = 0.
\]

Applying (21) and (12) to (20), we obtain

\[
\lim_{n \to \infty} n L_n^{(1)}(\psi(t; x_0)(t-x_0)^2; x_0) = 0.
\]

Now we immediately obtain (16) for a given \( x_0 \) and \( i = 1 \) from (18) by (19) and (22). This proves the desired assertion for \( i = 1 \). 

Similarly we can prove the following
Theorem 2. Suppose that $f \in C^r_r$ with some $r > 0$. Then
\[
\lim_{n \to \infty} n \left\{ L_n^{(i)} (f; x) - f(x) \right\} = f'(x) + \frac{x}{2} f''(x)
\]
for every $x \in \mathbb{R}_0$ and $i = 2, 4$.

References


Institute of Mathematics
Poznań University of Technology
Piotrowo 3A
60-965 Poznań
POLAND

Primera versió rebuda el 9 de Juliol de 1996,
darrera versió rebuda el 18 de Novembre de 1996