Publicacions Matemàtiques, Vol **41** (1997), 437–453.

G-STRUCTURES OF SECOND ORDER DEFINED BY LINEAR OPERATORS SATISFYING ALGEBRAIC RELATIONS

Demetra Demetropoulou-Psomopoulou

Abstract

The present work is based on a type of structures on a differential manifold V, called G-structures of the second kind, defined by endomorphism J on the second order tangent bundle $T^2(V)$. Our objective is to give conditions for a differential manifold to admit a real almost product and a generalised almost tangent structure of second order. The concepts of the second order frame bundle $H^2(V)$, its structural group L^2 and its associated tangent bundle of second order $T^2(V)$ of a differentiable manifold V, are used from the point of view that is described in papers [**5**] and [**6**]. Also, the almost tangent structure of order two is mentioned and its generalisation, the second order almost transverse structure, is defined.

Introduction

A special type of first kind G-structures on an n-dimensional differentiable manifold V_n, are those defined by a differentiable field of linear operators J_x, such that at each point $x \in V_n$, J_x maps the tangent space $T_x(V_n)$ into itself and satisfies algebraic relations.

Thus if $J_x^2 = 0$ and J_x is of rank p $(2p \leq n)$ everywhere, then there is defined a G_1-structure ([**12**], [**14**]), the so called generalised almost tangent structure ([**2**]). In particular, if $n = 2p$, then the manifold V_n has an almost tangent structure, briefly $(a.t.)$-structure ([**4**], [**11**]).

If $J^2x = I$, then there is, defined on V_n, a real almost product structure, briefly π_R-structure, of dimension (n_1, n_2) with $n_1 + n_2 = n$ ([**13**]).

A generalisation to the second order of the π_R-structure and the G_1-structure, respectively π_R^2-structure and G_1^2-structure, is given in [**5**] and [**6**], by means of a differential field of linear operators J_x, acting on the space $T_x^2(V_n)$ of the second order tangent vectors and satisfying the same algebraic equations.

For this purpose the concepts of the frame bundle and tangent bundle of second order are given with the use of the jet theory discussed in works of C. Ehresmann ([**7**], [**8**], [**9**], [**10**]). The notions of the second order tangent vector coincides with that defined by [**3**]. However there must be noticed that there are other, different definitions of the tangent bundle of higher order, in general ([**15**], [**17**]).

From the same standpoint in the present paper, there is given on V_n, at first, a $(a.p.)^2_R$-structure of dimension

$$\left(n + \binom{n+1}{2} - \left(p + \binom{p+1}{2}\right), p + \binom{p+1}{2}\right),$$

(that is an almost product structure of second order), induced from an G^2_1-structure of rank $p + \binom{p+1}{2}$.

Additionally, the second order almost tangent structure, briefly $(a.t.)^2$-structure, a special case of the G^2_1-structure ([**6**]), is mentioned. This structure is a particular case of the $(a.tr.)^2$-structure too, that is also defined, and is a generalisation to the second order of the almost transverse structure ([**16**]).

Finally, with the help of the previous structures, to a G^2_1-structure of rank $p + \binom{p+1}{2}$, there is given a compatible $(a.p.)^2_R$-structure of dimension

$$\left(n + \binom{n+1}{2} - 2\left(p + \binom{p+1}{2}\right), 2\left(p + \binom{p+1}{2}\right)\right).$$

1. Preliminaries

We recall the following, from the concepts given by C. Ehresmann ([**7**], [**8**], [**9**], [**10**]) and used in [**5**] and [**6**]:

Let V_n be an n-dimensional differentiable manifold of class C^∞. The second order frame bundle $H^2(V_n) = \bigcup_{x \in V_n} H^2_x(V_n)$ is a principal fibre bundle with basis V_n and structural group L^2_n, where H^2_x is the set of all invertible 2-jets of R^n into V_n with source $0 \in R^n$ and target $x \in V_n$. It can be identified ([**5**]) with the space of bases of the vector space $T^2_x(V_n)$, at $x \in V_n$, of the second order tangent vectors, as those are defined by [**3**].

The Lie group L^2_n, that is the set $j^2_0 f$ of all invertible 2-jets with source and target $0 \in R^n$ of a 2-mapping f at the point $0 \in R^n$, can be identified

([**5**], [**6**]), with a subgroup of matrices $Gl(N,R)$, where $N = n + \frac{n(n+1)}{2}$. To the element $a \in L^2_n$,

$$a = (a^i_{j_1}, a^i_{j_1 j_2}), \quad i, j_1, j_2 = 1, 2, \dots n,\ \det(a^i_j) \neq 0 \text{ and } a^i_{j_1 j_2} \text{ symmetric with respect to } j_1, j_2, \tag{1.1}$$

corresponds, the matrix A of the form,

$$A = \begin{bmatrix} a^i_{j_1} & 0 \\ a^i_{j_1 j_2} & a^{i_1}_{j_1} a^{i_2}_{j_2} \end{bmatrix}. \tag{1.2}$$

We have, $\dim L^2_n = n^2 + n \binom{n+1}{2}$.

The tangent bundle of the second order, $T^2(V_n) = \bigcup_{x \in V_n} T^2_x(V_n)$ has ([**5**], [**6**]) basis V_n, structural group L^2_n and fibre $F^2 = (L^2_{1,n})^*$, where $L^2_{1,n}$ is the set $j^2_0 g$ of all 2-jets with source $0 \in R^n$ and target $0 \in R$ of a 2-mapping g at $0 \in R^n$. This bundle is ([**5**]) the dual vector bundle of $T^{2*}_1(V_n) = \bigcup_{x \in V_n} T^{2*}_{1,x}(V_n)$, with basis V_n, structural group L^2_n and fibre $L^2_{1,n}$.

Each element ω of the vector space $L^2_{1,n}$, isomorphic to the fibre $T^{2*}_x(V_n) = T^{2*}_{1,x}(V_n)$, can be written in the form,

$$\omega = \begin{bmatrix} \omega_{i_1} \\ \omega_{i_1 i_2} \end{bmatrix}, \quad i_1, i_2 = 1, 2, \dots, n \text{ and } \omega_{i_1 i_2} \text{ symmetric with respect to the indices } i_1, i_2.$$

Also, $\dim L^2_{1,n} = \dim T^{2*}_x(V_n) = n + \binom{n+1}{2}$.

For two given charts, the transformation law for the coordinates of an element of $T^{2*}(V_n)$, is given by the form,

$$x^{i'} = x^{i'}(x^j),$$
$$\begin{bmatrix} \omega_{j'_1} \\ \omega_{j'_1 j'_2} \end{bmatrix} = \begin{bmatrix} a^{i_1}_{j'_1} & 0 \\ a^{i_1}_{j'_1 j'_2} & a^{i_1}_{j'_1} a^{i_2}_{j'_2} \end{bmatrix} \begin{bmatrix} \omega_{i_1} \\ \omega_{i_1 i_2} \end{bmatrix}, \tag{1.3}$$

where

$$\left(a^{i_1}_{j'_1} = \frac{\partial x^{i_1}}{\partial x^{j'_1}},\ a^{i_1}_{j'_1 j'_2} = \frac{\partial^2 x^{i_1}}{\partial x^{j'_1} \partial x^{j'_2}} \right) \in L^2_n \text{ and } \{x^i\}_{i=1,2,\dots,n},\ \{x^{j'}\}_{j=1,2,\dots,n}$$

are two systems of local coordinates at $x \in V_n$.

Let the $n + \binom{n+1}{2}$ second order tangent vectors,

(1.4) $(e_{i_1}, e_{i_1 i_2})$, $i_1, i_2 = 1, 2, \ldots, n$ and $e_{i_1 i_2}$ symmetric in the indices i_1, i_2,

define the natural basis for T_x^2, for the local chart $\{x^i\}_{i=1,2,\ldots,n}$ at the point $x \in V_n$. Then, every $v \in T_x^2$ can be expressed uniquely in the form,

$$v = [v^{i_1} v^{i_1 i_2}], \quad i_1, i_2 = 1, 2, \ldots, n \text{ and } v^{i_1 i_2} \text{ symmetric in the indices } i_1, i_2.$$

For another system of local coordinates $\{x^{j'}\}_{j=1,2,\ldots,n}$ at $x \in V_n$ the second order tangent vectors $(e_{j'_1}, e_{j'_1 j'_2})$ of a new basis of T_x^2, are transformed to the basis (1.4) by the matrices,

$$\begin{bmatrix} e_{i_1} \\ e_{i_1 i_2} \end{bmatrix} = \begin{bmatrix} a_{i_1}^{j'_1} & 0 \\ a_{i_1 i_2}^{j'_1} & a_{i_1}^{j'_1} a_{i_2}^{j'_2} \end{bmatrix} \begin{bmatrix} e_{j'_1} \\ e_{j'_1 j'_2} \end{bmatrix},$$

and

$$\begin{bmatrix} a_{i_1}^{j'_1} & 0 \\ a_{i_1 i_2}^{j'_1} & a_{i_1}^{j'_1} a_{i_2}^{j'_2} \end{bmatrix}$$

is the corresponding matrix to the element $(a_{i_1}^{j'_1}, a_{i_1 i_2}^{j'_1}) \in L_n^2$ (inverse of that given in the relations (1.3)). The transformation law for the local coordinates of $v \in T_x^2$ is given by the matrices,

$$\begin{bmatrix} v^{j'_1} & v^{j'_1 j'_2} \end{bmatrix} = \begin{bmatrix} v^{i_1} & v^{i_1 i_2} \end{bmatrix} \begin{bmatrix} a_{i_1}^{j'_1} & 0 \\ a_{i_1 i_2}^{j'_1} & a_{i_1}^{j'_1} a_{i_2}^{j'_2} \end{bmatrix}.$$

If J_x is a differentiable field of linear operators acting on the space T_x^2, then the corresponding element F of $T_x^2 \otimes T_x^{2*}$ can be represented by the matrix,

$$(1.5) \quad F = \begin{bmatrix} F_{i_1}^{j_1} & F_{i_1}^{j_1 j_2} \\ F_{i_1 i_2}^{j_1} & F_{i_1 i_2}^{j_1 j_2} \end{bmatrix}, \quad i_1, i_2, j_1, j_2 = 1, 2, \ldots, n,$$

$F_{i_1}^{j_1 j_2}$ symmetric in j_1, j_2, $F_{i_1 i_2}^{j_1}$ symmetric in i_1, i_2 and $F_{i_1 i_2}^{j_1 j_2}$ symmetric in i_1, i_2 and j_1, j_2.

It is defined by the relations,

$$\begin{cases} (J_x v)^{j_1} = F_{i_1}^{j_1} v^{i_1} + F_{i_1 i_2}^{j_1} v^{i_1 i_2}, \\ (J_x v)^{j_1 j_2} = F_{i_1}^{j_1 j_2} v^{i_1} + F_{i_1 i_2}^{j_1 j_2} v^{i_1 i_2}, \end{cases}$$

where $v = (v^{i_1}, v^{i_1 i_2})$, is a 2-tangent vector at $x \in V_n$.

2. A $(a.p.)^2_R$-structure induced from a G^2_1-structure

I. A real almost product structure of second order is defined ([**5**]) on an n-dimensional differentiable manifold V_n of class C^∞, by a linear operator J_x acting on the space $T^2_x(V_n)$ of the second order tangent vectors at each point $x \in V_n$ and satisfying the equation

$$J^2_x = I. \tag{2.1}$$

Then J_x gives a decomposition of T^2_x in a direct sum of two complementary subspaces.

If we assume that,

(i) L and M are the two proper supplementary subspaces of T^2_x corresponding to the eigenvalues -1 and $+1$ respectively, according to the relation (2.1),

(ii) $\dim L = n + \binom{n+1}{2} - \left(p + \binom{p+1}{2}\right) = q + \binom{q+1}{2} + qp$, $q = n - p$, $\dim M = p + \binom{p+1}{2}$, then, there is defined on V_n an $(a.p.)^2_R$-structure of dimension $\left(q + \binom{q+1}{2} + qp, p + \binom{p+1}{2}\right)$, with $n - p = q$. It is, also a generalisation of the real almost product structure of the first order, briefly π_R-structure ([**13**]).

Remark. It must be noticed that in [**5**] there is discussed a real almost product structure of second order, the π^2_R-structure, with different dimension.

Its adapted basis can be defined by

$$\begin{aligned}&\{(e_{\alpha_1}, e_{A_1})(e_{\alpha_1\alpha_2}, e_{\alpha_1 A_2}, e_{A_1 A_2}),\\ &\qquad\qquad \alpha_1, \alpha_2 = 1, 2, \dots, q,\ A_1, A_2 = q+1, 2, \dots, q+p = n,\end{aligned}$$

where the 2-tangent vectors,

$$\{e_{\alpha_1}, e_{\alpha_1\alpha_2}, e_{\alpha_1 A_2}\} \quad \text{and} \quad \{e_{A_1}, e_{A_1 A_2}\},$$

form a basis of L and M respectively.

The matrix A of the transformation for the adapted bases is of the form,

$$(2.2) \qquad A = \begin{bmatrix} \begin{bmatrix} a_{\alpha_1}^{\beta'_1} & 0 \\ 0 & a_{A_1}^{B'_1} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} a_{\alpha_1\alpha_2}^{\beta'_1} & 0 \\ a_{\alpha_1 A_2}^{\beta'_1} & 0 \\ 0 & a_{A_1A_2}^{B'_1} \end{bmatrix} & \begin{bmatrix} a_{\alpha_1}^{\beta'_1} a_{\alpha_2}^{\beta'_2} & 0 & 0 \\ 0 & a_{\alpha_1}^{\beta'_1} a_{A_1}^{B'_1} & 0 \\ 0 & 0 & a_{A_1}^{B'_1} a_{A_2}^{B'_2} \end{bmatrix} \end{bmatrix},$$

with

$$\begin{bmatrix} a_{\alpha_1}^{\beta'_1} & 0 \\ 0 & a_{A_1}^{B'_1} \end{bmatrix} \in L_{(n-p,p)},\ a_{\alpha_1}^{\beta'_1} \in L_q,\ a_{A_1}^{B'_1} \in L_p,$$

where $L_{(n-p,p)}$ is ([**13**]) the structural group of the π_R-structure.

Thus, this $(a.p.)^2_R$-structure is a G-structure of second order whose structural group $L^2_{(n-p,p)}$ is consisting of matrices of the form (2.2), subgroup of L^2_n.

The corresponding tensor F to the operator J for the $(a.p.)^2_R$-adapted basis can be represented by the matrix,

$$(2.3) \qquad F = \begin{bmatrix} \begin{bmatrix} -\delta_{\alpha_1}^{\beta_1} & 0 \\ 0 & \delta_{A_1}^{B_1} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} -\delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} & 0 & 0 \\ 0 & -\delta_{\alpha_1}^{\beta_1} \delta_{A_1}^{B_1} & 0 \\ 0 & 0 & \delta_{A_1}^{B_1} \delta_{A_2}^{B_2} \end{bmatrix} \end{bmatrix},$$

and we have

$$FA = AF$$

for every element A of the structural group $L^2_{(n-p,p)}$ of the $(a.p.)^2_R$-structure.

II. An n dimensional differentiable manifold V_n admits ([**6**]) a generalised almost tangent structure of second order, briefly G^2_1-structure, if there exists a differentiable field of linear operators J_x of constant rank $p + \binom{p+1}{2}$, such that at each point $x \in V_n$,

$$(2.4) \qquad \begin{cases} J_x \text{ maps } T^2_x(V_n) \text{ into itself,} \\ J^2_x = 0. \end{cases}$$

The differentiable manifold V_n is called G_1^2-manifold, and

$$n + \binom{n+1}{n} > 2\left(p + \binom{p+1}{2}\right), \quad n \geq 2p.$$

Let

$$\begin{gathered}\{(e_{\alpha_1(1)}, e_{\alpha_1}, e_{\alpha_1(2)}), (e_{\alpha_1(1)\alpha_2(1)}, e_{\alpha_1(1)\alpha_2}, e_{\alpha_1(1)\alpha_2(2)}, \\ e_{\alpha_1\alpha_2(2)}, e_{\alpha_1\alpha_2}, e_{\alpha_1(2)\alpha_2(2)})\}, \\ \alpha(1) = 1, 2, \dots, p,\ \alpha = p+1, \dots, q,\ \alpha(2) = q+1, \dots, q+p = n,\end{gathered}$$

be a basis of $T_x^2(V_n)$, such that the 2-tangent vectors

$$\{(e_{\alpha_1(1)}, e_{\alpha_1}), (e_{\alpha_1(1)\alpha_2(1)}, e_{\alpha_1(1)\alpha_2}, e_{\alpha_1(1)\alpha_2(2)}, e_{\alpha_1\alpha_2}, e_{\alpha_1\alpha_2(2)})\}$$

form a basis of $\operatorname{Ker} J_x$ and the tangent vectors of second order basis of S_x (complement space of $\operatorname{Ker} J_x$ in $T_x^2(V)$)

$$\{e_{\alpha_1(2)}, e_{\alpha_1(2)\alpha_2(2)}\},$$

satisfy two conditions,

$$\left\{\begin{array}{l} J_x e_{\alpha_1(2)} = e_{\alpha_1(1)}, \\ J_x e_{\alpha_1(2)\alpha_2(2)} = e_{\alpha_1(1)\alpha_2(1)}. \end{array}\right. \tag{2.5}$$

Such a basis is called ([**6**]) an G_1^2-adapted basis. The matrix B of the transformation for the G_1^2-adapted bases is of the form,

$$B = \begin{bmatrix} b_{i_1}^{j_1'} & 0 \\ b_{i_1 i_2}^{j_1'} & b_{i_1}^{j_1'} b_{i_2}^{j_2'} \end{bmatrix}, \tag{2.6}$$

with

$$b_{i_1}^{j_1'} = \begin{bmatrix} b_{\alpha_1(1)}^{\beta_1'(1)} & 0 & 0 \\ b_{\alpha_1}^{\beta_1'(1)} & b_{\alpha_1}^{\beta_1'} & 0 \\ b_{\alpha_1(2)}^{\beta_1'(1)} & b_{\alpha_1(2)}^{\beta_1'} & b_{\alpha_1(1)}^{\beta_1'(1)} \end{bmatrix} \in G_1,$$

$b_{\alpha_1(1)}^{\beta_1'(1)} = b_{\alpha_1(2)}^{\beta_1'(2)} \in L_p$, $b_{\alpha_1}^{\beta_1'(1)} \in \operatorname{Hom}(R^p, R^{n-2p})$, $b_{\alpha_1}^{\beta_1'} \in L_{n-2p}$, $b_{\alpha_1(2)}^{\beta_1'(1)} \in \operatorname{End}(R^p, R^p)$, $b_{\alpha_1(2)}^{\beta_1'} \in \operatorname{Hom}(R^{n-2p}, R^p)$, G_1 is the structural group of the

G_1-structure ([**2**], [**12**], [**14**]), and

$$b^{j'_1}_{i_1 i_2} = \begin{bmatrix} b^{\beta'_1(1)}_{\alpha_1(1)\alpha_2(1)} & 0 & 0 \\ b^{\beta'_1(1)}_{\alpha_1(1)\alpha_2} & b^{\beta'_1}_{\alpha_1(1)\alpha_2} & 0 \\ b^{\beta'_1(1)}_{\alpha_1(1)\alpha_2(2)} & b^{\beta'_1}_{\alpha_1(1)\alpha_2(2)} & 0 \\ b^{\beta'_1(1)}_{\alpha_1\alpha_2} & b^{\beta'_1}_{\alpha_1\alpha_2} & 0 \\ b^{\beta'_1(1)}_{\alpha_1\alpha_2(2)} & b^{\beta'_1}_{\alpha_1\alpha_2(2)} & 0 \\ b^{\beta'_1(1)}_{\alpha_1(2)\alpha_2(2)} & b^{\beta'_1}_{\alpha_1(2)\alpha_2(2)} & b^{\beta'_1(1)}_{\alpha_1(1)\alpha_2(1)} \end{bmatrix}, \tag{2.8}$$

$$b^{\beta'_1(1)}_{\alpha_1(1)\alpha_2(1)} = b^{\beta'_1(2)}_{\alpha_1(2)\alpha_2(2)}.$$

It can be noticed that the matrices,

$$\begin{bmatrix} b^{\beta'_1(1)}_{\alpha_1(1)} & 0 \\ b^{\beta'_1(1)}_{\alpha_1} & b^{\beta'_1}_{\alpha_1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b^{\beta'_1(1)}_{\alpha_1(1)\alpha_2(1)} & 0 \\ b^{\beta'_1(1)}_{\alpha_1(1)\alpha_2} & b^{\beta'_1}_{\alpha_1(1)\alpha_2} \\ b^{\beta'_1(1)}_{\alpha_1(1)\alpha_2(2)} & b^{\beta'_1}_{\alpha_1(1)\alpha_2(2)} \\ b^{\beta'_1(1)}_{\alpha_1\alpha_2} & b^{\beta'_1}_{\alpha_1\alpha_2} \\ b^{\beta'_1(1)}_{\alpha_1\alpha_2(2)} & b^{\beta'_1}_{\alpha_1\alpha_2(2)} \end{bmatrix} \tag{2.9}$$

from (2.7) and (2.8) respectively, express the basis transformation of Ker J_x.

Thus, a G_1^2-structure is a G_1-structure of the second order whose structural group G_1^2 is consisting of matrices of the form (2.6) (with (2.7) and (2.8)), subgroup of L_n^2.

The tensor F for the G_1^2-adapted basis can be represented by the matrix

$$H = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \delta^{\beta_1(1)}_{\alpha_1(1)} & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \delta^{\beta_1(1)}_{\alpha_1(1)}\delta^{\beta_2(1)}_{\alpha_2(1)} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix},$$

and we have

$$HB = BH,$$

for every element B of the structural group of the G^2_1-structure.

On the other hand, we have ([**2**]), that a differentiable manifold V_n admits a distribution of dimension $n-p=q$ of the tangent space $T_x(V_n)$, if and only if, V_n admits a $G_{(n-p,p)}$-structure, with structural group $G_{(n-p,p)}$, consisting of matrices of the form,

$$\begin{bmatrix} c^{\beta'_1}_{\alpha_1} & 0 \\ c^{\beta'_1}_{A_1} & c^{B'_1}_{A_1} \end{bmatrix}, \quad c^{\beta'_1}_{\alpha_1} \in L_q,\ c^{\beta'_1}_{A_1} \in \mathrm{End}(R^q, R^p),\ c^{B'_1}_{A_1} \in L_p,$$
$$\alpha = 1, 2, \dots, q,\ A = q+1, \dots, q+p = n,$$

subgroup of L_n.

Similarly, a differentiable manifold V_n admits a distribution of dimension $n + \binom{n+1}{2} - \left(p + \binom{p+1}{2}\right) = q + \binom{q+1}{2} + qp$ of the second order tangent space $T^2_x(V_n)$, if and only if, V_n admits a $G_{(n-p,p)}$-structure of second order, that is a $G^2_{(n-p,p)}$-structure, with structural group $G^2_{(n-p,p)}$, consisting of matrices of the form,

$$C = \begin{bmatrix} \begin{bmatrix} c^{\beta'_1}_{\alpha_1} & 0 \\ c^{\beta'_1}_{A_1} & c^{B'_1}_{A_1} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} c^{\beta'_1}_{\alpha_1\alpha_2} & 0 \\ c^{\beta'_1}_{\alpha_1 A_2} & 0 \\ c^{\beta'_1}_{A_1 A_2} & c^{B'_1}_{A_1 A_2} \end{bmatrix} & \begin{bmatrix} c^{\beta'_1}_{\alpha_1} c^{\beta'_2}_{\alpha_2} & 0 & 0 \\ c^{\beta'_1}_{\alpha_1} c^{\beta'_2}_{A_2} & c^{\beta'_1}_{\alpha_1} c^{B'_2}_{A_2} & 0 \\ c^{\beta'_1}_{A_1} c^{\beta'_2}_{A_2} & c^{\beta'_1}_{A_1} c^{B'_2}_{A_2} & c^{B'_1}_{A_1} c^{B'_2}_{A_2} \end{bmatrix} \end{bmatrix}.$$

Obviously, the matrix B ((2.6) with (2.7) and (2.8)) of the group G_1^2 is a special case of the above matrix C. Therefore,

Proposition 2.1. *If there is a given on the differentiable manifold V_n a G_1^2-structure, there exist two distributions L and L_1 of the second order tangent space $T_x^2(V_n)$, of dimension $n+\binom{n+1}{2}-\left(p+\binom{p+1}{2}\right)$ and $p+\binom{p+1}{2}$ respectively, such that $L_1 \subseteq L$.*

Proof: The spaces $L(x) = \operatorname{Ker} J_x$ and $L_1 = J_x(S_x)$ satisfy the above conditions. ■

Thus, if the distribution L of dimension $n+\binom{n+1}{2}-\left(p+\binom{p+1}{2}\right)$ is given, it is sufficient to define a distribution M supplementary of L in $T_x^2(V_n)$ of dimension $p+\binom{p+1}{2}$.

Consequently,

Proposition 2.2. *If the differentiable manifold V_n admits a G_1^2-structure, then the tensor field F induces an $(a.p.)^2_R$-structure of dimension $\left(n+\binom{n+1}{2}-\left(p+\binom{p+1}{2}\right), p+\binom{p+1}{2}\right)$.*

3. $(a.t.)^2$ and $(a.tr.)^2$ structures

I. The almost tangent structure of the first order, briefly $(a.t.)$-structure, is a particular case of the G_1-structure ([**13**]), if the differentiable manifold V is of dimension $2n$ and the rank of J_x is equal to n. Then the group G_1 (relation (2.7)) reduces to the structural group $G({}^{n}{}_{n,n})$ of the $(a.t.)$-structure ([**4**], [**11**]), consisting of matrices of the form,

$$\begin{bmatrix} K & 0 \\ N & K \end{bmatrix}, \quad K \in L_n,\ N \in \operatorname{End}(R^n, R^n).$$

To the above structure there is a generalisation to the second order by means of a differentiable field of linear operators J_x, acting on the space $T_x^2(V_{2n})$ of the second order tangent vectors, with constant rank $n+\binom{n+1}{2}$. Obviously this second order almost tangent structure, briefly $(a.t.)^2$-structure, is a particular case of the G_1^2-structure, too.

Thus, the components of the tensor F are given by the matrix,

$$F = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ \delta^{\beta_1}_{\alpha_1} & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2} & 0 & 0 \end{bmatrix} \end{bmatrix}, \tag{3.1}$$

and the element $a = (a^{j'_1}_{i_1}, a^{j'_1}_{i_1 i_2}) \in L^2_{2n}$, by the matrix,

$$A = \begin{bmatrix} \begin{bmatrix} a^{\beta'_1}_{\alpha_1} & 0 \\ a^{\beta'_1}_{A_1} & a^{\beta'_1}_{\alpha_1} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} a^{\beta'_1}_{\alpha_1\alpha_2} & 0 \\ a^{\beta'_1}_{\alpha_1 A_2} & 0 \\ a^{\beta'_1}_{A_1 A_2} & a^{\beta'_1}_{\alpha_1\alpha_2} \end{bmatrix} & \begin{bmatrix} a^{\beta'_1}_{\alpha_1} a^{\beta'_2}_{\alpha_2} & 0 & 0 \\ a^{\beta'_1}_{\alpha_1} a^{\beta'_2}_{A_2} & a^{\beta'_1}_{\alpha_1} a^{\beta'_2}_{\alpha_2} & 0 \\ a^{\beta'_1}_{A_1} a^{\beta'_2}_{A_2} & a^{\beta'_1}_{A_1} a^{\beta'_2}_{\alpha_2} & a^{\beta'_1}_{\alpha_1} a^{\beta'_2}_{\alpha_2} \end{bmatrix} \end{bmatrix}, \tag{3.2}$$

$$\alpha_1, \alpha_2 = 1, 2, \ldots, n,\ A_1, A_2 = 1, 2, \ldots, 2n \text{ (that is } A = n + \alpha),$$

with

$$\begin{bmatrix} a^{\beta'_1}_{\alpha_1} & 0 \\ a^{\beta'_1}_{A_1} & a^{\beta'_1}_{\alpha_1} \end{bmatrix} \in G({}^{n}{}_{n,n}),\ a^{\beta'_1}_{\alpha_1} = a^{B'_1}_{A_1} \in L_n,\ a^{\beta'_1}_{A_1} \in \mathrm{End}(R^n, R^n).$$

We have,

$$AF = FA$$

for every element A of the structural group $G^2({}^{n}{}_{n,n})$, of the $(a.t.)^2$-structure.

II. The notion of the almost transverse structure of the first order, briefly $(a.tr.)$-structure, is defined ([**16**]) on an n-dimensional manifold V_n equipped with a foliation $\mathcal{L}$ of codimension $n-p$. That is, for an atlas $\{U, x^{\alpha}, x^{\alpha(1)}, x^{\alpha(2)}\}$, $\alpha = 1, 2, \ldots, n-p$, $\alpha(1), \alpha(2) = 1, 2, \ldots, p$, adapted to the foliation $\mathcal{L}$ the transformation functions verify the relation,

$$\frac{\partial x^{\beta'}}{\partial x^{\alpha(2)}} = 0. \tag{3.3}$$

Then the space L of the tangent vectors to the foliation $\mathcal{L}$, and the quotient space $Q = T(V_n)/L$, $(\dim Q = \operatorname{codim} L = n-p)$ define an $(a.tr.)$-structure with structural group formed by matrices:

$$\begin{bmatrix} T & S & 0 \\ 0 & K & 0 \\ 0 & N & K \end{bmatrix}, \quad T \in L_{n-p},\ S \in \operatorname{Hom}(R^p, R^{n-p}), \quad K \in L_p,\ N \in \operatorname{End}(R^p, R^p). \tag{3.4}$$

As it can be seen, this structure is a generalisation of the $(a.t.)$-structure, too.

Generalising the almost transverse structure to the second order, by adding the condition,

$$\frac{\partial^2 x^{\beta'}}{\partial x^{\alpha_1(2)} \partial x^{\alpha_2(2)}} = 0, \tag{3.5}$$

it can be defined an $(a.tr.)^2$-structure, whose structural group is of the form:

$$\begin{bmatrix} l_{i_1}^{j_1'} & 0 \\ l_{i_1 i_2}^{j_1'} & l_{i_1}^{j_1'} l_{i_2}^{j_2'} \end{bmatrix}, \tag{3.6}$$

with

$$l_{i_1}^{j_1'} = \begin{bmatrix} l_{\alpha_1}^{\beta_1'} & l_{\alpha_1}^{\beta_1'(1)} & 0 \\ 0 & l_{\alpha_1(1)}^{\beta_1'(1)} & 0 \\ 0 & l_{\alpha_1(2)}^{\beta_1'(1)} & l_{\alpha_1(1)}^{\beta_1'(1)} \end{bmatrix},$$

$$l_{i_1 i_2}^{j_1'} = \begin{bmatrix} l_{\alpha_1\alpha_2}^{\beta_1'} & l_{\alpha_1\alpha_2}^{\beta_1'} & 0 \\ l_{\alpha_1\alpha_2(1)}^{\beta_1'} & l_{\alpha_1\alpha_2(1)}^{\beta_1'(1)} & 0 \\ l_{\alpha_1\alpha_2(2)}^{\beta_1'} & l_{\alpha_1\alpha_2(2)}^{\beta_1'(1)} & 0 \\ 0 & l_{\alpha_1(1)\alpha_2(1)}^{\beta_1'(1)} & 0 \\ l_{\alpha_1(1)\alpha_2(2)}^{\beta_1'} & l_{\alpha_1(1)\alpha_2(2)}^{\beta_1'(1)} & 0 \\ 0 & l_{\alpha_1(2)\alpha_2(2)}^{\beta_1'(1)} & l_{\alpha_1(1)\alpha_2(1)}^{\beta_1'(1)} \end{bmatrix}. \tag{3.7}$$

The above structure is also a generalisation of the $(a.t.)^2$-structure.

4. Compatible G_2^1 and $(a.p.)_R^2$ structures

An n-dimensional differentiable manifold V_n of class C^∞ admitting a G_1-structure of rank p, with $n \geq 2p$, can also have ([**1**]) a structural group consisting of matrices of the form,

$$\begin{bmatrix} T & S & 0 \\ 0 & K & 0 \\ P & N & K \end{bmatrix}, \quad T \in L_{n-2p},\ S \in \operatorname{Hom}(R^p, R^{n-2p}),$$

$$K \in L_p,\ N \in \operatorname{End}(R^p, R^p),\ P \in \operatorname{Hom}(R^{n-2p}, R^p).$$

Then, the tensor J will be represented by the matrix,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \end{bmatrix}.$$

The G_1-adapted basis is of the form,

$$\{e_\alpha, e_{\alpha(1)}, e_{\alpha(2)}\}, \quad \alpha = 1, 2, \dots, n-2p,\ \alpha(1), \alpha(2) = 1, 2, \dots, p$$

where $\{e_\alpha, e_{\alpha(2)}\}$ and $\{e_{\alpha(1)}\}$ define a basis of $\operatorname{Ker} J_x$ and S_x, respectively.

Similarly, the G_1^2-structure of rank $p + \binom{p+1}{2}$ and $n + \binom{n+1}{2} \geq 2\left(p + \binom{p+1}{2}\right)$, can have a structural group G_1^2 with matrices of the form given by (3.6) where,

$$l_{i_1}^{j_1'} = \begin{bmatrix} l_{\alpha_1}^{\beta_1'} & l_{\alpha_1}^{\beta_1'(1)} & 0 \\ 0 & l_{\alpha_1(1)}^{\beta_1'(1)} & 0 \\ l_{\alpha_1(2)}^{\beta_1'} & l_{\alpha_1(2)}^{\beta_1'(1)} & l_{\alpha_1(1)}^{\beta_1'(1)} \end{bmatrix},$$

$$l_{i_1 i_2}^{j_1'} = \begin{bmatrix} l_{\alpha_1\alpha_2}^{\beta_1'} & l_{\alpha_1\alpha_2}^{\beta_1'(1)} & 0 \\ l_{\alpha_1\alpha_2(1)}^{\beta_1'} & l_{\alpha_1\alpha_2(1)}^{\beta_1'(1)} & 0 \\ l_{\alpha_1\alpha_2(2)}^{\beta_1'} & l_{\alpha_1\alpha_2(2)}^{\beta_1'(1)} & 0 \\ 0 & l_{\alpha_1(1)\alpha_2(1)}^{\beta_1'(1)} & 0 \\ l_{\alpha_1(1)\alpha_2(2)}^{\beta_1'} & l_{\alpha_1(1)\alpha_2(2)}^{\beta_1'(1)} & 0 \\ l_{\alpha_1(2)\alpha_2(2)}^{\beta_1'} & l_{\alpha_1(2)\alpha_2(2)}^{\beta_1'(1)} & l_{\alpha_1(1)\alpha_2(1)}^{\beta_1'(1)} \end{bmatrix}. \tag{4.1}$$

The tensor J can be represented by the matrix,

$$\begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \delta^{\beta_1(1)}_{\alpha_1(1)} 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta^{\beta_1(1)}_{\alpha_1(1)} \delta^{\beta_2(1)}_{\alpha_2(1)} 0 & 0 & 0 \end{bmatrix} \end{bmatrix}.$$

The G_1^2-adapted basis is of the form

$$\begin{gathered} \{(e_{\alpha_1}, e_{\alpha_1(1)}, e_{\alpha_1(2)}), (e_{\alpha_1\alpha_2}, e_{\alpha_1\alpha_2(1)}, e_{\alpha_1\alpha_2(2)}, \\ e_{\alpha_1(1)\alpha_2(1)}, e_{\alpha_1(1)\alpha_2(2)}, e_{\alpha_1(2)\alpha_2(2)})\}, \\ \alpha = 1, 2, \dots, n - 2p, \quad \alpha(1), \alpha(2) = 1, 2, \dots, p, \end{gathered}$$

where $\{(e_{\alpha_1}, e_{\alpha_1(1)}), (e_{\alpha_1\alpha_2}, e_{\alpha_1\alpha_2(1)}, e_{\alpha_1\alpha_2(2)}, e_{\alpha_1(1)\alpha_2(1)}, e_{\alpha_1(1)\alpha_2(2)})\}$ and $\{e_{\alpha_1(2)}, e_{\alpha_1(2)\alpha_2(2)}\}$ define a basis of $\operatorname{Ker} J_x$ and S_x, respectively.

Now, we assume that the manifold V_n admits a distribution Δ of the second order tangent space $T_x^2(V_n)$ of dimension $n + \binom{n+1}{2} - 2\left(p + \binom{p+1}{2}\right)$.

Definition 4.1. A distribution Δ is compatible with the G_1^2-structure defined by the tensor J, if Δ is supplementary of $J(T_x^2(V_n))$ in the $\operatorname{Ker} J_x$.

Comparing the matrices (3.7) and (4.1) we have:

Proposition 4.1. *The differentiable manifold V_n admits a distribution Δ compatible with the G_1^2-structure, if and only if, V_n is equiped with an* $(a.tr.)^2$*-structure.*

Definition 4.2. An almost product structure of second order is compatible with the generalised almost tangent structure of second order, if

i) $\operatorname{Ker} J_x$ contains Δ and

ii) J_x applies the $p+\binom{p+1}{2}$ second order base vectors of the complement space of Δ, on the rest, not in Δ, $p+\binom{p+1}{2}$ base vectors of $\operatorname{Ker} J_x$.

Proposition 2. *A necessary and sufficient condition, that an* $(a.p.)^2_R$-*structure of dimension* $n+\binom{n+1}{2}-2\left(p+\binom{p+1}{2}\right)$, *with* $2p+q=n$ *is compatible with the* G^2_1-*structure of rank* $p+\binom{p+1}{2}$, *is the* n *dimensional differentiable manifold* V_n *of class* C^∞ *to admit a structure defined by matrices of the form (3.6) with,*

$$l^{j_1'}_{i_1}=\begin{bmatrix} l^{\beta_1'}_{\alpha_1} & 0 & 0\\ 0 & l^{\beta_1'(1)}_{\alpha_1(1)} & 0\\ 0 & l^{\beta_1'(1)}_{\alpha_1(2)} & l^{\beta_1'(1)}_{\alpha_1(1)} \end{bmatrix},$$

$$l^{j_1'}_{i_1i_2}=\begin{bmatrix} l^{\beta_1'}_{\alpha_1\alpha_2} & 0 & 0\\ l^{\beta_1'}_{\alpha_1\alpha_2(1)} & 0 & 0\\ l^{\beta_1'}_{\alpha_1\alpha_2(2)} & 0 & 0\\ 0 & l^{\beta_1'(1)}_{\alpha_1(1)\alpha_2(1)} & 0\\ 0 & l^{\beta_1'(1)}_{\alpha_1(1)\alpha_2(2)} & 0\\ 0 & l^{\beta_1'(1)}_{\alpha_1(2)\alpha_2(2)} & l^{\beta_1'(1)}_{\alpha_1(1)\alpha_2(1)} \end{bmatrix}, \tag{4.2}$$

$$l^{\beta_1'}_{\alpha_1}\in L_{n-2p}, l^{\beta_1'(1)}_{\alpha_1(1)}\in L_p, l^{\beta_1'(1)}_{\alpha_1(2)}\in \operatorname{End}(R^p,R^p).$$

References

1. A. Carfagna d'Andrea, Une caractérisation du fibré tangent à un feuilletage, *C. R. Acad. Sci. Paris Sér. I Math.* **301(3)** (1985), 77–80.

2. A. CARFAGNA D'ANDREA AND R. DI FEBO MARINELLI, Su una generalizzazione delle structure quasi tangenti, *Rend. Mat. Appl. (7)* **2** (1982), 257–267.

3. B. CENKL, On the G-structures of higher order, *Câsopis. Pêst. Mat.* **90** (1965), 26–32.

4. R. S. CLARK AND M. R. BRUCKHEIMER, Sur les structures presque tangentes, *C. R. Acad. Sci. Paris Sér. I Math.* **251** (1960), 627–629.

5. D. DEMETROPOULOU-PSOMOPOULOU, A prolongation of the real almost product structure of a differentiable manifold, *Demonstratio Math.* **20(3-4)** (1987), 423–439.

6. D. DEMETROPOULOU-PSOMOPOULOU, G_1-structures of second order, *Publ. Mat.* **36** (1992), 51–64.

7. C. EHRESMANN, Les prolongements d'une variété différentiable. I Calcul des jets, prolongements principal, *C. R. Acad. Sci. Paris Sér. I Math.* **233** (1951), 598–600.

8. C. EHRESMANN, Les prolongements d'une variété différentiable, Cong. Un. Mat. It. Taormina (1951), 317–325.

9. C. EHRESMANN, Les prolongements d'une variété différentiable. V Convariants différentiels et prolongements d'une structure infinitesimale, *C. R. Acad. Sci. Paris Sér. I Math.* **234** (1952), 1424–1425.

10. C. EHREMSANN, Introduction à la théorie des structures infinitésimales et des pseudogroupes de Lie, Coll. Inter. du C.N.R.S., Geom. Diff. Strasbourg (1953), 97–100.

11. H. A. ELIOPOULOS, Structures presque tangents sur les variétés différentiables, *C. R. Acad. Sci. Paris Sér. I Math.* **255** (1962), 1563–1565.

12. H. A. ELIOPOULOS, On the connections and the holonomy group of certain G-structures, *Ann. Mat. Pura Appl.(4)* **78** (1968), 1–12.

13. G. LEGRAND, Ètude d'une géneralisation des structures presques complexes sur les variétés différentiables, Thése, Paris (1968).

14. J. LEHMANN-LEJEUNE, Sur l'integrabilité de certaines G-structures, *C. R. Acad. Sci. Paris Sér. I Math.* **258** (1964), 5326–5329.

15. M. DE LÉON AND P. R. RODRÍGUEZ, "*Generalized Classical Mechanics and Field Theory*," North-Holland Mathematical Studies **112**, North-Holland, Amsterdam, 1985.

16. TONG VAN DUE, Structures presques transverse, *J. Differential Geom.* **14** (1979), 215–219.

17. K. YANO AND S. ISHIHARA, Tangent and Cotangent Bundles, Pure and Appl. Math. Ser. **16**, Marcel Dekker, New York, 1973.

Department of Mathematics
Faculty of Sciences
University of Thessaloniki
54006 Thessaloniki
GREECE

Primera versió rebuda el 12 d'Abril de 1996,
darrera versió rebuda el 3 de Setembre de 1996