**P-NILPOTENT COMPLETION IS NOT IDEMPOTENT**

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**Abstract**

Let \( P \) be an arbitrary set of primes. The \( P \)-nilpotent completion of a group \( G \) is defined by the group homomorphism \( \eta : G \to G^{\wedge}_P \) where \( G^{\wedge}_P = \text{invlim}(G/\Gamma_i G)_P \). Here \( \Gamma_2 G \) is the commutator subgroup \([G,G]\) and \( \Gamma_i G \) the subgroup \([G, \Gamma_{i-1} G]\) when \( i > 2 \). In this paper, we prove that \( P \)-nilpotent completion of an infinitely generated free group \( F \) does not induce an isomorphism on the first homology group with \( \mathbb{Z}_P \) coefficients. Hence, \( P \)-nilpotent completion is not idempotent. Another important consequence of the result in homotopy theory (as in [4]) is that any infinite wedge of circles is \( R \)-bad, where \( R \) is any subring of rationals.

**1. Introduction**

For a group \( G \), we denote by \( \Gamma_2 G \) the commutator subgroup \([G,G]\) and \( \Gamma_i G \) the subgroup \([G, \Gamma_{i-1} G]\) when \( i > 2 \). A group \( G \) is nilpotent if \( \Gamma_i(G) \) is trivial for some \( i \). The nilpotency class \( \text{nil}(G) \) of \( G \) is the least \( c \) such that \( \Gamma_c(G) \) is trivial. Let \( P \) be a set of prime numbers. There is a well-known \( P \)-localization in the category of nilpotent groups, [7]. We denote this localization on a nilpotent group \( N \) by \( e : N \to N_P \).

The \( P \)-nilpotent completion or \( \mathbb{Z}_P \)-completion of a group \( G \) is defined to be the group homomorphism \( \eta : G \to G^{\wedge}_P \) where \( G^{\wedge}_P = \text{invlim}(G/\Gamma_i G)_P \), with \( i \) running through all finite ordinals. For each \( i \), the group homomorphism \( G \to (G/\Gamma_i G)_P \) defines a localization on the category \( G \) of groups. Its universal property gives rise to a natural map \((G^{\wedge}_P/\Gamma_i G^{\wedge}_P)_P \to (G/\Gamma_i G)_P \). Passing to inverse limit, we obtain a natural transformation \( \chi : (G^{\wedge}_P)_P \to G^{\wedge}_P \) so that \((\_^{\wedge}_P, \eta, \chi)\) is a monad on \( G \).

Let \( F \) be a free group on an infinitely countable set of generators. In [4, Proposition IV.5.4], it is proved that the abelianization of \( \eta : F \to F^{\wedge}_{\mathbb{Z}} = \text{invlim}(F/\Gamma_i F) \) is not an isomorphism. This result is used to verify that \( \mathbb{Z}_P \)-completion (which is \( P \)-nilpotent completion when \( P \) is the set of all primes) is not idempotent in [3].
We study these proofs closely and obtain a similar proof of the non-idempotence of $P$-nilpotent completion for any set $P$ of primes. We use results from orthogonal pairs, idempotent monads and the $P$-localization on the category of nilpotent groups.

Although the $P$-nilpotent completion is not idempotent on the category of groups, a procedure to obtain an idempotent monad from it is described in [5]. This turns out to be the minimal $P$-localization, which is also obtained in [2]. It is the “smallest” (in the sense that it provides the least local objects) idempotent monad which extends $P$-localization on the category of nilpotent groups to the category of groups. This minimal $P$-localization coincides with the $P$-nilpotent completion on groups which have finitely generated abelianization [3] and groups with stable lower central series [2].

2. The $P$-nilpotent completion is not idempotent

Let $C$ be a category, $X$ be an object of $C$ and $f : A \rightarrow B$ be a morphism of $C$. Then $X$ and $f$ are said to be orthogonal to each other, denoted by $X \perp f$ or $f \perp X$, if $f^*: C(B, X) \cong C(A, X)$. For a class $D$ of objects in $C$, the orthogonal complement of $D$ in $C$, denoted by $D^\perp$, is the class of morphisms orthogonal to every object in $D$. Dually, the orthogonal complement of $S$ can be defined for a class $S$ of morphisms. An orthogonal pair $(S, D)$ in $C$ comprises a collection $S$ of morphisms in $C$ and a collection $D$ of objects in $C$ satisfying $S = D^\perp$ and $D = S^\perp$. Every idempotent monad (see [8, p. 133]) (also known as localization in [6]) is associated with a unique orthogonal pair.

Let $a_1, a_2, \ldots$ be elements of a group $G$. We define $[a_1, a_2] = a_1^{-1}a_2^{-1}a_1a_2$ and $[a_1, a_2, \ldots, a_k] = [[a_1, \ldots, a_{k-1}], a_k]$ recursively for $k \geq 3$.

**Proposition 1.** Let $F$ be the free group on $a_1, \ldots, a_k$. For every positive integer $n$, $[a_1, \ldots, a_k]^n$ does not belong to the subgroup of $\Gamma_2 F$ that is generated by $\Gamma_{k+1} F$ and $\Gamma_2 \Gamma_2 F$.

**Proof:** Replacing $F$ by the quotient $F/\langle \Gamma_{k+1} F, \Gamma_2 \Gamma_2 F \rangle$, the proposition becomes: for each $n$, there exists a group $G$ with the following properties: (i) The commutator subgroup $\Gamma_2 G$ is abelian, (ii) $G$ is nilpotent of class $k + 1$, and (iii) there exists $x \in \Gamma_k G$ such that $x^n \neq 1$.

However, it is enough to pick a prime $p$ that does not divide $n$ and find a $p$-group $G$ such that $\Gamma_2 G$ is abelian and $G$ is nilpotent of class $k + 1$. For any positive integer $m$, consider the $\mathbb{Z}/p$ vector space $V$ on a
basis \( \{v_1, v_2, \ldots, v_{p^m}\} \) and let \( \sigma \in GL(V) \) where
\[
\sigma(v_i) = \begin{cases} 
  v_i + v_{i+1} & \text{if } i \leq p^m - 1 \\
  v_{p^m} & \text{if } i = p^m.
\end{cases}
\]

For each positive integer \( j \leq p^m \),
\[
\sigma^j(v_i) = \begin{cases} 
  \sum_{l=0}^{j} \binom{j}{l} v_{i+l} & \text{if } i \leq p^m - j \\
  \sum_{l=0}^{r} \binom{j}{l} v_{i+l} & \text{if } i > p^m - j, \text{ where } r = p^m - i
\end{cases}
\]
so that the order of \( \sigma \) is \( p^m \). The semi-direct product group of \( V \) and \( \langle \sigma \rangle \) is a \( p \)-group whose commutator subgroup is abelian and has nilpotency class \( p^m \) (see [1] and [9]). By choosing \( m \geq k + 1 \) and factoring this semi-direct product group by the \( k+1 \)-th lower central term we obtain a group \( G \) with the required properties. \( \blacksquare \)

Let \( P \) be a fixed set of prime numbers. We use the notation \( n \in P \) to mean all prime divisors of \( n \) are in \( P \) and \( P' \) to denote the complement of \( P \) in the set of all primes. A group \( G \) is said to be \( P \)-local if the map \( g \mapsto g^n \) is a bijection for all \( n \in P' \). A group homomorphism \( f : G \to K \) is said to be (i) \( P \)-injective if for any two elements \( g_1, g_2 \in G \) such that \( f(g_1) = f(g_2) \), there exists an integer \( n \in P' \) such that \( g_1^n = g_2^n \); (ii) \( P \)-surjective if for every \( k \in K \), there exists an integer \( n \in P' \) such that \( k^n \in \text{Im} f \); and (iii) \( P \)-bijective if \( f \) is both \( P \)-injective and \( P \)-surjective.

On the category of nilpotent groups, there is a well-known \( P \)-localization \( [7] \), which is denoted by \( e : N \to N_P \) for each nilpotent group \( N \), where \( N_P \) is \( P \)-local nilpotent and \( e \) is a \( P \)-bijection.

The \( P \)-nilpotent completion or \( \mathbb{Z}_P \)-completion of a group \( G \) is defined to be the group homomorphism \( \eta : G \to G_{\hat{P}} \) induced by the group homomorphisms \( G \to G/\Gamma_i G \xrightarrow{e} (G/\Gamma_i G)_P \), where \( G_{\hat{P}} = \text{invlim}(G/\Gamma_i G)_P \), with \( i \) running through all finite ordinals. For each \( i \), the above group homomorphism \( G \to (G/\Gamma_i G)_P \) defines an idempotent monad on \( \mathcal{G} \) whose universal property enables us to complete the following diagram
\[
\begin{array}{ccc}
G_{\hat{P}} & \longrightarrow & (G_{\hat{P}}/\Gamma_i G_{\hat{P}})_P \\
\downarrow & & \downarrow \\
(G/\Gamma_i G)_P & & \end{array}
\]
by a unique map \((G_p/G_{\Gamma_i}G_p)_P \rightarrow (G/G_{\Gamma_i}G)_P\). Passing to inverse limits, we obtain a natural transformation \(\chi : (G_p)_P \rightarrow G_p\) so that \((\chi, \eta, \chi)\) is a monad on \(G\).

Let \(G\) be the category of groups and \(G'\) be the full subcategory of groups \(G\) such that the natural homomorphism \(G_p \rightarrow (G_p)_P\) is an isomorphism. Then \((\chi)_P\) restricts to an idempotent monad on \(G'\). Let \((S', D')\) be the associated orthogonal pair. Since every abelian group \(A\) satisfies \(A_p \cong A_P\), all abelian groups are objects of \(G'\); moreover, all \(P\)-local abelian groups are in \(D'\).

For any group \(G\) in \(G'\), the completion homomorphism \(\eta : G \rightarrow G_p\) is in \(S'\) and hence it is orthogonal to all \(P\)-local abelian groups. From this fact it follows that, for all groups \(G\) in \(G'\), the natural map

\[(G/G_2 G)_P \rightarrow (G_p/G_2 G_p)_P\]

induced by \(\eta\) is an isomorphism. Thus, if \(G\) is in \(G'\), then \(H_1(G;Z_P) \cong H_1(G_p;Z_P)\).

For any group \(G\), we denote by \(\gamma_i\) the projection of \(G\) onto \(G/G_i G\), by \(\theta_i\) the natural epimorphism from \(G_p\) onto \((G/G_i G)_P\), by \(\bar{\eta}\) the abelianization of \(\eta : G \rightarrow G_p\), and by \(e\) the \(P\)-localization homomorphism. Since \((G/G_2 G)_P\) is abelian, there is a unique homomorphism \(\bar{\theta}_2 : G_p/G_2 G_p \rightarrow (G/G_2 G)_P\) such that \(\bar{\theta}_2 \gamma_2 = \theta_2\). Now we have

\[\bar{\theta}_2 \bar{\eta} \gamma_2 = \theta_2 \gamma_2 \eta = \theta_2 \eta = e \gamma_2.\]

Since \(\gamma_2\) is surjective, we infer that \(\bar{\theta}_2 \bar{\eta} = e\). Under the assumption that the group \(G\) is in the subcategory \(G'\), both \(\bar{\eta}\) and \(e\) are \(P\)-bijections. It follows that \(\bar{\theta}_2\) is a \(P\)-bijection as well. Hence, we have proved the following result.

**Proposition 2.** For a group \(G\), if the natural homomorphism \(G_p \rightarrow (G_p)_P\) is an isomorphism, then the homomorphism \(H_1(G;Z_P) \rightarrow H_1(G_p;Z_P)\) induced by the \(P\)-completion map \(G \rightarrow G_p\) and the homomorphism \(H_1(G_p;Z_P) \rightarrow H_1(G;Z_P)\) induced by the projection \(G_p \rightarrow (G/G_2 G)_P\) are isomorphisms, and they are inverse to each other.

We next prove that if \(F\) is a free group on an infinite set of generators, then \(\bar{\theta}_2\) is not \(P\)-injective. This implies that \(F\) is not in \(G'\), as desired.

Thus, we shall assume that \(\bar{\theta}_2\) is \(P\)-injective and arrive at a contradiction. Pick a countable subset of free generators of \(F\) and label them as \(\{a_{ij}\}\), where \(1 \leq j \leq i\). Denote by \(F_m\) the free group generated by \(a_{m1}, \ldots, a_{mm}\). Let \(\pi_m\) be the projection of \(F\) onto \(F_m\) sending all other
The nilpotent completion is not idempotent

Consider the element $b = (b_2, b_3, \ldots) \in \hat{F}_Z$, where $b_2 = 1$ and, for $m \geq 2$, $b_{m+1}$ is the class of

\[ [a_{21}, a_{22}][a_{31}, a_{32}, a_{33}] \cdots [a_{m1}, \ldots, a_{mm}] \]

in $F/\Gamma_{m+1}F$. Since the natural map $\hat{F}_Z \to \hat{F}_P$ is injective, we may view $b$ as an element of $\hat{F}_P$ as well. In fact we have

\[ \hat{\pi}_m(b) = \eta_m([a_{m1}, \ldots, a_{mm}]) \]

Since $1 = \theta_2(b) = \bar{\theta}_2(\gamma_2(b))$ and $\bar{\theta}_2$ is assumed to be $P$-injective, it follows that $\gamma_2(b)^n = 1$ for some $n \in P'$. Hence, $b^n \in \Gamma_2 \hat{F}_P$. Therefore we may write

\[ b^n = [u_1, u_2] \cdots [u_{2k-1}, u_{2k}] \]

with $u_i \in \hat{F}_P$ for all $i$. Now, for each $i$, we have $\theta_2(u_i)^{t_i} = e(\gamma_2(z_i))$, for some $t_i \in P'$ and $z_i \in F$ because $e$ is $P$-surjective and $\gamma_2$ is surjective.

Since only a finite number of generators of $F$ are involved in $z_i$, we have $\pi_m(z_i) = 1$ for all $i$ and all $m$ except for a finite number of indices $m_1, \ldots, m_r$. Choose any $m \neq m_1, \ldots, m_r$, which will remain fixed in the rest of the argument.

Let $\psi_m$ be the unique homomorphism that renders the following diagram commutative:

\[
\begin{array}{cccccc}
F_P & \xrightarrow{\theta_2} & (F/\Gamma_2 F)_P & \xrightarrow{e} & (F/\Gamma_2 F) & \xrightarrow{\gamma_2} & F \\
\downarrow{\hat{\pi}_m} & & \downarrow{\psi_m} & & \downarrow{\pi_m} & & \\
(F_m)_P & \xrightarrow{\theta_2} & (F_m/\Gamma_2 F_m)_P & \xrightarrow{e} & (F_m/\Gamma_2 F_m) & \xrightarrow{\gamma_2} & F_m
\end{array}
\]

For each $i$, we have

\[ \psi_m(\theta_2(u_i))^{t_i} = \psi_m(e(\gamma_2(z_i))) = e(\gamma_2(\pi_m(z_i))) = 1. \]

Since the target of $\psi_m$ is a $P$-local group, we infer that $\psi_m(\theta_2(u_i)) = 1$, and hence $\theta_2(\pi_m(u_i)) = 1$. Therefore, $\theta_{m+1}(\pi_m(u_i))$ belongs to the kernel of the reduction map $(F_m/\Gamma_{m+1} F_m)_P \to (F_m/\Gamma_2 F_m)_P$, that is,

\[ \theta_{m+1}(\pi_m(u_i)) \in (\Gamma_2 F_m/\Gamma_{m+1} F_m)_P \]
for all $i$. Now observe that

$$\theta_{m+1}(\eta_m([a_{m1}, \ldots , a_{mm}]^n)) = \theta_{m+1}(\pi_m(b^n))$$

$$=[\theta_{m+1}(\pi_m(u_1)), \theta_{m+1}(\pi_m(u_2))], \ldots , [\theta_{m+1}(\pi_m(u_{2k-1})), \theta_{m+1}(\pi_m(u_{2k}))],$$

which is an element of $(\Gamma_2 \Gamma_2 F_m/\Gamma_{m+1} F_m)_P$. Hence, there is an integer $q \in P'$ and an element $x \in \Gamma_2 \Gamma_2 F_m/\Gamma_{m+1} F_m$ such that

$$\theta_{m+1}(\eta_m([a_{m1}, \ldots , a_{mm}]^n))^q = e(x).$$

Since we have the commutative diagram

$$\begin{array}{ccc}
F_m & \xrightarrow{\eta_m} & (F_m)\widehat{P} \\
\gamma_{m+1} \downarrow & & \downarrow \theta_{m+1} \\
F_m/\Gamma_{m+1} F_m & \xrightarrow{e} & (F_m/\Gamma_{m+1} F_m)_P
\end{array}$$

where $F_m/\Gamma_{m+1} F_m$ is torsion-free and hence the localization map $e$ is injective, we infer that

$$\gamma_{m+1}([a_{m1}, \ldots , a_{mm}]^nq) = x.$$ 

It follows that $[a_{m1}, \ldots , a_{mm}]^nq$ belongs to the subgroup of $F_m$ generated by $\Gamma_2 \Gamma_2 F_m$ and $\Gamma_{m+1} F_m$. This contradicts Proposition 1. We have thus shown

**Theorem 3.** Let $F$ be a free group on an infinite set of generators. Then, for any set of primes $P$, the natural homomorphisms $H_1(F; Z_P) \rightarrow H_1(F_\widehat{P}; Z_P)$ and $\eta : F_\widehat{P} \rightarrow (F_\widehat{P})_\widehat{P}$ both fail to be isomorphisms.

We thus conclude that $P$-nilpotent completion is not idempotent on the category of groups. As in [4, Proposition IV.5.4], it follows from our theorem and [4, Proposition IV.5.3] that

**Corollary 4.** Any infinite wedge of circles is $R$-bad, where $R$ is any subring of the rationals.

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