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COMPOSITION OF MAXIMAL OPERATORS

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Abstract

Consider the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy.$$

It is known that M applied to f twice is pointwise comparable to the maximal operator $M_{L \log L} f$, defined by replacing the mean value of $|f|$ over the cube Q by the $L \log L$-mean, namely

$$M_{L \log L} f(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \log \left(e + \frac{|f|}{|f|_Q} \right)(y)\, dy,$$

where $|f|_Q = \frac{1}{|Q|} \int_Q |f|$ (see [**L**], [**LN**], [**P**]).

In this paper we prove that, more generally, if $\Phi(t)$ and $\Psi(t)$ are two Young functions, there exists a third function $\Theta(t)$, whose explicit form is given as a function of $\Phi(t)$ and $\Psi(t)$, such that the composition $M_\Psi \circ M_\Phi$ is pointwise comparable to M_Θ. Through the paper, given an Orlicz function $A(t)$, by $M_A f$ we mean

$$M_A f(x) = \sup_{Q \ni x} ||f||_{A,Q}$$

where $||f||_{A,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q A\left(\frac{|f|}{\lambda}\right)(x)\, dx \le 1 \right\}$.

1. Introduction.

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator Mf of f is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy.$$

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A well-known result of Coifman and Rochberg, (see [**CR**], [**T**]), states that if $Mf < \infty$ a.e. and if $\delta \in (0,1)$, then $(Mf)^\delta \in A_1$, where A_1 is the Muckenhoupt class of the non negative weights w such that

$$A_1(w) = \sup_Q \frac{\displaystyle\fint_Q w}{\operatorname{ess\,inf}_Q w} < \infty,$$

where $\fint_Q w$ stands for the average of w over Q and the supremum being taken over all cubes Q of $\mathbb{R}^n$.

Setting

$$M_r f = \sup_{Q \ni x} \left(\fint_Q |f|^r \right)^{\frac{1}{r}} \qquad r > 1,$$

from the mentioned result, with $\delta = \frac{1}{r}$, it follows that $M \circ M_r \sim M_r$. This means that there exists a constant c, such that

$$M_r f(x) \leq M(M_r f(x)) \leq c M_r f(x) \quad \text{a.e. in} \quad \mathbb{R}^n.$$

For $r = 1$ the situation is different, namely we have that $M \circ M \sim M_{L \log L}$, i.e.

$$c_1 M_{L \log L} f(x) \leq M(Mf(x)) \leq c_2 M_{L \log L} f(x) \quad \text{a.e. in} \quad \mathbb{R}^n$$

(see [**L**], [**LN**], [**P**]), and this corresponds to Stein's result, i.e. for f supported in a cube Q

$$f \in L \log L(Q) \Longleftrightarrow Mf \in L^1(Q)$$

(see [**S**]). The maximal operator $M_{L \log L} f$ is defined by replacing the mean value of $|f|$ over the cube Q by the $L \log L$-mean, namely

$$M_{L \log L} f(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \log \left(e + \frac{|f|}{|f|_Q} \right)(y)\, dy, \tag{1.1}$$

where $|f|_Q = \frac{1}{|Q|} \int_Q |f|$.

The previous results justify the introduction of a maximal operator in an Orlicz space such as $L \log L$.

More precisely, let Ω be a cube of $\mathbb{R}^n$. A continuosly increasing function on $[0,\infty]$, say $\Psi : [0,\infty] \to [0,\infty]$ such that $\Psi(0) = 0$, $\Psi(1) = 1$ and $\Psi(\infty) = \infty$, will be referred to as an Orlicz function.

The generalized Orlicz space denoted by $L^\Psi(\Omega)$ consists of all functions $g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ such that

$$\int_\Omega \Psi\left(\frac{|g|}{\lambda}\right)(x)\,dx < \infty$$

for some $\lambda > 0$.

Let us define the Ψ-average of g over a cube Q contained in Ω by

$$||g||_{\Psi,Q} = \inf\left\{\lambda > 0 : \fint_Q \Psi\left(\frac{|g|}{\lambda}\right)(x)\,dx \le 1\right\}. \tag{1.2}$$

When $\Psi(t)$ is a Young function, i.e. a convex Orlicz function, the quantity

$$||g||_\Psi = \inf\left\{\lambda > 0 : \int_\Omega \Psi\left(\frac{|g|}{\lambda}\right)(x)\,dx \le 1\right\}$$

is the well known Luxemburg norm in the space $L^\Psi(\Omega)$ (see [**KR**], [**RR**]).

If $f \in L^\Psi(\mathbb{R}^n)$, the maximal function of f with respect to Ψ is defined by setting

$$M_\Psi f(x) = \sup_{x\in Q} ||f||_{\Psi,Q} \tag{1.3}$$

where the supremum is taken over all cubes Q of $\mathbb{R}^n$ containing x with sides parallel to the coordinate axes.

Let us remark that if we choose $\Psi(t) = t\log(e+t)$, the maximal operator $M_\Psi f$ defined by (1.3) is equivalent to the $M_{L\log L}$ operator defined by (1.1) (see [**IS**]).

In this paper we generalize the mentioned results: namely, given two Young functions $\Phi(t)$ and $\Psi(t)$, we get a third Young function $\Theta(t)$, such that the composition, $M_\Psi \circ M_\Phi$, between M_Φ and M_Ψ is equivalent to the operator M_Θ.

As an application, we reobtain, in a simple way, the Herz type inequality for the nonincreasing rearrangement of the maximal operator in $L\log L$ (see [**B**]).

Moreover, we obtain a pointwise estimate for the maximal function of the jacobian of a function f such that $|Df|^n$ belongs to L^1.

2. The main result.

Let Ω be a cube of $\mathbb{R}^n$ and set

$$\overline{\mathcal{M}}_\Phi f(x) = \sup_{x\in Q\subseteq\Omega} ||f||_{\Phi,Q}.$$

First, let us prove a result which will be useful in the following.

Theorem 1. *Let $\Psi(t)$ be an Orlicz function and $\Phi(t)$ be a Young one. For*

$$\Theta(t) = \int_0^t \Psi'(s)\Phi\left(\frac{t}{s}\right)\,ds,$$

there exist two positive constants c_1, c_2 such that

$$c_1||\overline{\mathcal{M}}_\Phi f||_{\Psi,\Omega} \le ||f||_{\Theta,\Omega} \le c_2||\overline{\mathcal{M}}_\Phi f||_{\Psi,\Omega} \tag{2.1}$$

for every $f \in L^\Theta(\Omega)$.

Proof: In order to prove that

$$||\overline{\mathcal{M}}_\Phi f||_{\Psi,\Omega} \le c||f||_{\Theta,\Omega}, \tag{2.2}$$

we use the following equality:

$$\int_\Omega \Psi\left(\frac{\overline{\mathcal{M}}_\Phi f(x)}{\lambda}\right)\,dx = \int_0^\infty \Psi'(t)|\{x\in\Omega : \overline{\mathcal{M}}_\Phi f(x) > t\lambda\}|\,dt.$$

Let us set

$$E_{t\lambda} = \{x\in\Omega : \overline{\mathcal{M}}_\Phi f(x) > t\lambda\}.$$

Thanks to Proposition 4.1 in [**BP**], we can consider a sequence of cubes $\{Q_k\}$ such that

$$E_{t\lambda} = \cup_k Q_k \quad\text{and}\quad \int_{Q_k} \Phi\left(\frac{|f|}{\lambda t}\right)(x)\,dx > |Q_k|.$$

Now, we observe that

$$\begin{aligned}
|Q_k| &< \int_{Q_k} \Phi\left(\frac{|f|}{\lambda t}\right)(x)\,dx \\
&= \int_{\left\{x\in\Omega:|f|>\frac{\lambda t}{2}\right\}\cap Q_k} \Phi\left(\frac{|f|}{\lambda t}\right)(x)\,dx + \int_{\left\{x\in\Omega:|f|\le\frac{\lambda t}{2}\right\}\cap Q_k} \Phi\left(\frac{|f|}{\lambda t}\right)(x)\,dx \\
&\le \int_{\left\{x\in\Omega:|f|>\frac{\lambda t}{2}\right\}\cap Q_k} \Phi\left(\frac{|f|}{\lambda t}\right)(x)\,dx + \Phi\left(\frac{1}{2}\right)|Q_k|.
\end{aligned}$$

Without loss of generality, we may assume $\Phi\left(\frac{1}{2}\right) < 1$, then we have

$$\begin{aligned}
|Q_k| &< c\int_{\left\{x\in\Omega:|f|>\frac{\lambda t}{2}\right\}\cap Q_k} \Phi\left(\frac{|f|}{\lambda t}\right)(x)\,dx \\
&< c\int_{\left\{x\in\Omega:|f|>\frac{\lambda t}{2}\right\}\cap Q_k} \Phi\left(\frac{2|f|}{\lambda t}\right)(x)\,dx
\end{aligned}$$

by monotonicity of Φ.

We get

$$|E_{t\lambda}| \leq c \int_{\{x\in\Omega : |f| > \frac{\lambda t}{2}\}\cap E_{t\lambda}} \Phi\left(\frac{2|f|}{\lambda t}\right)(x)\, dx.$$

After that, we obtain

$$\begin{aligned}\int_\Omega \Psi\left(\frac{\overline{\mathcal{M}}_\Phi f(x)}{\lambda}\right) dx &\leq c\int_0^\infty \Psi'(t)\int_{\left\{x\in\Omega : |f| > \frac{\lambda t}{2}\right\}} \Phi\left(\frac{2|f|}{\lambda t}\right)(x)\, dx\, dt \\ &= c\int_\Omega \int_0^{\frac{2|f|}{\lambda}} \Psi'(t)\Phi\left(\frac{2|f|}{\lambda t}\right)(x)\, dt\, dx \qquad (2.3) \\ &= c\int_\Omega \Theta\left(\frac{2|f|}{\lambda}(x)\right) dx.\end{aligned}$$

By estimate above, we have (2.2). Now, we have to prove that

$$||f||_{\Theta,\Omega} \leq c||\overline{\mathcal{M}}_\Phi f||_{\Psi,\Omega}. \qquad (2.4)$$

By Calderon-Zygmund lemma, we may cover $E_{t\lambda} = \{x\in\Omega : \overline{\mathcal{M}}_\Phi f(x) > t\lambda\}$ by a sequence of nonoverlapping cubes Q_k, each having the property

$$2^{-n}|Q_k| \leq |Q_k\cap E_{t\lambda}| < |Q_k|$$

and such that

$$2^n|E_{t\lambda}| \geq \sum |Q_k| \geq \sum\int_{Q_k}\Phi\left(\frac{|f|}{\lambda t}\right) dx \geq \int_{E_{t\lambda}}\Phi\left(\frac{|f|}{\lambda t}\right) dx.$$

We have that

$$\int_\Omega \Psi\left(\frac{\overline{\mathcal{M}}_\Phi f(x)}{\lambda}\right) dx \geq \tilde{c}\int_\Omega \Theta\left(\frac{|f|}{\lambda}(x)\right) dx. \qquad (2.5)$$

In fact

$$\begin{aligned}\int_\Omega \Psi\left(\frac{\overline{\mathcal{M}}_\Phi f(x)}{\lambda}\right) dx &= \int_0^\infty \Psi'(t)|\{x\in\Omega : \overline{\mathcal{M}}_\Phi f(x) > t\lambda\}|\, dt \\ &\geq c\int_0^\infty \Psi'(t)\int_{E_{t\lambda}}\Phi\left(\frac{|f|}{t\lambda}\right) dx\, dt \\ &= c\int_\Omega\int_0^{\frac{\overline{\mathcal{M}}_\Phi(f)}{\lambda}} \Psi'(t)\Phi\left(\frac{|f|}{t\lambda}\right) dt\, dx \\ &\geq c\int_\Omega\int_0^{\frac{f(x)}{\lambda}} \Psi'(t)\Phi\left(\frac{|f|}{t\lambda}\right) dt\, dx\end{aligned}$$

since $\Phi(t)$ is convex.

Finally, we get

$$\int_\Omega \Psi\left(\frac{\overline{\mathcal{M}}_\Phi f(x)}{\lambda}\right) dx \geq c \int_\Omega \Theta\left(\frac{|f|}{\lambda}\right) dx,$$

which implies (2.4), then the theorem is proved ■.

Remark 1. Theorem 1 with Φ and Ψ both Young functions, is proved in [**BP**].

Moreover, in the particular case $\Phi(t) = t$ and $\Psi(t)$ any Orlicz function, Theorem 1 gives Proposition 3.1 of [**GIM**].

Using the previous result, we develop a useful estimate for the composition $M_\Psi \circ M_\Phi$, where Φ and Ψ are Young functions.

Theorem 2. *Let $\Psi(t)$ and $\Phi(t)$ be two Young functions. For*

$$\Theta(t) = \int_0^t \Psi'(s)\Phi\left(\frac{t}{s}\right) ds,$$

there exist two positive constants, c_1 and c_2, such that for every $f \in L^\Theta_{\mathrm{loc}}(\mathbb{R}^n)$ we have

$$c_1 M_\Theta f(x) \leq M_\Psi\left(M_\Phi f(x)\right) \leq c_2 M_\Theta f(x) \tag{2.6}$$

almost everywhere.

Proof: Let us fix $x \in \mathbb{R}^n$ and a cube Q containing x. Put $f = f_1 + f_2$ with $f_1 = f\chi_{3Q}$, we have, by triangle inequality of the Luxemburg norm $||\quad||_\Psi$,

$$||M_\Phi f||_{\Psi,Q} \leq ||M_\Phi f_1||_{\Psi,Q} + ||M_\Phi f_2||_{\Psi,Q} = I + II. \tag{2.7}$$

In order to estimate I, consider

$$\overline{\mathcal{M}}_\Phi f(x) = \sup\{||f||_{\Phi,\bar{Q}} : x \in \bar{Q},\, \bar{Q} \subseteq 3Q\}$$

and we observe that there exists a constant $c(n)$ such that

$$M_\Phi f_1(x) \leq c(n)\overline{\mathcal{M}}_\Phi f_1(x). \tag{2.8}$$

Namely, for every cube $\tilde{Q} \subseteq \mathbb{R}^n$, $\tilde{Q} \ni x$, $\tilde{Q} \cap \mathcal{C}(3Q) \neq \emptyset$ the following inequality holds

$$\fint_{\tilde{Q}} \Phi(|f_1|) = \frac{1}{|\tilde{Q}|} \int_{\tilde{Q} \cap 3Q} \Phi(|f_1|) \leq 3^n \fint_{3Q} \Phi(|f_1|)$$

and then, if $\lambda > 0$ is such that

$$\fint_{3Q} \Phi\left(\frac{|f_1|}{\lambda}\right) \leq 1$$

we have

$$\frac{1}{3^n} \fint_{\tilde{Q}} \Phi\left(\frac{|f_1|}{\lambda}\right) \leq 1.$$

By convexity of Φ, we get

$$\fint_{\tilde{Q}} \Phi\left(\frac{|f_1|}{3^n \lambda}\right) \leq 1$$

and this implies

$$||f_1||_{\Phi,\tilde{Q}} \leq 3^n ||f_1||_{\Phi,3Q}. \tag{2.9}$$

Note that (2.9) is trivial if $\tilde{Q} \subseteq 3Q$.

Observing that $||f_1||_{\Phi,3Q} \leq \overline{\mathcal{M}}_\Phi f_1(x)$ and taking the supremum over all cubes $\tilde{Q}$ of $\mathbb{R}^n$ containing x on the left hand side of (2.9), we have (2.8). By formulas (2.8) and (2.2), applied with $\overline{\mathcal{M}}$ and $\Omega = 3Q$, we deduce

$$I = ||M_\Phi f_1||_{\Psi,Q} \leq C ||f||_{\Theta,Q}.$$

To estimate II it suffices to observe that

$$M_\Phi f_2(y) \leq C \inf_Q M_\Phi f_2 \quad \forall y \in Q. \tag{2.10}$$

In fact, let us fix a point $y \in Q$ and a cube $\bar{Q} \ni y$ such that $\bar{Q} \cap \mathcal{C}(3Q) \neq \emptyset$; the cube $3\bar{Q}$ contains every point $x \in Q$. Reasoning as before, we obtain

$$||f_2||_{\Phi,\bar{Q}} \leq 3^n ||f_2||_{\Phi,3\bar{Q}} \leq C M_\Phi f_2(x)$$

and so (2.10). Now we observe that there are positive constants c_1, c_2, t_0, depending on Φ and Ψ, such that $\Phi(c_1 t) \leq c_2 \Theta(t)$, for $t \geq t_0$. Namely,

$$\begin{aligned}
\Theta(t) &= \int_0^t \Psi'(s) \Phi\left(\frac{t}{s}\right) ds \\
&\geq \int_0^t \Psi(s) \Phi'\left(\frac{t}{s}\right) \frac{t}{s^2}\, ds \\
&\geq c_0 \int_{t_0}^t \Phi'\left(\frac{t}{s}\right) \frac{t}{s^2}\, ds \\
&= c_0 \int_1^{\frac{t}{t_0}} \Phi'(\sigma)\, d\sigma = c_2^{-1} \Phi\left(\frac{t}{t_0}\right)
\end{aligned}$$

and then there exists a positive constant c_3 such that $M_\Phi f(x) \leq c_3 M_\Theta f(x)$ a.e. .

By (2.7), (2.9) and (2.10), we conclude that

$$||M_\Phi f||_{\Psi,Q} \leq C_1 ||f||_{\Theta,Q} + C_2 M_\Theta f(x). \tag{2.11}$$

Taking the supremum over the cubes Q containing x in (2.11), we get

$$M_\Psi(M_\Phi f(x)) \leq c_2 M_\Theta f(x) \quad \text{a.e. .} \tag{2.12}$$

On the other hand, formula (2.4) implies that

$$M_\Psi(M_\Phi f(x)) \geq c_1 M_\Theta f(x) \quad \text{a.e. .} \tag{2.13}$$

Formulas (2.12) and (2.13) give the thesis. ■

Let us give some example of such compositions.

Example 1. Let us consider

$$\Psi(t) = t^p$$
$$\Phi(t) = \begin{cases} 1 & t \leq 1 \\ t^q & t > 1, \end{cases} \qquad p, q \geq 1$$

recalling that

$$M_r f(x) = \sup_{x \in Q} \left(\fint_Q f^r \right)^{\frac{1}{r}}$$

we have

$$M_p f(x) = M_\Psi f(x)$$
$$M_q f(x) = M_\Phi f(x).$$

Theorem 2 implies that

$$M_p \circ M_q \sim \begin{cases} M_r & r = \max\{p, q\},\ p \neq q \\ M_{L^q \log L} & p = q. \end{cases}$$

Let us note that in some very special cases one can determine the constants c_1, c_2.

Example 2. For $r > 1$, if f is a nonincreasing function $f : (0, \infty) \to (0, \infty)$ then

$$M_r f(x) \leq M(M_r f(x)) \leq \frac{r}{r-1} M_r f(x),$$

by Kolmogorov inequality (see [**BDS**]). Moreover

$$\begin{aligned}\int_0^x f(t)\left[1+\log\frac{xf(t)}{\int_0^x f(s)\,ds}\right]dt &\le M(Mf(x))\\ &\le \frac{e}{e-1}\int_0^x f(t)\left[1+\log\frac{xf(t)}{\int_0^x f(s)\,ds}\right]dt.\end{aligned}$$

Let us consider Bagby's formula, (see [**B**]), for such f. If $\Phi_\alpha(t)=t[1+(\log^+ t)^\alpha]$, $\alpha>0$, then

$$c_1 M_{\Phi_\alpha} f(t) \le \frac{1}{t}\int_0^t \left(\log\frac{t}{s}\right)^\alpha f(s)\,ds \le c_2 M_{\Phi_\alpha} f(t),$$

and observe that $M_{\Phi_\alpha}\circ M_{\Phi_\beta}\sim M_{\Phi_{\alpha+\beta+1}}$.

Remark 2. If in Theorem 2 $\Phi(t)=t$, we get

$$\frac{\Theta(t)}{t}=\int_0^t \frac{\Psi'(s)}{s}\,ds.$$

It is easy to verify that

$$c_1 M_\Psi f(x) \le M_\Psi(Mf)(x) \le c_2 M_\Psi f(x)$$

if and only if $\Psi(t)=A(t)t^p$, where $A(t)$ is a continuous increasing function and $p>1$.

Moreover, we have that

$$c_1 M_\Phi f(x) \le M(M_\Phi f)(x) \le c_2 M_\Phi f(x)$$

if and only if $\Phi(t)=t^p$, where $p>1$. So, we reobtain the mentioned result of [**CR**].

3. Some consequences.

Corollary 1. *Let $A(t)$ be a Young function in $(0,\infty)$. The n-composition, $M_A\circ M_A\circ\cdots\circ M_A$, of the maximal operator M_A, is equivalent to the maximal operator M_{A_n}, where*

$$A_n(t)=\int_0^t A'(s)A_{n-1}\left(\frac{t}{s}\right)ds. \tag{3.1}$$

Proof: Put $A_1(t) = A(t)$. For $n = 2$, (3.1) follows by Theorem 1. By an induction argument, we have the assertion for every positive integer n. ■

In particular, the n-iterate of the Hardy-Littlewood maximal operator is equivalent to the operator

$$M_{L\log^{n-1}L}$$

(see [**L**], [**LN**], [**P**]).

Let f be a measurable function defined on $\mathbb{R}^n$. We denote by μ the distribution function of f, namely, for $t > 0$ we set

$$\mu(t) = |\{x \in \mathbb{R}^n : |f(x)| > t\}|.$$

Then we define the decreasing rearrangement f^* of f:

$$f^*(s) = \sup\{t > 0 : \mu(t) > s\} \quad (s > 0).$$

The following theorem states the equivalence between $(Mf)^*$ and $M(f^*)$.

Theorem 3 (Herz)**.** *If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ then $\forall t > 0$ it holds that*

$$4^{-n}(Mf)^*(t) \le M(f^*)(t) \le (2^n + 1)(Mf)^*(t).$$

Proof: See [**BS**]. ■

Another corollary of Theorem 2 is the following Herz type inequality for the $L\log^n L$-maximal operator.

Corollary 2. *There exist $c_1(n)$, $c_2(n) > 0$ such that, if f belongs to $L\log^k L$, $k \in \mathbb{N}$, then $\forall t > 0$*

$$c_1(M_{L\log^k L}f)^*(t) \le M_{L\log^k L}(f^*)(t) \le c_2(M_{L\log^k L}f)^*(t).$$

Proof: Let us prove the thesis for $k = 1$. Herz inequality states that

$$c_1(Mg)^*(t) \le M(g^*)(t) \le c_2(Mg)^*(t). \tag{3.2}$$

Formula (3.2), with g replaced by Mf, becomes

$$c_1(M \circ M(f))^*(t) \le M((Mf)^*)(t) \le c_2(M \circ M(f))^*(t) \tag{3.3}$$

and using (3.2) again in the right hand side of (3.3), we get

$$(M \circ M(f))^*(t) \sim M((Mf)^*)(t) \sim M \circ M(f^*). \tag{3.4}$$

Applying Theorem 1 in (3.4), with $\Phi(t) = \Psi(t) = t$, we get

$$(M_{L\log L}f)^*(t) \sim (M \circ M(f))^* \sim M_{L\log L}(f^*)(t). \tag{3.5}$$

The thesis follows arguing by induction. ■

As another application of Theorem 2, we obtain a pointwise estimate about the maximal function of the jacobian of a function f (see [**IS**], [**M**]). Namely we have the following

Theorem 4. *If* $|Df|^n \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, *then we have*

$$M_{L\log L}J(x) \le c(n)M(|Df|^n)(x) \quad a.e. \quad x \in \mathbb{R}^n \tag{3.6}$$

where $J = J_f(x) \ge 0$ *is the jacobian of* f.

Proof: In [**IS**] is proved that if $|Df|^n \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for any cube $Q \subset \mathbb{R}^n$, $0 < \sigma < 1$, then we have

$$\fint_{\sigma Q} J\,dy \le c(n)\left[\fint_Q |Df|^{\frac{n^2}{n+1}}\,dy\right]^{\frac{n+1}{n}},$$

and so

$$MJ(x) \le c(n)[M(|Df|^{n\frac{n}{n+1}})]^{\frac{n+1}{n}}(x). \tag{3.7}$$

Now, thanks to Theorem 2, applying the maximal function M to both sides in (3.7), we get

$$\begin{aligned} M_{L\log L}(J)(x) &\le cM[(M(|Df|^{\frac{n}{n+1}}))^{\frac{n+1}{n}}](x) \\ &= c\left(M_{\frac{n+1}{n}}(M[|Df|^{\frac{n^2}{n+1}}])(x)\right)^{\frac{n+1}{n}} \\ &\le cM(|Df|^n) \end{aligned} \tag{3.8}$$

as desired. ■

Remark 3. Let us observe that, in general, the following estimate

$$M_\Theta J(x) \le c(n)M_\Psi(|Df|^n)(x) \quad \text{a.e.} \quad x \in \mathbb{R}^n$$

holds, where $\Psi(t)$ is a Young function, $|Df|^n \in L^\Psi_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\Theta(t) = \Psi(t) + \frac{1}{n+1}t\int_0^t \frac{\Psi(s)}{s^2}\,ds.$$

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