ESTIMATES OF SOLUTIONS OF IMPULSIVE PARABOLIC EQUATIONS AND APPLICATIONS TO THE POPULATION DYNAMICS

Drumi Bainov and Emil Minchev

Abstract

A theorem on estimates of solutions of impulsive parabolic equations by means of solutions of impulsive ordinary differential equations is proved. An application to the population dynamics is given.

Introduction

The impulsive differential equations can be successfully used for mathematical simulation of processes and phenomena which are subject to short-term perturbations during their evolution. The duration of the perturbations is negligible in comparison with the duration of the process considered, and they can be thought of as momentary. The theory of impulsive ordinary differential equations started with the pioneer paper of V. Mil’man and A. Myshkis [17] and it was an object of intensive investigations during the last three decades. Detailed bibliographical information can be found in the monographs [2], [8]-[10], [16].

The theory of impulsive partial differential equations (PDE) marked its beginning with the paper [14]. The impulsive PDE provide natural framework for mathematical simulation of many processes and phenomena in theoretical physics, population dynamics, bio-technologies, chemistry, impulse technique and economics. We would like to note the applications of the impulsive PDE in the quantum mechanics. In 1992 it was introduced a model of impulsive moving mirror [19], [20] presented by the apparatus of the impulsive PDE.

It must be pointed out that this theory is an object of many lectures delivered at international meetings. Note the lectures of C. Y. Chan, L. Ke [12] delivered at the First International Conference on Dynamic
At the present time the theory of impulsive PDE undergoes rapid development \cite{3}-\cite{6}, \cite{11}-\cite{15}, \cite{19}, \cite{20}.

In this paper we give estimates of the solutions of impulsive parabolic equations and consider their applications to a model in the population dynamics. The estimates obtained can be used successfully in the qualitative theory of the impulsive parabolic equations. These estimates can be applied for obtaining of sufficient conditions for stability of the solutions of the equations investigated as well as they are useful for the entire formulation of the fundamental theory of the impulsive parabolic equations.

2. Preliminary notes

First we propose two models describing processes in the population dynamics.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial \Omega$ and $\mathcal{O} = \Omega \cup \partial \Omega$. Suppose that

$$0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$$

are given numbers such that $\lim_{k \to \infty} t_k = +\infty$.

We define

$$E = \{(t,x) \in \mathbb{R}^{1+n} : t \in \mathbb{R}_+, x \in \mathcal{O}\}, \quad \mathbb{R}_+ = [0, +\infty),$$

$$\Gamma_k = \{(t,x) \in E : t \in (t_k, t_{k+1}), x \in \Omega\}, \quad k = 0, 1, \ldots;$$

$$\Gamma = \bigcup_{k=0}^{\infty} \Gamma_k,$$

$$B_k = \{(t,x) \in E : t \in (t_k, t_{k+1}), x \in \partial \Omega\}, \quad k = 0, 1, \ldots.$$

Let $\text{C}_{\text{imp}}[E, \mathbb{R}]$ be the class of all functions $u : E \to \mathbb{R}$ such that:

(i) The functions $u|_{\Gamma_k \cup B_k}$, $k = 0, 1, \ldots$, are continuous.

(ii) For each $k$, $k = 1, 2, \ldots$, $t = t_k$, there exists

$$\lim_{(s,q) \to (t,x), s < t} u(s,q) = u(t^-, x), \quad x \in \Omega.$$

(iii) For each $k$, $k = 0, 1, \ldots$, $t = t_k$, there exists

$$\lim_{(s,q) \to (t,x), s > t} u(s,q) = u(t^+, x), \quad x \in \Omega,$$

and $u(t, x) = u(t^+, x), \quad x \in \Omega$. 
2.1. Impulsive single species model. Suppose that \( f_0 : \mathbb{R} \to \mathbb{R} \), \( u_0 : \overline{\Omega} \to \mathbb{R}, g : \{ t_k \}_{k=1}^\infty \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) are given functions.

Consider the reaction-diffusion equation (see [18])

\[
 u_t(t, x) = \kappa \Delta u(t, x) + u(t, x) f_0(u(t, x)) + c_0, \quad (t, x) \in \Gamma,
\]

subject to the initial condition

\[
 u(0, x) = u_0(x), \quad x \in \overline{\Omega},
\]

the boundary condition

\[
 u(t, x) = 0, \quad (t, x) \in \mathbb{R}_+ \times \partial \Omega
\]

and the impulses at fixed moments

\[
 u(t_k, x) = u(t_{k-}, x) + g(t_k, x, u(t_{k-}, x)), \quad x \in \overline{\Omega}, \quad k = 1, 2, \ldots
\]

The initial-boundary value problem (IBVP) (1)-(4) describes a single species population in a bounded environment. The function \( u(t, x) \) represents the population density at the point \( x \in \overline{\Omega} \) and time \( t \geq 0 \), \( f_0(u) \) is the specific growth rate of \( u \), \( \kappa > 0 \) is the diffusion coefficient, \( c_0 \geq 0 \) is a constant. Condition (4) describes instantaneous changes in the population density due to phenomena as: harvesting, disasters, immigration, emigration, etc. Particularly, the case \( g(t_k, x, u(t_{k-}, x)) < 0 \) corresponds to instantaneous harvesting of a plant population \( (c_0 = 0) \) at times \( t_k, k = 1, 2, \ldots \), while the case \( g(t_k, x, u(t_{k-}, x)) > 0 \) describes heavy immigration of a human population \( (c_0 > 0) \).

2.2. Impulsive predator-prey system. Suppose that \( v_0^{(1)}, v_0^{(2)} : \overline{\Omega} \to \mathbb{R}, g^{(1)}, g^{(2)} : \{ t_k \}_{k=1}^\infty \times \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R} \) are given functions.

Consider a system describing predator-prey interaction

\[
 v_t^{(i)} = \Delta v^{(i)} + v^{(i)}(\tilde{a} - \tilde{v}^{(i)} - \tilde{c} v^{(2)}) \text{ in } \Gamma, \quad i = 1, 2,
\]

\[
 v_{t}^{(1)}(0, x) = v_{0}^{(1)}(x), \quad v_{t}^{(2)}(0, x) = v_{0}^{(2)}(x), \quad x \in \overline{\Omega},
\]

\[
 v^{(i)}(t, x) = v^{(i)}(t, x) = 0, \quad (t, x) \in \mathbb{R}_+ \times \partial \Omega,
\]

\[
 v^{(1)}(t_k, x) = v^{(1)}(t_{k-}, x) + g^{(1)}(t_k, x, v^{(1)}(t_{k-}, x), v^{(2)}(t_{k-}, x)),
\]

\[
 v^{(2)}(t_k, x) = v^{(2)}(t_{k-}, x) + g^{(2)}(t_k, x, v^{(1)}(t_{k-}, x), v^{(2)}(t_{k-}, x))
\]

\[
 x \in \overline{\Omega}, \quad k = 1, 2, \ldots
\]
In the IBVP (5)-(10) \( v^{(1)}(t, x) \) denotes the population density of a prey and \( v^{(2)}(t, x) \) that of a predator; \( \bar{a}, \bar{c}, \bar{d}, \bar{e} \) are positive constants. Conditions (9) and (10) represent instantaneous changes in the population density of the prey and the predator, respectively. For example, the case when \( g^{(1)} \equiv 0 \) and \( g^{(2)} < 0 \) describes killing of the predators by hunters at the moments \( t_k \), \( k = 1, 2, \ldots \). Other possible situation is \( g^{(1)} > 0, g^{(2)} \equiv 0 \) which corresponds to heavy immigration of the prey due to human interference.

Motivated by the above models we consider initial boundary value problem for impulsive nonlinear parabolic equations.

Suppose that \( M[n] \) be the class of all matrices \( A = [a_{ij}]_{1 \leq i, j \leq n} \) with real entries. Let \( f : \Gamma \times \mathbb{R} \times \mathbb{R}^n \times M[n] \rightarrow \mathbb{R} \), \( \varphi : \mathbb{R}_+ \times \partial \Omega \rightarrow \mathbb{R} \), \( u_0 : \overline{\Omega} \rightarrow \mathbb{R} \), \( g : \{t_k\}_{k=1}^{\infty} \times \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R} \) are given functions.

Consider the initial-boundary value problem

\[
\begin{align*}
(11) & \quad u_t(t, x) = f(t, x, u(t, x), u_x(t, x), u_{xx}(t, x)), \quad (t, x) \in \Gamma, \\
(12) & \quad u(0, x) = u_0(x), \quad x \in \overline{\Omega}, \\
(13) & \quad u(t, x) = \varphi(t, x), \quad (t, x) \in \mathbb{R}_+ \times \partial \Omega, \\
(14) & \quad u(t_k, x) = u(t_k^-, x) + g(t_k, x, u(t_k^-, x)), \quad x \in \overline{\Omega}, \quad k = 1, 2, \ldots,
\end{align*}
\]

where \( u_x = (u_{x_1}, \ldots, u_{x_n}), \quad u_{xx} = [u_{x_i x_j}]_{1 \leq i, j \leq n} \).

**Definition 1.** A function \( u : E \rightarrow \mathbb{R} \) is a solution of the IBVP (11)-(14) if:

(i) \( u \in C_{imp}[E, \mathbb{R}] \), there exist continuous partial derivatives \( u_t(t, x), \ u_x(t, x), \ u_{xx}(t, x) \) for \( (t, x) \in \Gamma \) and \( u \) satisfies (11) on \( \Gamma \),

(ii) \( u \) satisfies (12)-(14).

**Definition 2.** A function \( f : \Gamma \times \mathbb{R} \times \mathbb{R}^n \times M[n] \rightarrow \mathbb{R} \) is said to be elliptic at \( \Gamma \) if for each point \( (t, x) \in \Gamma \) and any \( Q, S \in M[n] \) the quadratic form

\[
\sum_{i,j=1}^{n} (Q_{ij} - S_{ij}) \lambda_i \lambda_j \leq 0
\]

for arbitrary vector \( \lambda \in \mathbb{R}^n \) implies

\[
f(t, x, u, P, Q) \leq f(t, x, u, P, S)
\]

for fixed \( (t, x, u, P) \in \Gamma \times \mathbb{R} \times \mathbb{R}^n \).

We introduce the following assumption:

**H1.** The function \( f \) is elliptic at \( \Gamma \).
3. Main results

**Theorem 1.** Let the following conditions hold:

1. Assumption H1 is fulfilled.
2. There exist functions \( f_1, f_2 \in C((\mathbb{R}_+ \setminus \{t_k\}_{k=1}^{\infty}) \times \mathbb{R}, \mathbb{R}) \) such that
   \[
   f_1(t, p) \leq f(t, x, p, 0, 0) \leq f_2(t, p)
   \]
   for \((t, x) \in \Gamma, \ p \in \mathbb{R}\).
3. There exist functions \( g_1, g_2 \in C(\{t_k\}_{k=1}^{\infty} \times \mathbb{R}, \mathbb{R}) \) such that
   \[
   g_1(t_k, p) \leq g(t_k, x, p) \leq g_2(t_k, p),
   \]
   \(x \in \overline{\Omega}, \ p \in \mathbb{R}, \ k = 1, 2, \ldots\).
4. The functions \( p + g_1(t_k, p) \) and \( p + g_2(t_k, p) \) are nondecreasing on \( \mathbb{R} \) for each \( k, \ k = 1, 2, \ldots \).
5. There exist functions \( r(t) \) and \( \gamma(t) \) which are minimal and maximal solutions of the problems
   \[
   r'(t) = f_1(t, r(t)), \quad t \neq t_k ,
   \]
   \[
   \begin{align*}
   r(0) &= r_0 ,
   r(t_k) &= r(t_k^-) + g_1(t_k, r(t_k^-)), \quad k = 1, 2, \ldots
   \end{align*}
   \]
   and
   \[
   \gamma'(t) = f_2(t, \gamma(t)), \quad t \neq t_k ,
   \]
   \[
   \begin{align*}
   \gamma(0) &= \gamma_0 ,
   \gamma(t_k) &= \gamma(t_k^-) + g_2(t_k, \gamma(t_k^-)), \quad k = 1, 2, \ldots,
   \end{align*}
   \]
   respectively, where
   \[
   \begin{align*}
   r_0 &\leq u_0(x) \leq \gamma_0 , \quad x \in \overline{\Omega},
   \end{align*}
   \]
   \[
   \begin{align*}
   r(t) &\leq \varphi(t, x) \leq \gamma(t) , \quad (t, x) \in \mathbb{R}_+ \times \partial \Omega.
   \end{align*}
   \]

Then for any solution \( u \) of the IBVP (11)-(14) we have that

\[
\begin{align*}
   &r(t) \leq u(t, x) \leq \gamma(t) \quad \text{on } E.
   \end{align*}
\]

**Proof:** Let \( T_0 > 0, \ E_{T_0} = [0, T_0] \times \overline{\Omega} \) and there exists a positive integer \( m \) such that \( t_m < T_0 < t_{m+1} \). We prove that

\[
\begin{align*}
   &r(t) \leq u(t, x) \leq \gamma(t) \quad \text{on } E_{T_0}.
   \end{align*}
\]
There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the solution $\gamma(\cdot; \varepsilon)$ of the problem

\begin{align}
\gamma'(t; \varepsilon) &= f_2(t, \gamma(t; \varepsilon)) + \varepsilon, \quad t \neq t_k, \\
\gamma(0; \varepsilon) &= \gamma(0) + \varepsilon, \\
\gamma(t_k; \varepsilon) &= \gamma(t_k^-; \varepsilon) + g_2(t_k, \gamma(t_k^-; \varepsilon)) + \varepsilon, \quad k = 1, 2, \ldots, m,
\end{align}

is defined on $[0, T_0]$ and $\lim_{\varepsilon \to 0} \gamma(t; \varepsilon) = \gamma(t)$, uniformly on $[0, T_0]$. We prove that

\begin{align}
0 < u(t, x) < \gamma(t; \varepsilon) \quad \text{on } E_{T_0}.
\end{align}

Suppose (24) is not true. Then the set $Z = \{ t \in [0, T_0] : \text{there exists } x \in \Omega \text{ such that } u(t, x) \geq \gamma(t; \varepsilon) \}$ is non-empty. Defining $\bar{t} = \inf Z$. It follows from (19) and (20) that $\bar{t} > 0$ and there exists a point $\bar{x} \in \Omega$ such that:

\begin{align}
0 < u(t, x) < \gamma(t; \varepsilon), \quad (t, x) \in [0, \bar{t}] \times \Omega, \\
\gamma(\bar{t}, \bar{x}) = \gamma(\bar{t}; \varepsilon).
\end{align}

There are two cases to be distinguished:

**Case 1.** $(\bar{t}, \bar{x}) \in \Gamma$. Then we have

\begin{align*}
u_t(\bar{t}, \bar{x}) &= \gamma'(\bar{t}; \varepsilon), \\
u_x(\bar{t}, \bar{x}) &= 0, \\
\sum_{i,j=1}^n u_{x_ix_j}(\bar{t}, \bar{x})\lambda_i\lambda_j &= 0, \quad \lambda \in \mathbb{R}^n.
\end{align*}

From H1 and (15) we obtain that

\begin{align*}
0 &\leq u_t(\bar{t}, \bar{x}) - \gamma'(\bar{t}; \varepsilon) \\
&\leq f(\bar{t}, \bar{x}, u(\bar{t}, \bar{x}), 0, 0) - f_2(\bar{t}, \gamma(\bar{t}; \varepsilon)) - \varepsilon < 0,
\end{align*}

which is a contradiction.

**Case 2.** $\bar{t} = t_k$ for some $k$, $1 \leq k \leq m$. Then we have from (25) that

\begin{align}
0 &= u(t_k, \bar{x}) - \gamma(t_k; \varepsilon) \\
&= u(t_k^-, \bar{x}) + g(t_k, \bar{x}, u(t_k^-, \bar{x})) - \gamma(t_k^-; \varepsilon) - g_2(t_k, \gamma(t_k^-; \varepsilon)) - \varepsilon \\
&\leq u(t_k^-, \bar{x}) + g_2(t_k, u(t_k^-, \bar{x})) - \gamma(t_k^-; \varepsilon) - g_2(t_k, \gamma(t_k^-; \varepsilon)) - \varepsilon < 0,
\end{align}

From (16), (26) and Condition 4 of the theorem we conclude that
Estimates of solutions of impulsive parabolic equations

which is a contradiction.

Hence \( Z \) is empty and (24) follows. Since \( \lim_{\varepsilon \to 0} \gamma(t; \varepsilon) = \gamma(t) \) uniformly on \([0, T_0]\) we conclude that

\[
    u(t, x) \leq \gamma(t) \text{ on } E_{T_0}.
\]

Analogously we can prove that

\[
    r(t) \leq u(t, x) \text{ on } E_{T_0}.
\]

Since \( T_0 > 0 \) was arbitrary we get the estimates (21). \( \blacksquare \)

**Remark 1.** Existence and uniqueness results for impulsive parabolic equations are considered in [5], [11] and [15].

4. Effective estimates of the population density
in the impulsive single species model

Particular interest for the mathematical biology is the special case of IBVP (1)-(4) when \( p f_0(p) = p(a - bp), a > 0, b > 0 \) are constants and \( c_0 = 0 \). Then the IBVP (1)-(4) describing single species model takes on the form

(27) \( u_t(t, x) = \kappa \Delta u(t, x) + u(t, x)(a - bu(t, x)), \quad (t, x) \in \Gamma, \)

(28) \( u(0, x) = u_0(x), \quad x \in \Omega, \)

(29) \( u(t, x) = 0, \quad (t, x) \in \mathbb{R}_+ \times \partial \Omega, \)

(30) \( u(t_k, x) = u(t_{k-}, x) + g(t_k, x, u(t_{k-}, x)), \quad x \in \Omega, k = 1, 2, \ldots. \)

Suppose that \( g(t_k, x, p) \leq g_2(t_k, p) = \beta_k p, \beta_k > -1, x \in \Omega, k = 1, 2, \ldots, \) and

\[
    \prod_{s < t_k \leq t} \left( \frac{1}{1 + \beta_k} \right) \geq L e^{-\beta(t-s)},
\]

where \( L > 0, \beta > 0 \) are constants, \( \gamma_0 = \max_{x \in \Omega} u_0(x) > 0 \). Let \( u \) be a solution of the IBVP (27)-(30). Then we consider the problem

\[
\begin{align*}
\gamma'(t) &= \gamma(t)(a - b\gamma(t)), \quad t \neq t_k, \\
\gamma(0) &= \gamma_0, \\
\gamma(t_k) &= \gamma(t_{k-}) + \beta_k \gamma(t_{k-}), \quad k = 1, 2, \ldots.
\end{align*}
\]
We substitute \( \rho(t) = \frac{1}{\gamma(t)} \) and obtain the problem
\[
\rho'(t) = -a \rho(t) + b, \quad t \neq t_k, \\
\rho(0) = \rho_0 = \frac{1}{\gamma_0}, \\
\rho(t_k) = \frac{1}{1 + \beta_k} \rho(t_k^-), \quad k = 1, 2, \ldots. 
\]
Then we have
\[
\rho(t) = \rho_0 \prod_{0 < t_k \leq t} \left( \frac{1}{1 + \beta_k} \right) e^{-at} + \int_0^t \prod_{s < t_k \leq t} \left( \frac{1}{1 + \beta_k} \right) e^{-a(t-s)b} ds \\
\geq L \rho_0 e^{-(a+\beta)t} + Lb \int_0^t e^{-\beta(t-s)} e^{-a(t-s)} ds \\
= \left( L \rho_0 - \frac{Lb}{a+\beta} \right) e^{-(a+\beta)t} + \frac{Lb}{a+\beta}. 
\]
Therefore
\[
\gamma(t) \leq \left[ L \left( \frac{1}{\gamma_0} - \frac{b}{a+\beta} \right) e^{-(a+\beta)t} + \frac{Lb}{a+\beta} \right]^{-1} 
\]
and by Theorem 1 we conclude that
\[
u(t, x) \leq \left[ L \left( \frac{1}{\gamma_0} - \frac{b}{a+\beta} \right) e^{-(a+\beta)t} + \frac{Lb}{a+\beta} \right]^{-1}. 
\]
In the case without impulsive perturbations we have the above inequality with \( L = 1 \) and \( \beta = 0. \)

**Acknowledgements.** The authors express their deep gratitude to the referee for his valuable advices and helpful suggestions.

The present investigation was partially supported by the Bulgarian Ministry of Education, Science and Technologies under grant MM-422.

**References**


Higher Medical Institute
P.O. Box 45
1504 Sofia
BULGARIA

Primera versió rebuda el 2 de Març de 1995,
darrera versió rebuda el 2 de Setembre de 1995