ON FINITE ABELIAN GROUPS
REALIZABLE AS MISLIN GENERA

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Abstract

We study the realizability of finite abelian groups as Mislin genera of finitely generated nilpotent groups with finite commutator subgroup. In particular, we give criteria to decide whether a finite abelian group is realizable as the Mislin genus of a direct product of nilpotent groups of a certain specified type. In the case of a positive answer, we also give an effective way of realizing that abelian group as a genus. Further, we obtain some non-realizability results.

1. Introduction

The (Mislin) genus ([5]) of a finitely generated nilpotent group N, denoted \( G(N) \), is the set of isomorphism classes of finitely generated nilpotent groups M having, at each prime \( p \), a localization isomorphic with that of N, i.e. \( M_p \cong N_p \) for all \( p \). It was shown in [2], [5] that \( G(N) \) may be given the structure of a finite abelian group, with the isomorphism class of N as identity element, if the commutator subgroup \([N, N]\) is finite. Thus we are led to study the class \( N_0 \) of finitely generated nilpotent groups with finite commutator subgroup.

No general method has yet been discovered for calculating \( G(N) \) when \( N \in N_0 \). However, in [1], a general method was given if \( N \in N_1 \), where \( N_1 \) is the following subclass of \( N_0 \). Here we describe \( N_1 \) by introducing the short exact sequence

\[
TN \rightarrowtail N \twoheadrightarrow FN
\]

associated with the nilpotent group N, where \( TN \) is the torsion subgroup of N and \( FN \) is the torsion-free quotient. Plainly the class \( N_0 \) is given by the conditions that \( TN \) be finite and \( FN \) free abelian of finite rank.
Then the class $\mathcal{N}_1 \subseteq \mathcal{N}_0$ is given by the supplementary conditions

1. $TN$ is abelian;
2. $TN \to N \to FN$ splits on the right, so that $N$ is the semidirect product for an action $\omega : FN \to \text{Aut} TN$;
3. $\omega(FN)$ lies in the centre of $\text{Aut} TN$.

Moreover, in the presence of (1), condition (3) is equivalent to

(3') given $\xi \in FN$, there exists a positive integer $u$ such that the action of $\xi$ on $TN$ is given by $\xi \cdot a = ua$ for all $a \in TN$ (here, $TN$ is written additively).

Let $t$ be the height of $\ker \omega$ in $FN$, that is,

$$t = \max\{h \in \mathbb{N} \mid \ker \omega \subseteq hFN\}$$

(here, $FN$ is written additively). Then it is shown in [1] that

(1.1) \[ \mathcal{G}(N) \cong (\mathbb{Z}/t)^*/\{\pm 1\}, \]

where $(\mathbb{Z}/t)^*$ is the multiplicative group of units of $\mathbb{Z}/t$. It was further shown how to associate with every unit $m$ of $\mathbb{Z}/t$ a group $N_m$ in the genus of $N$ such that

$N_m \leftrightarrow [m]$ provides an isomorphism (1.1). Moreover, an algorithm was given for calculating $t$, knowing the exponent of $TN$ and the positive integers $u$ referred to in (3').

Unfortunately, the class $\mathcal{N}_1$ is very restricted; indeed, it was shown in [4] that, if a group $N$ in $\mathcal{N}_1$ has non-trivial genus, then $FN$ is cyclic. However, in [3], the systematic calculation of $\mathcal{G}(N)$ was extended from $\mathcal{N}_1$ to the class $\mathcal{N}_2$ consisting of direct products of groups in $\mathcal{N}_1$. It is plain that conditions (1) and (2) for membership of $\mathcal{N}_1$ are inherited by direct products, but, in general, condition (3) is not. Thus the class $\mathcal{N}_2$ is substantially larger than $\mathcal{N}_1$. Of course, membership of $\mathcal{N}_0$ is inherited by direct products.

The calculation of $\mathcal{G}(N)$, for $N$ in $\mathcal{N}_2$, is somewhat technical, but, from our point of view in this paper, the salient facts are the following. First if $N = N_1 \times \cdots \times N_k$ (all $N_i \in \mathcal{N}_1$) and if $FN_i$ is not cyclic for some $i$, then $\mathcal{G}(N)$ is trivial; indeed, we will generalize this result below (see Corollary 2.2). Now assume that $FN_i$ is cyclic for all $i$ ($1 \leq i \leq k$), and, in accordance with (1.1), suppose

\[ \mathcal{G}(N_i) \cong (\mathbb{Z}/t_i)^*/\{\pm 1\}. \]
Let \( t = \gcd(t_1, \ldots, t_k) \) and let \( T_i = \{p_1, \ldots, p_{\lambda_i}\} \), where

\[
(1.2) \quad t = p_1^{\ell_1} \cdots p_{\lambda}^{\ell_\lambda}, \quad \ell_j \geq 1
\]

is the prime power factorization of \( t \). Then we determine from the ‘fine structure’ of the groups \( N_1, \ldots, N_k \) a subset \( P \) of \( T_i \) —any subset of \( T_i \) may arise— so that \( \mathcal{G}(N) \) is determined as follows:

**Theorem 1.1.** The genus \( \mathcal{G}(N) \) is obtained from \((\mathbb{Z}/t)^*\) by factoring out \(-1\) and those residues \( m \mod t \) such that (see (1.2))

\[
m \equiv \begin{cases} 
  \pm 1 \mod p_j^{\ell_j} & \text{if } p_j \in P \\
  1 \mod p_j^{\ell_j} & \text{if } p_j \notin P.
\end{cases}
\]

Notice that \( \mathcal{G}(N) \) is entirely determined by the two invariants \((t, P)\). It is not difficult to show that, given \((t, P)\) with \( P \subseteq T_i \), there is always a group \( N \in \mathcal{N}_2 \) yielding the invariants \((t, P)\) —see Section 4. We remark that if \( k = 1 \), so that \( N \in \mathcal{N}_1 \), then \( P \) is empty. For full details see [3, Theorem 1.6].

Our principal aim in this paper is to describe those finite abelian groups which arise as described in Theorem 1.1 and which can therefore be realized as a (Mislin) genus group \( \mathcal{G}(N) \), for some nilpotent group \( N \) in \( \mathcal{N}_2 \). We will thereby also obtain some non-realizability results.

In Section 2 we obtain some preliminary results which are of independent interest. We adopt the convention that \( \mathbb{Z}/n \) is written \( C_n \) when thought of as a multiplicative group.

**2. Preliminary results**

As in Section 1, let \( N = N_1 \times N_2 \times \cdots \times N_k \), where each \( N_i \in \mathcal{N}_1 \). There is then a function \( \Psi_i : \mathcal{G}(N_i) \rightarrow \mathcal{G}(N) \), given by

\[
(2.1) \quad \Psi_i(M_i) = N_1 \times \cdots \times N_{i-1} \times M_i \times N_{i+1} \times \cdots \times N_k.
\]

**Proposition 2.1.** The function \( \Psi_i \) of (2.1) is a surjective homomorphism.

**Proof:** In accordance with (1.1) we have the short exact sequence

\[
\{\pm 1\} \rightarrow (\mathbb{Z}/t_i)^* \rightarrow \mathcal{G}(N_i)
\]
that can be embedded in the commutative diagram
\[
\begin{array}{c}
\{\pm 1\} \xrightarrow{\alpha} (\mathbb{Z}/t_i)^* \rightarrow G(N_i) \\
\downarrow \quad \downarrow \\
H \xrightarrow{} (\mathbb{Z}/t)^* \rightarrow G(N)
\end{array}
\]

(2.2)

where \( H \) is the subgroup factored out of \( (\mathbb{Z}/t)^* \) to yield \( G(N) \), and \( \alpha(m) = m \mod t \). Then (2.2) may be completed by a homomorphism \( \Psi_i : G(N_i) \rightarrow G(N) \) which will be surjective since \( \alpha \) is surjective. It remains to show that \( \Psi_i \) is given by (2.1). Of course, we may assume that \( FN_i \) is cyclic, say, \( FN_i = \langle \xi_i \rangle \). Then (see [1]) we have a commutative diagram

\[
\begin{array}{c}
TN_i \xrightarrow{} M_i \rightarrow FN_i \\
\| \quad \downarrow \phi_i \quad \downarrow \psi_i \\
TN_i \xrightarrow{} N_i \rightarrow FN_i
\end{array}
\]

where \( \psi_i(\xi_i) = m\xi_i, m \) being given by \( M_i \leftrightarrow [m] \) under the isomorphism (1.1) between \( G(N_i) \) and \( (\mathbb{Z}/t_i)^*/\{\pm 1\} \). We may, and shall, choose \( m \) from its residue class mod \( t_i \) to be prime to the order of \( TN \).

Let \( T \) be the set of primes \( p \) such that \( TN \) has \( p \)-torsion. Set \( M = N_1 \times \cdots \times N_{i-1} \times M_i \times N_{i+1} \times \cdots \times N_k \). The homomorphisms \( \phi_i, \psi_i \) determine, in an obvious way, homomorphisms

\[
\phi : M \rightarrow N, \quad \psi : FN \rightarrow FN,
\]

yielding a commutative diagram

\[
\begin{array}{c}
TN \xrightarrow{} M \rightarrow FN \\
\| \quad \downarrow \phi \quad \downarrow \psi \\
TN \xrightarrow{} N \rightarrow FN
\end{array}
\]

and \( \psi \) is a \( T \)-automorphism with \( \det \psi = m \). Thus, in the bottom row of (2.2), \([m] \in (\mathbb{Z}/t)^*\) goes to \( M \) in \( G(N) \), completing the proof. \( \blacksquare \)

**Corollary 2.2.** Let \( N_1, \ldots, N_k \in \mathbb{N} \) and set \( N = N_1 \times \cdots \times N_k \). Then \( G(N) = 0 \) if \( G(N_i) = 0 \) for any \( i \).

From the explicit description in [3] of the set of primes \( P \) which appears in our statement of Theorem 1.1 the following conclusions are plain.
Proposition 2.3. Any finite abelian group realizable as $G(N)$ with $N \in N_2$, is realizable as $G(N_1 \times N_2)$, where $N_1, N_2 \in N_1$.

This will simplify our choice of examples in Section 3.

Proposition 2.4. Let $N_1, \ldots, N_k \in N_1$ and set $N = N_1 \times \cdots \times N_k$. Then $G(N \times N_j)$ for $1 \leq j \leq k$ is obtained from Theorem 1.1 (applied to $G(N)$) by taking $P$ to be $T_i$ itself. In particular, $G(N \times N_j)$ is independent of $j$.

3. Realizing an abelian group as a Mislin genus

We first enunciate two relevant lemmas on finite abelian groups. For these lemmas we will adopt additive notation; and $p$ will always denote a prime.

Lemma 3.1. Let $G = \bigoplus_{i=1}^\lambda \mathbb{Z}/m_i$, where $m_i = p^{r_i+1}n_i$, $r_i \geq 0$, $p \nmid n_i$, and let $r_1 = \min_i r_i$. Let $a_i$ be a generator of $\mathbb{Z}/m_i$. If $G'$ is obtained from $G$ by adding the relation $\sum_{i=1}^\lambda p^{r_i}a_i = 0$, then

$$G' \cong \bigoplus_{i=1}^\lambda \mathbb{Z}/\overline{m}_i = \langle \overline{a}_1, a_2, \ldots, a_\lambda \rangle,$$

where

$$\overline{m}_i = \begin{cases} p^{r_i}n_1, & i = 1 \\ m_i, & i \geq 2. \end{cases}$$

Proof: We have $\mathbb{Z}/m_1 = \mathbb{Z}/p^{r_1+1} \oplus \mathbb{Z}/n_1 = \langle b_1, c_1 \rangle$, where $b_1 = n_1a_1$, $c_1 = p^{r_1+1}a_1$. Then $p^{r_1}a_1 = p^{r_1}b_1$, so the new relation is given by $p^{r_1}b_1 + \sum_{i=2}^\lambda p^{r_i}a_i = 0$, or

$$p^{r_1} \left( b_1 + \sum_{i=2}^\lambda p^{r_i-r_1}n_ia_i \right) = 0.$$

Set $b' = b_1 + \sum_{i=2}^\lambda p^{r_i-r_1}n_ia_i$. Then

$$G = \langle b', a_1, a_2, \ldots, a_\lambda \rangle = \mathbb{Z}/p^{r_1+1} \oplus \mathbb{Z}/n_1 \oplus \mathbb{Z}/m_2 \oplus \cdots \oplus \mathbb{Z}/m_\lambda,$$

and the new relation is $p^{r_1}b' = 0$. Thus

$$G' = \langle \overline{b'}, a_1, a_2, \ldots, a_\lambda \rangle = \mathbb{Z}/p^{r_1} \oplus \mathbb{Z}/n_1 \oplus \mathbb{Z}/m_2 \oplus \cdots \oplus \mathbb{Z}/m_\lambda,$$

and we set $\overline{a}_1 = \overline{b'} + c_1$. 

Our second lemma is very elementary; the proof will be omitted.
Lemma 3.2. Let $G = \mathbb{Z}/p \oplus B$, where the first summand is generated by $a$, and let $b \in B$ with $pb = 0$. If we obtain $G$ from $G$ by adding the relation $a + b = 0$, then $G \cong B$.

Both these lemmas will be applied with $p = 2$. We now apply Theorem 1.1 to prove our main theorem. We denote the Euler totient function by $\Phi$.

Theorem 3.3. The finite abelian groups which are realizable as the genus of a group in $N_2$ are precisely the groups of the form

$$C_{2^\ell} \times \prod_{p_i \in P} C_{\frac{1}{2}\Phi(p_i^{\ell_i})} \times \prod_{p_j \in Q} C_{\Phi(p_j^{\ell_j})},$$

where $\ell \geq 0$, $\ell_i \geq 1$, $\ell_j \geq 1$ and $P$, $Q$ are disjoint (finite) sets of odd primes.

Proof: We will prove that the finite abelian groups which are realizable as the genus of a group in $N_2$ are precisely those groups which, in multiplicative notation, are obtained through the following process:

Step 1: Take $\prod_{i=1}^{\lambda} C_{\Phi(p_i^{\ell_i})}$, where the $p_i$ are distinct odd primes and $\ell_i \geq 1$.

Step 2: Reduce the order of $\mu$ of the factors $C_{\phi(p_i^{\ell_i})}$ to $\frac{1}{2}\Phi(p_i^{\ell_i})$, $0 \leq \mu \leq \lambda$.

Step 3: Take the direct product of the result of Step 2 with $C_{2^\ell}$, $\ell \geq 0$.

We recall from Theorem 1.1 that $N = N_1 \times \cdots \times N_k$ determines a certain natural number $t$ and that $G(N)$ is obtained from $\mathbb{Z}/t^*$ by factoring out the residue class $-1$ and residue classes $m$ such that $m \equiv \pm 1 \mod p_i^{\ell_i}$ for $p_i \in P$, where $P$ is a certain subset (perhaps empty) of $T_i$, the set of prime divisors of $t = \prod_{i=1}^{\lambda} p_i^{\ell_i}$, and $m \equiv 1 \mod p_i^{\ell_i}$ for $p_i \in T_i - P$. Obviously, this is equivalent to factoring out $-1$ and the residue classes $m_i$, where $p_i \in P$ and

$$m_i \equiv \begin{cases} -1 \mod p_i^{\ell_i} \\ 1 \mod p_j^{\ell_j}, \quad j \neq i. \end{cases}$$

Assume first that $t$ is odd, so that each $p_i$ is odd. Then $\mathbb{Z}/t^*$ is given by Step 1. Factoring out $m_i$ simply reduces $C_{\Phi(p_i^{\ell_i})}$ to $C_{\frac{1}{2}\Phi(p_i^{\ell_i})}$; it follows from Lemma 3.1 that factoring out $-1$ reduces $C_{\phi(p_i^{\ell_i})}$ to $C_{\frac{1}{2}\Phi(p_i^{\ell_i})}$, where
$p_j$ is chosen among the primes in $T_i - P$ to be such that the 2-valuation of $p_j - 1$ is minimal. If $P = T_i$, then this last part of Step 2 is void (because then $-1 = \prod_{p_i \in T_i} m_i$). Step 3 is also void if $t$ is odd, that is, we take $\ell = 0$.

Assume now that $t$ is even. Notice that if $t = 2t'$, with $t'$ odd, then $(\mathbb{Z}/t)^* \cong (\mathbb{Z}/t')^*$ and the process proceeds just as above with $(\mathbb{Z}/t')^*$, using the same subset $P$ and ignoring the prime 2. Thus we may assume that $4 \mid t$; and we change notation to write

$$t = 2^{\ell+2} \prod_{i=1}^\lambda p_i^{\ell_i}, \quad \ell \geq 0.$$ 

Then

$$(3.2) \quad (\mathbb{Z}/t)^* \cong C_2 \times C_2^\ell \times \prod_{i=1}^\lambda C_{\Phi(p_i^{\ell_i})}.$$ 

To pass to $G(N)$, we first factor out the $m_i$ defined as in (3.1) with $p_i \in P$. This is achieved by a partial Step 2 of the process, applied to $\prod_{i=1}^\lambda C_{\Phi(p_i^{\ell_i})}$. If $2 \in P$, we erase $C_2$ on the right of (3.2) and then factor out $-1$ (if $P \neq T_i$) just as in the case of $t$ odd, by reducing the order of a suitable $C_{\Phi(p_i^{\ell_i})}$ with $p_j \in T_i - P$. If $2 \not\in P$, then we apply Lemma 3.2, factoring out $-1$ by effectively erasing $C_2$. We are thus left with the direct product of $C_{2^\ell}, \ell \geq 0$, and the result of Step 2 applied to $\prod_{i=1}^\lambda C_{\Phi(p_i^{\ell_i})}$.

We see, conversely, that every group achieved by executing the three steps is realizable as $G(N)$ with $N \in \mathcal{N}_1$ — but certainly not uniquely. There is not even always a unique pair $(t, P)$ giving rise to a given finite abelian group. However, if the group we want to realize is

$$A = C_2 \phi \times \prod_{p_i \in P} C_{\phi(p_i^{\ell_i})} \times \prod_{p_j \in Q} C_{\phi(p_j^{\ell_j})},$$

where $P, Q$ are disjoint finite sets of odd primes, then we realize $A$ by the pair $(t, P)$, where

$$t = 2^{\ell+2} \prod_{p_i \in P} p_i^{\ell_i} \prod_{p_j \in Q} p_j^{\ell_j},$$

and $2 \not\in P$ (of course, $P$ or $Q$ may be empty). This completes the proof.

We close this section with two observations supplementary to Theorem 3.3. First we characterize those finite abelian groups which can be realized as $G(N)$ for $N \in \mathcal{N}_2$. We recall that this is equivalent to characterizing the finite abelian groups which can be realized as $G(N)$ for $N \in \mathcal{N}_2$ with $P$ empty. This provides the proof of the following.
Proposition 3.4. The finite abelian groups which are realizable as the genus of a group in $\mathcal{N}_1$ are precisely those groups which, in multiplicative notation, are obtained through the following process:

**Step 1:** Take a group $\prod_{i=1}^{\lambda} C_{\Phi(p_i^{\ell_i})}$, where the $p_i$ are distinct odd primes and $\ell_i \geq 1$.

**Step 2:**Either (i) reduce the order of some $C_{\Phi(p_i^{\ell_i})}$ to $\frac{1}{2} \Phi(p_i^{\ell_i})$, where $p_i$ is chosen so that the 2-valuation of $p_i - 1$ is minimal; or (ii) take the direct product with $C_{2^t}$, $t \geq 0$.

Notice that we may simply stop at Step 1.

Our second observation relates to Step 2 in Theorem 3.3. Obviously Step 2 involves factoring out of the group taken in Step 1 an elementary abelian 2-subgroup. An easy extension of Lemma 3.1 establishes

**Theorem 3.5.** If $H$ is any elementary abelian 2-subgroup of the group described in Step 1 of Theorem 3.3, then the quotient of this group by $H$ may be achieved by a suitably chosen Step 2.

4. Examples and supplementary results

We first give some examples of realizability and non-realizability.

**Example 4.1.** We may realize the group $G = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$ as $\mathcal{G}(N)$ for $N \in \mathcal{N}_1$. For, in multiplicative notation, $G = C_4 \times C_{\Phi(3)}$ so $G \cong (\mathbb{Z}/t)^* / \{\pm 1\}$ for $t = 144$. Of course, other values of $t$ will also serve, e.g. $t = 104, 112$. It is shown in [1] or [4] how any $t$ may be realized by a group $N$ in $\mathcal{N}_1$.

**Example 4.2.** We cannot realize the group $\mathbb{Z}/5 \oplus \mathbb{Z}/9$ as $\mathcal{G}(N)$ for $N \in \mathcal{N}_1$. This follows from the fact that 90 is not a value taken by the Euler totient function $\Phi$. For if $\Phi(t) = 90$, then we easily eliminate $t = p, p^2, p^3$ ($p$ odd); but if $t = 2^{\ell+2}p^m$ ($\ell \geq 0$) or $t = mpq$ ($q$ odd), then $4 \mid \Phi(t)$.

On the other hand, we can realize $\mathbb{Z}/5 \oplus \mathbb{Z}/9$ as $\mathcal{G}(N)$ for $N \in \mathcal{N}_2$. For if we start with $C_{\Phi(11)} \times C_{\Phi(19)} = C_{10} \times C_{18}$, we reduce the order of both factors to get $C_5 \times C_9$ and Step 3 is void. This realization amounts to choosing $N$ so that $t = 836$ and $P = \{11, 19\}$. (We will see later how to realize any $(t, P)$ by a group $N$ in $\mathcal{N}_2$).

We next prove a theorem on the realizability of cyclic groups of prime power order.
Theorem 4.3. Let $p$ be a prime number and $m \geq 1$. Then $C_{p^m}$ may be realized as $G(N)$, $N \in \mathcal{N}_2$, if and only if $p = 2$, $p = 3$ or $2p^m + 1$ is prime.

Proof: It is plain that if $p = 2$, $p = 3$, or $2p^m + 1$ is prime, then $C_{p^m}$ may even be realized as $G(N)$ for some $N$ in $\mathcal{N}_1$. To prove the converse, suppose that $C_{p^m}$ is obtained from $(\mathbb{Z}/t)^*$ by factoring out some elementary abelian 2-subgroup $H$. We assume henceforth that $p \neq 2$. Let $t = 2^\ell \prod_{i=1}^{\lambda} p_i^{\ell_i}$, where each $p_i$ is odd and $\ell_i \geq 1$. Since $|G(N)|$ is to be odd, it is clear that $\ell = 0, 1$ or 2 (the case $\ell = 1$ can be ignored in practice) and that all possible reductions of order must take place. Thus

$$C_{p^m} \cong \prod_{i=1}^{\lambda} C_{\frac{2}{2}(p_i - 1)p_i^{\ell_i - 1}}.$$

If any $\ell_i \geq 2$ then (4.1) implies that $p_i = p$ and $\frac{2}{2}(p_i - 1) = 1$, so $p = p_i = 3$. If each $\ell_i = 1$, then each group on the right of (4.1) is a $p$-group, so there can be only one non-trivial factor, say the $i$th factor, yielding $\frac{2}{2}(p_i - 1) = p^m$. Thus $2p^m + 1 = p_i$ is prime. ■

Remarks.

(a) Notice that, in fact, $t$ can only have, at most, two odd prime factors, namely 3 and $2p^m + 1$.

(b) We find a source of genera which are cyclic 2-groups by taking $t$ to be a Fermat prime. Of course, we may take $t$ to be a product of distinct Fermat primes to yield genera which are non-cyclic 2-groups.

(c) Mendelsohn (see [6]) has proved that there exist infinitely many primes $p$ such that $2^np$ is not a value of the $\Phi$-function, for any $n \geq 1$. For such primes $p$, no group of order $2^mp$, $m > 0$, can be realizable.

We close by showing how to realize a pair $(t, P)$, where $P \subseteq T_t$, by a group $N$ in $\mathcal{N}_2$. We first take $P = \emptyset$ and realize $t$ by a group $N$ in $\mathcal{N}_1$. The procedure given in [1] or [4] is as follows. Let $t$ be odd, say, $t = p_1^{\ell_1} \ldots p_\lambda^{\ell_\lambda}$. Set $TN = \mathbb{Z}/n$, where $n = p_1^{\ell_1 + 1} \ldots p_\lambda^{\ell_\lambda + 1}$ and let $FN = \langle \xi \rangle$ act on $TN$ by $\xi \cdot a = ua$, where $u = 1 + p_1 \ldots p_\lambda$. If $N$ is the semidirect product for this action, then $N$ is nilpotent and $G(N) = (\mathbb{Z}/t)^*/\{ \pm 1 \}$. Now let $t$ be even, say $t = 2^\ell p_1^{\ell_1} \ldots p_\lambda^{\ell_\lambda}$. Set $TN = \mathbb{Z}/n$, where $n = 2^{\ell + 2}p_1^{\ell_1 + 1} \ldots p_\lambda^{\ell_\lambda + 1}$ and let $FN = \langle \xi \rangle$ act on $TN$.
by $\xi \cdot a = ua$, where $u = 1 + 4p_1 \ldots p_\lambda$. If $N$ is the semidirect product for this action, then $N$ is nilpotent and $G(N) = (\mathbb{Z}/t)^*/\{\pm 1\}$. Certainly, in both cases, $N \in N_1$.

We now pass to the general case; as indicated earlier, we will be able to realize $(t, P)$ by a group in $N_2$ of the form $N_1 \times N_2$, with $N_1$, $N_2$ in $N_1$. We first realize $t$ just as above by a group $N_1$ in $N_1$. The group $N_2$ is constructed just as $N_1$ except that, for the order $n'$ of $TN_2 = \mathbb{Z}/n'$, we raise the power of those primes outside $P$ (including, perhaps, the prime 2) by 1. Of course, if $P = T$, then this recipe yields $N_2 = N_1$.

**Example 4.4.** Let $t = 165$, $P = \emptyset$. Then we construct $N$ in $N_1$ by taking $TN = \mathbb{Z}/n$, $n = 27225$; and $FN = (\xi)$ acts on $TN$ by $\xi \cdot a = 166a$. We may describe $N$ as

$$N = \langle x, y \mid x^{27225} = 1, yxy^{-1} = x^{166}\rangle.$$ 

Then $G(N) = (\mathbb{Z}/165)^*/\{\pm 1\} = C_4 \times C_{10}$.

Now take $P = \{5\}$. Then we construct $N_1$ as $N$ was constructed above. However, for $N_2$, we replace 27225 by $27225 \cdot 33 = 898425$. Then

$$G(N_1 \times N_2) = (\mathbb{Z}/165)^*/\{-1, 34\} = C_2 \times C_{10}.$$ 

**References**