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P-NILPOTENT COMPLETION IS NOT IDEMPOTENT

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Abstract

Let P be an arbitrary set of primes. The P-nilpotent completion of a group G is defined by the group homomorphism $\eta : G \to G_{\widehat{P}}$ where $G_{\widehat{P}} = \operatorname{invlim}(G/\Gamma_i G)_P$. Here $\Gamma_2 G$ is the commutator subgroup $[G, G]$ and $\Gamma_i G$ the subgroup $[G, \Gamma_{i-1} G]$ when $i > 2$. In this paper, we prove that P-nilpotent completion of an infinitely generated free group F does not induce an isomorphism on the first homology group with $\mathbf{Z}_P$ coefficients. Hence, P-nilpotent completion is not idempotent. Another important consequence of the result in homotopy theory (as in [**4**]) is that any infinite wedge of circles is R-bad, where R is any subring of rationals.

1. Introduction

For a group G, we denote by $\Gamma_2 G$ the commutator subgroup $[G, G]$ and $\Gamma_i G$ the subgroup $[G, \Gamma_{i-1} G]$ when $i > 2$. A group G is *nilpotent* if $\Gamma_i(G)$ is trivial for some i. The *nilpotency class nil*(G) of G is the least c such that $\Gamma_c(G)$ is trivial. Let P be a set of prime numbers. There is a well-known P-localization in the category of nilpotent groups, [**7**]. We denote this localization on a nilpotent group N by $e : N \to N_P$.

The P-nilpotent completion or $\mathbf{Z}_P$-completion of a group G is defined to be the group homomorphism $\eta : G \to G_{\widehat{P}}$ where $G_{\widehat{P}} = \operatorname{invlim}(G/\Gamma_i G)_P$, with i running through all finite ordinals. For each i, the group homomorphism $G \to (G/\Gamma_i G)_P$ defines a localization on the category $\mathcal{G}$ of groups. Its universal property gives rise to a natural map $(G_{\widehat{P}}/\Gamma_i G_{\widehat{P}})_P \to (G/\Gamma_i G)_P$. Passing to inverse limit, we obtain a natural transformation $\chi : (G_{\widehat{P}})_{\widehat{P}} \to G_{\widehat{P}}$ so that $((\ )_{\widehat{P}}, \eta, \chi)$ is a monad on $\mathcal{G}$.

Let F be a free group on an infinitely countable set of generators. In [**4**, Proposition IV.5.4], it is proved that the abelianization of $\eta : F \to F_{\hat{\mathbf{Z}}} = \operatorname{invlim}(F/\Gamma_i F)$ is not an isomorphism. This result is used to verify that $\mathbf{Z}$-completion (which is P-nilpotent completion when P is the set of all primes) is not idempotent in [**3**].

We study these proofs closely and obtain a similar proof of the non-idempotence of P-nilpotent completion for any set P of primes. We use results from orthogonal pairs, idempotent monads and the P-localization on the category of nilpotent groups.

Although the P-nilpotent completion is not idempotent on the category of groups, a procedure to obtain an idempotent monad from it is described in [**5**]. This turns out to be the minimal P-localization, which is also obtained in [**2**]. It is the "smallest" (in the sense that it provides the least local objects) idempotent monad which extends P-localization on the category of nilpotent groups to the category of groups. This minimal P-localization coincides with the P-nilpotent completion on groups which have finitely generated abelianization [**3**] and groups with stable lower central series [**2**].

2. The P-nilpotent completion is not idempotent

Let $\mathcal{C}$ be a category, X be an object of $\mathcal{C}$ and $f : A \to B$ be a morphism of $\mathcal{C}$. Then X and f are said to be *orthogonal* to each other, denoted by $X \perp f$ or $f \perp X$, if $f^* : \mathcal{C}(B, X) \cong \mathcal{C}(A, X)$. For a class D of objects in $\mathcal{C}$, the *orthogonal complement* of D in $\mathcal{C}$, denoted by $D^\perp$, is the class of morphisms orthogonal to every object in D. Dually, the orthogonal complement of S can be defined for a class S of morphisms. An *orthogonal pair* (S, D) in $\mathcal{C}$ comprises a collection S of morphisms in $\mathcal{C}$ and a collection D of objects in $\mathcal{C}$ satisfying $S = D^\perp$ and $D = S^\perp$. Every idempotent monad (see [**8**, p. 133]) (also known as localization in [**6**]) is associated with a unique orthogonal pair.

Let $a_1, a_2, \ldots$ be elements of a group G. We define $[a_1, a_2] = a_1^{-1}a_2^{-1}a_1a_2$ and $[a_1, a_2, \ldots, a_k] = [[a_1, \ldots, a_{k-1}], a_k]$ recursively for $k \geq 3$.

Proposition 1. *Let F be the free group on $a_1, \ldots, a_k$. For every positive integer n, $[a_1, \ldots, a_k]^n$ does not belong to the subgroup of $\Gamma_2 F$ that is generated by $\Gamma_{k+1}F$ and $\Gamma_2\Gamma_2 F$.*

Proof: Replacing F by the quotient $F/\langle \Gamma_{k+1}F, \Gamma_2\Gamma_2F\rangle$, the proposition becomes: for each n, there exists a group G with the following properties: (i) The commutator subgroup $\Gamma_2 G$ is abelian, (ii) G is nilpotent of class $k+1$, and (iii) there exists $x \in \Gamma_k G$ such that $x^n \neq 1$.

However, it is enough to pick a prime p that does not divide n and find a p-group G such that $\Gamma_2 G$ is abelian and G is nilpotent of class $k+1$. For any positive integer m, consider the $\mathbf{Z}/p$ vector space V on a basis $\{v_1, v_2, \ldots, v_{p^m}\}$ and let $\sigma \in GL(V)$ where

$$\sigma(v_i) = \begin{cases} v_i + v_{i+1} & \text{if } i \leq p^m - 1 \\ v_{p^m} & \text{if } i = p^m. \end{cases}$$

For each positive integer $j \leq p^m$,

$$\sigma^j(v_i) = \begin{cases} \sum_{l=0}^{j} \binom{j}{l} v_{i+l} & \text{if } i \leq p^m - j \\ \\ \sum_{l=0}^{r} \binom{j}{l} v_{i+l} & \text{if } i > p^m - j, \text{ where } r = p^m - i \end{cases}$$

so that the order of σ is p^m. The semi-direct product group of V and $\langle\sigma\rangle$ is a p-group whose commutator subgroup is abelian and has nilpotency class p^m (see [**1**] and [**9**]). By choosing $m \geq k + 1$ and factoring this semi-direct product group by the $k+1$-th lower central term we obtain a group G with the required properties. ∎

Let P be a fixed set of prime numbers. We use the notation $n \in P$ to mean all prime divisors of n are in P and P' to denote the complement of P in the set of all primes. A group G is said to be *P-local* if the map $g \mapsto g^n$ is a bijection for all $n \in P'$. A group homomorphism $f : G \to K$ is said to be (i) *P-injective* if for any two elements $g_1, g_2 \in G$ such that $f(g_1) = f(g_2)$, there exists an integer $n \in P'$ such that $g_1^n = g_2^n$; (ii) *P-surjective* if for every $k \in K$, there exists an integer $n \in P'$ such that $k^n \in \mathrm{Imf}$; and (iii) *P-bijective* if f is both P-injective and P-surjective.

On the category of nilpotent groups, there is a well-known P-localization [**7**], which is denoted by $e : N \to N_P$ for each nilpotent group N, where N_P is P-local nilpotent and e is a P-bijection.

The P-nilpotent completion or $\mathbf{Z}_P$-completion of a group G is defined to be the group homomorphism $\eta : G \to G_{\widehat{P}}$ induced by the group homomorphisms $G \twoheadrightarrow G/\Gamma_i G \xrightarrow{e} (G/\Gamma_i G)_P$, where $G_{\widehat{P}} = \mathrm{invlim}(G/\Gamma_i G)_P$, with i running through all finite ordinals. For each i, the above group homomorphism $G \to (G/\Gamma_i G)_P$ defines an idempotent monad on $\mathcal{G}$ whose universal property enables us to complete the following diagram

$$\begin{array}{ccc} G_{\widehat{P}} & \longrightarrow & (G_{\widehat{P}}/\Gamma_i G_{\widehat{P}})_P \\ \downarrow & & \\ (G/\Gamma_i G)_P & & \end{array}$$

by a unique map $(G_{\widehat{P}}/\Gamma_i G_{\widehat{P}})_P \to (G/\Gamma_i G)_P$. Passing to inverse limits, we obtain a natural transformation $\chi : (G_{\widehat{P}})_{\widehat{P}} \to G_{\widehat{P}}$ so that $((\ )_{\widehat{P}}, \eta, \chi)$ is a monad on $\mathcal{G}$.

Let $\mathcal{G}$ be the category of groups and $\mathcal{G}'$ be the full subcategory of groups G such that the natural homomorphism $G_{\widehat{P}} \to (G_{\widehat{P}})_{\widehat{P}}$ is an isomorphism. Then $(\ )_{\widehat{P}}$ restricts to an idempotent monad on $\mathcal{G}'$. Let (S', D') be the associated orthogonal pair. Since every abelian group A satisfies $A_{\widehat{P}} \cong A_P$, all abelian groups are objects of $\mathcal{G}'$; moreover, all P-local abelian groups are in D'.

For any group G in $\mathcal{G}'$, the completion homomorphism $\eta : G \to G_{\widehat{P}}$ is in S' and hence it is orthogonal to all P-local abelian groups. From this fact it follows that, for all groups G in $\mathcal{G}'$, the natural map

$$(G/\Gamma_2 G)_P \to (G_{\widehat{P}}/\Gamma_2 G_{\widehat{P}})_P$$

induced by η is an isomorphism. Thus, if G is in $\mathcal{G}'$, then $H_1(G; \mathbf{Z}_P) \cong H_1(G_{\widehat{P}}; \mathbf{Z}_P)$.

For any group G, we denote by γ_i the projection of G onto $G/\Gamma_i G$, by θ_i the natural epimorphism from $G_{\widehat{P}}$ onto $(G/\Gamma_i G)_P$, by $\bar{\eta}$ the abelianization of $\eta : G \to G_{\widehat{P}}$, and by e the P-localization homomorphism. Since $(G/\Gamma_2 G)_P$ is abelian, there is a unique homomorphism $\bar{\theta}_2 : G_{\widehat{P}}/\Gamma_2 G_{\widehat{P}} \to (G/\Gamma_2 G)_P$ such that $\bar{\theta}_2 \gamma_2 = \theta_2$. Now we have

$$\bar{\theta}_2 \bar{\eta} \gamma_2 = \bar{\theta}_2 \gamma_2 \eta = \theta_2 \eta = e \gamma_2.$$

Since γ_2 is surjective, we infer that $\bar{\theta}_2 \bar{\eta} = e$. Under the assumption that the group G is in the subcategory $\mathcal{G}'$, both $\bar{\eta}$ and e are P-bijections. It follows that $\bar{\theta}_2$ is a P-bijection as well. Hence, we have proved the following result.

Proposition 2. *For a group G, if the natural homomorphism $G_{\widehat{P}} \to (G_{\widehat{P}})_{\widehat{P}}$ is an isomorphism, then the homomorphism $H_1(G; \mathbf{Z}_P) \to H_1(G_{\widehat{P}}; \mathbf{Z}_P)$ induced by the P-completion map $G \to G_{\widehat{P}}$ and the homomorphism $H_1(G_{\widehat{P}}; \mathbf{Z}_P) \to H_1(G; \mathbf{Z}_P)$ induced by the projection $G_{\widehat{P}} \to (G/\Gamma_2 G)_P$ are isomorphisms, and they are inverse to each other.*

We next prove that if F is a free group on an infinite set of generators, then $\bar{\theta}_2$ is not P-injective. This implies that F is not in $\mathcal{G}'$, as desired.

Thus, we shall assume that $\bar{\theta}_2$ is P-injective and arrive at a contradiction. Pick a countable subset of free generators of F and label them as $\{a_{ij}\}$, where $1 \le j \le i$. Denote by F_m the free group generated by $a_{m1}, \ldots, a_{mm}$. Let π_m be the projection of F onto F_m sending all other generators to 1. Likewise, we denote by $\hat{\pi}_m$, the induced homomorphism $F_{\widehat{P}} \to (F_m)_{\widehat{P}}$ and by η_m, the completion map $F_m \to (F_m)_{\widehat{P}}$.

Consider the element $b = (b_2, b_3, \dots) \in F_{\hat{\mathbf{Z}}}$, where $b_2 = 1$ and, for $m \geq 2$, b_{m+1} is the class of

$$[a_{21}, a_{22}][a_{31}, a_{32}, a_{33}] \cdots [a_{m1}, \dots, a_{mm}]$$

in $F/\Gamma_{m+1}F$. Since the natural map $F_{\hat{\mathbf{Z}}} \to F_{\widehat{P}}$ is injective, we may view b as an element of $F_{\widehat{P}}$ as well. In fact we have

$$\hat{\pi}_m(b) = \eta_m([a_{m1}, \dots, a_{mm}]).$$

Since $1 = \theta_2(b) = \bar{\theta}_2(\gamma_2(b))$ and $\bar{\theta}_2$ is assumed to be P-injective, it follows that $\gamma_2(b)^n = 1$ for some $n \in P'$. Hence, $b^n \in \Gamma_2 F_{\widehat{P}}$. Therefore we may write

$$b^n = [u_1, u_2] \cdots [u_{2k-1}, u_{2k}],$$

with $u_i \in F_{\widehat{P}}$ for all i. Now, for each i, we have $\theta_2(u_i)^{t_i} = e(\gamma_2(z_i))$, for some $t_i \in P'$ and $z_i \in F$ because e is P-surjective and γ_2 is surjective. Since only a finite number of generators of F are involved in z_i, we have $\pi_m(z_i) = 1$ for all i and all m except for a finite number of indices $m_1, \dots, m_r$. Choose any $m \neq m_1, \dots, m_r$, which will remain fixed in the rest of the argument.

Let ψ_m be the unique homomorphism that renders the following diagram commutative:

$$\begin{tikzcd}
F_{\widehat{P}} \arrow[r, tail, "\theta_2"] \arrow[d, "\hat{\pi}_m"] & (F/\Gamma_2 F)_P \arrow[d, "\psi_m"] & (F/\Gamma_2 F) \arrow[l, "e"'] \arrow[d] & F \arrow[l, "\gamma_2"'] \arrow[d, "\pi_m"'] \\
(F_m)_{\widehat{P}} \arrow[r, tail, "\theta_2"] & (F_m/\Gamma_2 F_m)_P & (F_m/\Gamma_2 F_m) \arrow[l, "e"'] & F_m \arrow[l, "\gamma_2"']
\end{tikzcd}$$

For each i, we have

$$\psi_m(\theta_2(u_i))^{t_i} = \psi_m(e(\gamma_2(z_i))) = e(\gamma_2(\pi_m(z_i))) = 1.$$

Since the target of ψ_m is a P-local group, we infer that $\psi_m(\theta_2(u_i)) = 1$, and hence $\theta_2(\hat{\pi}_m(u_i)) = 1$. Therefore, $\theta_{m+1}(\hat{\pi}_m(u_i))$ belongs to the kernel of the reduction map $(F_m/\Gamma_{m+1}F_m)_P \to (F_m/\Gamma_2 F_m)_P$, that is,

$$\theta_{m+1}(\hat{\pi}_m(u_i)) \in (\Gamma_2 F_m/\Gamma_{m+1}F_m)_P$$

for all i. Now observe that

$$\begin{aligned}&\theta_{m+1}(\eta_m([a_{m1},\dots,a_{mm}]^n))=\theta_{m+1}(\hat{\pi}_m(b^n))\\&=[\theta_{m+1}(\hat{\pi}_m(u_1)),\theta_{m+1}(\hat{\pi}_m(u_2))]\cdots[\theta_{m+1}(\hat{\pi}_m(u_{2k-1})),\theta_{m+1}(\hat{\pi}_m(u_{2k}))],\end{aligned}$$

which is an element of $(\Gamma_2\Gamma_2F_m/\Gamma_{m+1}F_m)_P$. Hence, there is an integer $q\in P'$ and an element $x\in\Gamma_2\Gamma_2F_m/\Gamma_{m+1}F_m$ such that

$$\theta_{m+1}(\eta_m([a_{m1},\dots,a_{mm}]^n))^q=e(x).$$

Since we have the commutative diagram

$$\begin{tikzcd} F_m \arrow[r, "\eta_m"] \arrow[d, "\gamma_{m+1}"'] & (F_m)_{\widehat{P}} \arrow[d, "\theta_{m+1}"] \\ F_m/\Gamma_{m+1}F_m \arrow[r, "e"] & (F_m/\Gamma_{m+1}F_m)_P \end{tikzcd}$$

where $F_m/\Gamma_{m+1}F_m$ is torsion-free and hence the localization map e is injective, we infer that

$$\gamma_{m+1}([a_{m1},\dots,a_{mm}]^{nq})=x.$$

It follows that $[a_{m1},\dots,a_{mm}]^{nq}$ belongs to the subgroup of F_m generated by $\Gamma_2\Gamma_2F_m$ and $\Gamma_{m+1}F_m$. This contradicts Proposition 1. We have thus shown

Theorem 3. *Let F be a free group on an infinite set of generators. Then, for any set of primes P, the natural homomorphisms $H_1(F;\mathbf{Z}_P)\to H_1(F_{\widehat{P}};\mathbf{Z}_P)$ and $\eta: F_{\widehat{P}}\to(F_{\widehat{P}})_{\widehat{P}}$ both fail to be isomorphisms.*

We thus conclude that P-nilpotent completion is not idempotent on the category of groups. As in [**4**, Proposition IV.5.4], it follows from our theorem and [**4**, Proposition IV.5.3] that

Corollary 4. *Any infinite wedge of circles is R-bad, where R is any subring of the rationals.*

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