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NEW CHARACTERIZATIONS OF VON NEUMANN REGULAR RINGS AND A CONJECTURE OF SHAMSUDDIN

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Abstract

A theorem of Utumi states that if R is a right self-injective ring such that every maximal ideal has nonzero annihilator, then R modulo the Jacobson radical J is a finite product of simple rings and is a von Neuman regular ring. We prove two theorems and a conjecture of Shamsuddin that characterize when R itself is a von Neumann ring, using a splitting theorem of the author on when the maximal regular ideal of a ring splits off.

Introduction.

Using splitting off theorems for the maximal Von Neumann regular ideal $M(R)$ of a ring R in the author's paper [**F**], we verify a conjecture of Shamsuddin's as a consequence of the following:

Theorem 1. *Let R be a ring with Jacobson radical J. The f.a.e.c.'s:*

(1) *R is von Neumann regular (=VNR).*
(2) *R is semiprime, $\bar{R} = R/J$ is VNR, $M(R)$ splits off, and J is an annihilator (left or right) ideal.*

As another consequence of $M(R)$ splitting we deduce:

Theorem 2. *Let R be a two-sided continuous (e.g., two-sided self-injective) ring. Then, R is VNR iff R is a semiprime ring whose Jacobson radical is an annihilator ideal.*

We now state:

Shamsuddin's Conjecture. *Let R be a ring such that R/J is a VNR and a finite product of simple ideals $A_1, \dots, A_n$. Then R is VNR iff R is a semiprime ring and J is an annihilator ideal.*

Proof of Theorem 1: The necessity if trivial. Conversely, by assumption, $R = M \times R_2$, where $M = M(R)$ and R_2 has zero regular ideal, that is, $M(R_2) = 0$. Moreover, $J \subseteq R_2$. We shall show that $R_2 = 0$, and then $R = M(R)$ is von Neumann regular. Since R is semiprime each nonzero ideal I of R satisfies

$$^{\perp}I \bigcap I = I^{\perp} \bigcap I = 0$$

where the exponent $\perp$ denotes the annihilator on the appropriate side. Clearly $^{\perp}J \supseteq M$ hence $^{\perp}J = M \oplus (^{\perp}J \bigcap R_2)$, hence assuming J is a right annihilator, then

$$(*) \qquad J = (^{\perp}J)^{\perp} = M^{\perp} \bigcap (^{\perp}J \bigcap R_2)^{\perp}.$$

Since $^{\perp}J \bigcap R_2 = \ell_{R_2}(J)$ is the left annihilator of J in R_2, then

$$\left(^{\perp}J \bigcup R_2\right)^{\perp} = \ell_{R_2}(J)^{\perp} = M + r\ell_{R_2}(J)$$

where $R()$ denotes right annihilation in R_2. Using $(*)$ and the fact that $M^{\perp} = R_2$, then

$$(**) \qquad \begin{aligned} J &= M^{\perp} \bigcap (M + r\ell_{R_2}(J)) \\ &= R_2 \bigcap (M + r\ell_{R_2}(J)). \end{aligned}$$

Since $R_2 \bigcap M = 0$, and $R_2 \supseteq r\ell_{R_2}(M)$, then $(**)$ implies that $J = r\ell_{R_2}(J)$. If $R_2 \neq 0$, then $J \neq R$, hence $\ell_2(J) \neq 0$. But R_2 is semiprime (along with R), hence $\ell_2(J) \bigcap J = 0$ which shows that $L = \ell_2(J)$ embeds canonically in $\bar{R}_2 = R_2/J$. Since

$$\bar{R} = R/J = M \times (R_2/J)$$

is VNR, then $\bar{R}_2 = R_2/J$ is also VNR, and hence L is a VNR ideal. This shows that L is an VNR ideal of R_2 contrary to the fact that $M(R_2) = 0$, that is R_2 has no nonzero regular ideals. This contradiction shows that $R_2 = 0$, and hence $R = M(R)$ is VNR as asserted. ■

Proof of Conjecture: By Theorem 2 of [**F**] (and its Corollary), $M = M(R)$ splits off in R as a ring direct summand iff the canonical image $\bar{M}$ of M in $\bar{R} = R/J$ splits off as a right or left ideal. Under the assumption of the conjecture, the only ideals of $\bar{R}$ are the finite products of a subset of $\{0, A_1, \dots, A_n\}$ without repetitions, and all of these are necessarily ring direct summands of $\bar{R}$. Thus $\bar{M}$ splits off in $\bar{R}$, hence M splits off in R, so Theorem 2 applies to prove that R is VNR. ■

Proof of Theorem 2: By the main theorem of [**F**], $M(R)$ splits off in any two-sided continuous (eg. self-injective) ring, so Theorem 1 applies. ■

We conclude with another corollary as applications of two theorems of Utumi. In

Corollary 3. *If R is a right self-injective ring in which maximal ideals have nonzero annihilators, then R is semiprime iff R is VNR. In this case, R is a finite product of simple rings.*

Proof: By Theorem 2.3 of [**U1**], R is a finite product of simple rings, and J is an annihilator ideal. Moreover, R/J is VNR by Theorem 4.8 of [**U2**]. Thus the truth of Shamsuddin's conjecture completes the proof. ■

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References

[F] C. FAITH, The maximal regular ideal of self-injective and continuous rings splits off, *Arch. Math.* **4** (1985), 511–521.

[U1] Y. UTUMI, Self-injective rings, *J. Algebra* **6** (1967), 59–64.

[U2] Y. UTUMI, Continuous rings and self-injective rings, *Trans. Amer. Math. Soc.* **118** (1965), 158–173.

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