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ON FINITE ABELIAN GROUPS REALIZABLE AS MISLIN GENERA

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Abstract

We study the realizability of finite abelian groups as Mislin genera of finitely generated nilpotent groups with finite commutator subgroup. In particular, we give criteria to decide whether a finite abelian group is realizable as the Mislin genus of a direct product of nilpotent groups of a certain specified type. In the case of a positive answer, we also give an effective way of realizing that abelian group as a genus. Further, we obtain some non-realizability results.

1. Introduction

The (Mislin) genus ([**5**]) of a finitely generated nilpotent group N, denoted $\mathcal{G}(N)$, is the set of isomorphism classes of finitely generated nilpotent groups M having, at each prime p, a localization isomorphic with that of N, i.e. $M_p \cong N_p$ for all p. It was shown in [**2**], [**5**] that $\mathcal{G}(N)$ may be given the structure of a finite abelian group, with the isomorphism class of N as identity element, if the commutator subgroup $[N, N]$ is finite. Thus we are led to study the class $\mathcal{N}_0$ of finitely generated nilpotent groups with finite commutator subgroup.

No general method has yet been discovered for calculating $\mathcal{G}(N)$ when $N \in \mathcal{N}_0$. However, in [**1**], a general method was given if $N \in \mathcal{N}_1$, where $\mathcal{N}_1$ is the following subclass of $\mathcal{N}_0$. Here we describe $\mathcal{N}_1$ by introducing the short exact sequence

$$TN \rightarrowtail N \twoheadrightarrow FN$$

associated with the nilpotent group N, where TN is the torsion subgroup of N and FN is the torsion-free quotient. Plainly the class $\mathcal{N}_0$ is given by the conditions that TN be finite and FN free abelian of finite rank.

Then the class $\mathcal{N}_1 \subseteq \mathcal{N}_0$ is given by the supplementary conditions

(1) TN is abelian;

(2) $TN \rightarrowtail N \twoheadrightarrow FN$ splits on the right, so that N is the semidirect product for an action $\omega : FN \to \operatorname{Aut} TN$;

(3) $\omega(FN)$ lies in the centre of $\operatorname{Aut} TN$.

Moreover, in the presence of (1), condition (3) is equivalent to

(3') given $\xi \in FN$, there exists a positive integer u such that the action of ξ on TN is given by $\xi \cdot a = ua$ for all $a \in TN$ (here, TN is written additively).

Let t be the height of $\ker \omega$ in FN, that is,

$$t = \max\{h \in \mathbb{N} \mid \ker \omega \subseteq hFN\}$$

(here, FN is written additively). Then it is shown in [**1**] that

$$\mathcal{G}(N) \cong (\mathbb{Z}/t)^*/\{\pm 1\}, \tag{1.1}$$

where $(\mathbb{Z}/t)^*$ is the multiplicative group of units of $\mathbb{Z}/t$. It was further shown how to associate with every unit m of $\mathbb{Z}/t$ a group N_m in the genus of N such that

$$N_m \leftrightarrow [m]$$

provides an isomorphism (1.1). Moreover, an algorithm was given for calculating t, knowing the exponent of TN and the positive integers u referred to in (3').

Unfortunately, the class $\mathcal{N}_1$ is very restricted; indeed, it was shown in [**4**] that, if a group N in $\mathcal{N}_1$ has non-trivial genus, then FN is cyclic. However, in [**3**], the systematic calculation of $\mathcal{G}(N)$ was extended from $\mathcal{N}_1$ to the class $\mathcal{N}_2$ consisting of direct products of groups in $\mathcal{N}_1$. It is plain that conditions (1) and (2) for membership of $\mathcal{N}_1$ are inherited by direct products, but, in general, condition (3) is not. Thus the class $\mathcal{N}_2$ is substantially larger than $\mathcal{N}_1$. Of course, membership of $\mathcal{N}_0$ is inherited by direct products.

The calculation of $\mathcal{G}(N)$, for N in $\mathcal{N}_2$, is somewhat technical, but, from our point of view in this paper, the salient facts are the following. First if $N = N_1 \times \cdots \times N_k$ (all $N_i \in \mathcal{N}_1$) and if FN_i is not cyclic for some i, then $\mathcal{G}(N)$ is trivial; indeed, we will generalize this result below (see Corollary 2.2). Now assume that FN_i is cyclic for all i ($1 \le i \le k$), and, in accordance with (1.1), suppose

$$\mathcal{G}(N_i) \cong (\mathbb{Z}/t_i)^*/\{\pm 1\}.$$

Let $t = \gcd(t_1, \dots, t_k)$ and let $T_t = \{p_1, \dots, p_\lambda\}$, where

$$t = p_1^{\ell_1} \dots p_\lambda^{\ell_\lambda}, \quad \ell_j \geq 1 \tag{1.2}$$

is the prime power factorization of t. Then we determine from the 'fine structure' of the groups $N_1, \dots, N_k$ a subset P of T_t —any subset of T_t may arise— so that $\mathcal{G}(N)$ is determined as follows:

Theorem 1.1. *The genus $\mathcal{G}(N)$ is obtained from $(\mathbb{Z}/t)^*$ by factoring out -1 and those residues $m \bmod t$ such that (see (1.2))*

$$m \equiv \begin{cases} \pm 1 \mod p_j^{\ell_j} & \text{if } p_j \in P \\ 1 \mod p_j^{\ell_j} & \text{if } p_j \notin P. \end{cases}$$

Notice that $\mathcal{G}(N)$ is entirely determined by the two invariants (t, P). It is not difficult to show that, given (t, P) with $P \subseteq T_t$, there is always a group N in $\mathcal{N}_2$ yielding the invariants (t, P) —see Section 4. We remark that if $k = 1$, so that $N \in \mathcal{N}_1$, then P is empty. For full details see [**3**, Theorem 1.6].

Our principal aim in this paper is to describe those finite abelian groups which arise as described in Theorem 1.1 and which can therefore be realized as a (Mislin) genus group $\mathcal{G}(N)$, for some nilpotent group N in $\mathcal{N}_2$. We will thereby also obtain some non-realizability results.

In Section 2 we obtain some preliminary results which are of independent interest. We adopt the convention that $\mathbb{Z}/n$ is written C_n when thought of as a multiplicative group.

2. Preliminary results

As in Section 1, let $N = N_1 \times N_2 \times \cdots \times N_k$, where each $N_i \in \mathcal{N}_1$. There is then a function $\Psi_i : \mathcal{G}(N_i) \longrightarrow \mathcal{G}(N)$, given by

$$\Psi_i(M_i) = N_1 \times \cdots \times N_{i-1} \times M_i \times N_{i+1} \times \cdots \times N_k. \tag{2.1}$$

Proposition 2.1. *The function Ψ_i of (2.1) is a surjective homomorphism.*

Proof: In accordance with (1.1) we have the short exact sequence

$$\{\pm 1\} \rightarrowtail (\mathbb{Z}/t_i)^* \twoheadrightarrow \mathcal{G}(N_i)$$

that can be embedded in the commutative diagram

$$
\begin{tikzcd}
\{\pm 1\} \arrow[r, tail] \arrow[d] & (\mathbb{Z}/t_i)^* \arrow[r, two heads] \arrow[d, "\alpha"'] & \mathcal{G}(N_i) \\
H \arrow[r, tail] & (\mathbb{Z}/t)^* \arrow[r, two heads] & \mathcal{G}(N)
\end{tikzcd}
\tag{2.2}
$$

where H is the subgroup factored out of $(\mathbb{Z}/t)^*$ to yield $\mathcal{G}(N)$ and $\alpha(m) = m \mod t$. Then (2.2) may be completed by a homomorphism $\Psi_i : \mathcal{G}(N_i) \longrightarrow \mathcal{G}(N)$ which will be surjective since α is surjective. It remains to show that Ψ_i is given by (2.1). Of course, we may assume that FN_i is cyclic, say, $FN_i = \langle \xi_i \rangle$. Then (see [**1**]) we have a commutative diagram

$$
\begin{tikzcd}
TN_i \arrow[r, tail] \arrow[d, equal] & M_i \arrow[r, two heads] \arrow[d, "\phi_i"] & FN_i \arrow[d, "\psi_i"] \\
TN_i \arrow[r, tail] & N_i \arrow[r, two heads] & FN_i
\end{tikzcd}
$$

where $\psi_i(\xi_i) = m\xi_i$, m being given by $M_i \leftrightarrow [m]$ under the isomorphism (1.1) between $\mathcal{G}(N_i)$ and $(\mathbb{Z}/t_i)^*/\{\pm 1\}$. We may, and shall, choose m from its residue class mod t_i to be prime to the order of TN. Let T be the set of primes p such that TN has p-torsion. Set $M = N_1 \times \cdots \times N_{i-1} \times M_i \times N_{i+1} \times \cdots \times N_k$. The homomorphisms ϕ_i, ψ_i determine, in an obvious way, homomorphisms

$$\phi : M \longrightarrow N, \quad \psi : FN \longrightarrow FN,$$

yielding a commutative diagram

$$
\begin{tikzcd}
TN \arrow[r, tail] \arrow[d, equal] & M \arrow[r, two heads] \arrow[d, "\phi"] & FN \arrow[d, "\psi"] \\
TN \arrow[r, tail] & N \arrow[r, two heads] & FN
\end{tikzcd}
$$

and ψ is a T-automorphism with $\det \psi = m$. Thus, in the bottom row of (2.2), $[m] \in (\mathbb{Z}/t)^*$ goes to M in $\mathcal{G}(N)$, completing the proof. ∎

Corollary 2.2. *Let $N_1, \ldots, N_k \in \mathcal{N}_1$ and set $N = N_1 \times \cdots \times N_k$. Then $\mathcal{G}(N) = 0$ if $\mathcal{G}(N_i) = 0$ for any i.*

From the explicit description in [**3**] of the set of primes P which appears in our statement of Theorem 1.1 the following conclusions are plain.

Proposition 2.3. *Any finite abelian group realizable as $\mathcal{G}(N)$ with $N \in \mathcal{N}_2$, is realizable as $\mathcal{G}(N_1 \times N_2)$, where $N_1, N_2 \in \mathcal{N}_1$.*

This will simplify our choice of examples in Section 3.

Proposition 2.4. *Let $N_1, \dots, N_k \in \mathcal{N}_1$ and set $N = N_1 \times \cdots \times N_k$. Then $\mathcal{G}(N \times N_j)$ for $1 \le j \le k$ is obtained from Theorem 1.1 (applied to $\mathcal{G}(N)$) by taking P to be T_t itself. In particular, $\mathcal{G}(N \times N_j)$ is independent of j.*

3. Realizing an abelian group as a Mislin genus

We first enunciate two relevant lemmas on finite abelian groups. For these lemmas we will adopt additive notation; and p will always denote a prime.

Lemma 3.1. *Let $G = \bigoplus_{i=1}^{\lambda} \mathbb{Z}/m_i$, where $m_i = p^{r_i+1} n_i$, $r_i \ge 0$, $p \nmid n_i$, and let $r_1 = \min_i r_i$. Let a_i be a generator of $\mathbb{Z}/m_i$. If $\overline{G}$ is obtained from G by adding the relation $\sum_{i=1}^{\lambda} p^{r_i} n_i a_i = 0$, then*

$$\overline{G} \cong \bigoplus_{i=1}^{\lambda} \mathbb{Z}/\overline{m}_i = \langle \overline{a}_1, a_2, \dots, a_\lambda \rangle,$$

where

$$\overline{m}_i = \begin{cases} p^{r_1} n_1, & i = 1 \\ m_i, & i \ge 2. \end{cases}$$

Proof: We have $\mathbb{Z}/m_1 = \mathbb{Z}/p^{r_1+1} \oplus \mathbb{Z}/n_1 = \langle b_1, c_1 \rangle$, where $b_1 = n_1 a_1$, $c_1 = p^{r_1+1} a_1$. Then $p^{r_1} n_1 a_1 = p^{r_1} b_1$, so the new relation is given by $p^{r_1} b_1 + \sum_{i=2}^{\lambda} p^{r_i} n_i a_i = 0$, or

$$p^{r_1} \left(b_1 + \sum_{i=2}^{\lambda} p^{r_i - r_1} n_i a_i \right) = 0.$$

Set $b' = b_1 + \sum_{i=2}^{\lambda} p^{r_i - r_1} n_i a_i$. Then

$$G = \langle b', c_1, a_2, \dots, a_\lambda \rangle = \mathbb{Z}/p^{r_1+1} \oplus \mathbb{Z}/n_1 \oplus \mathbb{Z}/m_2 \oplus \cdots \oplus \mathbb{Z}/m_\lambda,$$

and the new relation is $p^{r_1} b' = 0$. Thus

$$\overline{G} = \langle \overline{b'}, c_1, a_2, \dots, a_\lambda \rangle = \mathbb{Z}/p^{r_1} \oplus \mathbb{Z}/n_1 \oplus \mathbb{Z}/m_2 \oplus \cdots \oplus \mathbb{Z}/m_\lambda,$$

and we set $\overline{a}_1 = \overline{b'} + c_1$. ■

Our second lemma is very elementary; the proof will be omitted.

Lemma 3.2. *Let $G = \mathbb{Z}/p \oplus B$, where the first summand is generated by a, and let $b \in B$ with $pb = 0$. If we obtain $\overline{G}$ from G by adding the relation $a + b = 0$, then $\overline{G} \cong B$.*

Both these lemmas will be applied with $p = 2$. We now apply Theorem 1.1 to prove our main theorem. We denote the Euler totient function by Φ.

Theorem 3.3. *The finite abelian groups which are realizable as the genus of a group in $\mathcal{N}_2$ are precisely the groups of the form*

$$C_{2^\ell} \times \prod_{p_i \in P} C_{\frac{1}{2}\Phi(p_i^{\ell_i})} \times \prod_{p_j \in Q} C_{\Phi(p_j^{\ell_j})},$$

where $\ell \geq 0$, $\ell_i \geq 1$, $\ell_j \geq 1$ and P, Q are disjoint (finite) sets of odd primes.

Proof: We will prove that the finite abelian groups which are realizable as the genus of a group in $\mathcal{N}_2$ are precisely those groups which, in multiplicative notation, are obtained through the following process:

> Step 1: Take $\prod_{i=1}^{\lambda} C_{\Phi(p_i^{\ell_i})}$, where the p_i are distinct odd primes and $\ell_i \geq 1$.
> Step 2: Reduce the order of μ of the factors $C_{\Phi(p_i^{\ell_i})}$ to $\frac{1}{2}\Phi(p_i^{\ell_i})$, $0 \leq \mu \leq \lambda$.
> Step 3: Take the direct product of the result of Step 2 with C_{2^ℓ}, $\ell \geq 0$.

We recall from Theorem 1.1 that $N = N_1 \times \cdots \times N_k$ determines a certain natural number t and that $\mathcal{G}(N)$ is obtained from $(\mathbb{Z}/t)^*$ by factoring out the residue class -1 and residue classes m such that $m \equiv \pm 1 \mod p_i^{\ell_i}$ for $p_i \in P$, where P is a certain subset (perhaps empty) of T_t, the set of prime divisors of $t = \prod_{i=1}^{\lambda} p_i^{\ell_i}$, and $m \equiv 1 \mod p_i^{\ell_i}$ for $p_i \in T_t - P$. Obviously, this is equivalent to factoring out -1 and the residue classes m_i, where $p_i \in P$ and

$$m_i \equiv \begin{cases} -1 \mod p_i^{\ell_i} \\ 1 \mod p_j^{\ell_j}, \quad j \neq i. \end{cases} \tag{3.1}$$

Assume first that t is odd, so that each p_i is odd. Then $(\mathbb{Z}/t)^*$ is given by Step 1. Factoring out m_i simply reduces $C_{\Phi(p_i^{\ell_i})}$ to $C_{\frac{1}{2}\Phi(p_i^{\ell_i})}$; it follows from Lemma 3.1 that factoring out -1 reduces $C_{\Phi(p_j^{\ell_j})}$ to $C_{\frac{1}{2}\Phi(p_j^{\ell_j})}$, where

p_j is chosen among the primes in $T_t - P$ to be such that the 2-valuation of $p_j - 1$ is minimal. If $P = T_t$, then this last part of Step 2 is void (because then $-1 = \prod_{p_i \in T_t} m_i$). Step 3 is also void if t is odd, that is, we take $\ell = 0$.

Assume now that t is even. Notice that if $t = 2t'$, with t' odd, then $(\mathbb{Z}/t)^* \cong (\mathbb{Z}/t')^*$ and the process proceeds just as above with $(\mathbb{Z}/t')^*$, using the same subset P and ignoring the prime 2. Thus we may assume that $4 \mid t$; and we change notation to write

$$t = 2^{\ell+2} \prod_{i=1}^{\lambda} p_i^{\ell_i}, \quad \ell \geq 0.$$

Then

$$(\mathbb{Z}/t)^* \cong C_2 \times C_{2^\ell} \times \prod_{i=1}^{\lambda} C_{\Phi(p_i^{\ell_i})}. \tag{3.2}$$

To pass to $\mathcal{G}(N)$, we first factor out the m_i defined as in (3.1) with $p_i \in P$. This is achieved by a partial Step 2 of the process, applied to $\prod_{i=1}^{\lambda} C_{\Phi(p_i^{\ell_i})}$. If $2 \in P$, we erase C_2 on the right of (3.2) and then factor out -1 (if $P \neq T_t$) just as in the case of t odd, by reducing the order of a suitable $C_{\Phi(p_j^{\ell_j})}$ with $p_j \in T_t - P$. If $2 \notin P$, then we apply Lemma 3.2, factoring out -1 by effectively erasing C_2. We are thus left with the direct product of C_{2^ℓ}, $\ell \geq 0$, and the result of Step 2 applied to $\prod_{i=1}^{\lambda} C_{\Phi(p_i^{\ell_i})}$.

We see, conversely, that every group achieved by executing the three steps is realizable as $\mathcal{G}(N)$ with $N \in \mathcal{N}_2$ —but certainly not uniquely. There is not even always a unique pair (t, P) giving rise to a given finite abelian group. However, if the group we want to realize is

$$A = C_{2^\ell} \times \prod_{p_i \in P} C_{\frac{1}{2}\Phi(p_i^{\ell_i})} \times \prod_{p_j \in Q} C_{\Phi(p_j^{\ell_j})},$$

where P, Q are disjoint finite sets of odd primes, then we realize A by the pair (t, P), where

$$t = 2^{\ell+2} \prod_{p_i \in P} p_i^{\ell_i} \prod_{p_j \in Q} p_j^{\ell_j},$$

and $2 \notin P$ (of course, P or Q may be empty). This completes the proof. ∎

We close this section with two observations supplementary to Theorem 3.3. First we characterize those finite abelian groups which can be realized as $\mathcal{G}(N)$ for N in $\mathcal{N}_1$. We recall that this is equivalent to characterizing the finite abelian groups which can be realized as $\mathcal{G}(N)$ for $N \in \mathcal{N}_2$ with P empty. This provides the proof of the following.

Proposition 3.4. *The finite abelian groups which are realizable as the genus of a group in $\mathcal{N}_1$ are precisely those groups which, in multiplicative notation, are obtained through the following process:*

Step 1: Take a group $\prod_{i=1}^{\lambda} C_{\Phi(p_i^{\ell_i})}$, where the p_i are distinct odd primes and $\ell_i \geq 1$.

Step 2: Either (i) reduce the order of some $C_{\Phi(p_i^{\ell_i})}$ to $\frac{1}{2}\Phi(p_i^{\ell_i})$, where p_i is chosen so that the 2-valuation of $p_i - 1$ is minimal; or (ii) take the direct product with C_{2^ℓ}, $\ell \geq 0$.

Notice that we may simply stop at Step 1.

Our second observation relates to Step 2 in Theorem 3.3. Obviously Step 2 involves factoring out of the group taken in Step 1 an elementary abelian 2-subgroup. An easy extension of Lemma 3.1 establishes

Theorem 3.5. *If H is any elementary abelian 2-subgroup of the group described in Step 1 of Theorem 3.3, then the quotient of this group by H may be achieved by a suitably chosen Step 2.*

4. Examples and supplementary results

We first give some examples of realizability and non-realizability.

Example 4.1. We may realize the group $G = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$ as $\mathcal{G}(N)$ for $N \in \mathcal{N}_1$. For, in multiplicative notation, $G = C_4 \times C_{\Phi(3^2)}$ so $G \cong (\mathbb{Z}/t)^*/\{\pm 1\}$ for $t = 144$. Of course, other values of t will also serve, e.g. $t = 104, 112$. It is shown in [**1**] or [**4**] how any t may be realized by a group N in $\mathcal{N}_1$.

Example 4.2. We cannot realize the group $\mathbb{Z}/5 \oplus \mathbb{Z}/9$ as $\mathcal{G}(N)$ for $N \in \mathcal{N}_1$. This follows from the fact that 90 is not a value taken by the Euler totient function Φ. For if $\Phi(t) = 90$, then we easily eliminate $t = p, p^2, p^3$ (p odd); but if $t = 2^{\ell+2}p^m$ ($\ell \geq 0$) or $t = mpq$ (q odd), then $4 \mid \Phi(t)$.

On the other hand, we can realize $\mathbb{Z}/5 \oplus \mathbb{Z}/9$ as $\mathcal{G}(N)$ for $N \in \mathcal{N}_2$. For if we start with $C_{\Phi(11)} \times C_{\Phi(19)} = C_{10} \times C_{18}$, we reduce the order of both factors to get $C_5 \times C_9$ and Step 3 is void. This realization amounts to choosing N so that $t = 836$ and $P = \{11, 19\}$. (We will see later how to realize any (t, P) by a group N in $\mathcal{N}_2$).

We next prove a theorem on the realizability of cyclic groups of prime power order.

Theorem 4.3. *Let p be a prime number and $m \geq 1$. Then C_{p^m} may be realized as $\mathcal{G}(N)$, $N \in \mathcal{N}_2$, if and only if $p = 2$, $p = 3$ or $2p^m + 1$ is prime.*

Proof: It is plain that if $p = 2$, $p = 3$, or $2p^m + 1$ is prime, then C_{p^m} may even be realized as $\mathcal{G}(N)$ for some N in $\mathcal{N}_1$. To prove the converse, suppose that C_{p^m} is obtained from $(\mathbb{Z}/t)^*$ by factoring out some elementary abelian 2-subgroup H. We assume henceforth that $p \neq 2$. Let $t = 2^\ell \prod_{i=1}^{\lambda} p_i^{\ell_i}$, where each p_i is odd and $\ell_i \geq 1$. Since $|\mathcal{G}(N)|$ is to be odd, it is clear that $\ell = 0, 1$ or 2 (the case $\ell = 1$ can be ignored in practice) and that all possible reductions of order must take place. Thus

$$C_{p^m} \cong \prod_{i=1}^{\lambda} C_{\frac{1}{2}(p_i-1)p_i^{\ell_i-1}}. \tag{4.1}$$

If any $\ell_i \geq 2$ then (4.1) implies that $p_i = p$ and $\frac{1}{2}(p_i - 1) = 1$, so $p = p_i = 3$. If each $\ell_i = 1$, then each group on the right of (4.1) is a p-group, so there can be only one non-trivial factor, say the ith factor, yielding $\frac{1}{2}(p_i - 1) = p^m$. Thus $2p^m + 1 = p_i$ is prime. ∎

Remarks.

(a) Notice that, in fact, t can only have, at most, two odd prime factors, namely 3 and $2p^m + 1$.

(b) We find a source of genera which are cyclic 2-groups by taking t to be a Fermat prime. Of course, we may take t to be a product of distinct Fermat primes to yield genera which are non-cyclic 2-groups.

(c) Mendelsohn (see [**6**]) has proved that there exist infinitely many primes p such that $2^n p$ is not a value of the Φ-function, for any $n \geq 1$. For such primes p, no group of order $2^m p$, $m > 0$, can be realizable.

We close by showing how to realize a pair (t, P), where $P \subseteq T_t$, by a group N in $\mathcal{N}_2$. We first take $P = \varnothing$ and realize t by a group N in $\mathcal{N}_1$. The procedure given in [**1**] or [**4**] is as follows. Let t be odd, say, $t = p_1^{\ell_1} \dots p_\lambda^{\ell_\lambda}$. Set $TN = \mathbb{Z}/n$, where $n = p_1^{\ell_1+1} \dots p_\lambda^{\ell_\lambda+1}$ and let $FN = \langle \xi \rangle$ act on TN by $\xi \cdot a = ua$, where $u = 1 + p_1 \dots p_\lambda$. If N is the semidirect product for this action, then N is nilpotent and $\mathcal{G}(N) = (\mathbb{Z}/t)^*/\{\pm 1\}$. Now let t be even, say $t = 2^\ell p_1^{\ell_1} \dots p_\lambda^{\ell_\lambda}$. Set $TN = \mathbb{Z}/n$, where $n = 2^{\ell+2} p_1^{\ell_1+1} \dots p_\lambda^{\ell_\lambda+1}$ and let $FN = \langle \xi \rangle$ act on TN

by $\xi \cdot a = ua$, where $u = 1 + 4p_1 \dots p_\lambda$. If N is the semidirect product for this action, then N is nilpotent and $\mathcal{G}(N) = (\mathbb{Z}/t)^*/\{\pm 1\}$. Certainly, in both cases, $N \in \mathcal{N}_1$.

We now pass to the general case; as indicated earlier, we will be able to realize (t, P) by a group in $\mathcal{N}_2$ of the form $N_1 \times N_2$, with N_1, N_2 in $\mathcal{N}_1$. We first realize t just as above by a group N_1 in $\mathcal{N}_1$. The group N_2 is constructed just as N_1 except that, for the order n' of $TN_2 = \mathbb{Z}/n'$, we raise the power of those primes outside P (including, perhaps, the prime 2) by 1. Of course, if $P = T_t$, then this recipe yields $N_2 = N_1$.

Example 4.4. Let $t = 165$, $P = \varnothing$. Then we construct N in $\mathcal{N}_1$ by taking $TN = \mathbb{Z}/n$, $n = 27225$; and $FN = \langle \xi \rangle$ acts on TN by $\xi \cdot a = 166a$. We may describe N as

$$N = \langle x, y \mid x^{27225} = 1, yxy^{-1} = x^{166} \rangle.$$

Then $\mathcal{G}(N) = (\mathbb{Z}/165)^*/\{\pm 1\} = C_4 \times C_{10}$.

Now take $P = \{5\}$. Then we construct N_1 as N was constructed above. However, for N_2, we replace 27225 by $27225 \cdot 33 = 898425$. Then

$$\mathcal{G}(N_1 \times N_2) = (\mathbb{Z}/165)^*/\langle -1, 34 \rangle = C_2 \times C_{10}.$$

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