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NON-FREE TWO-GENERATOR SUBGROUPS OF $\mathbf{SL}_2(\mathbb{Q})$

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Abstract

The question of whether two parabolic elements A, B of $\mathrm{SL}_2(\mathbb{C})$ are a free basis for the group they generate is considered. Some known results are generalized, using the parameter $\tau = \mathrm{tr}(AB)-2$. If $\tau = a/b \in \mathbb{Q}$, $|\tau| < 4$, and $|a| \leq 16$, then the group is not free. If the subgroup generated by b in $\mathbb{Z}/a\mathbb{Z}$ has a set of representatives, each of which divides one of $b \pm 1$, then the subgroup of $\mathrm{SL}_2(\mathbb{C})$ will not be free.

1. Background

Several papers have been written on the question of when two noncommuting parabolic elements A, B of $\mathrm{SL}_2(\mathbb{C})$ generate a free group. Two groups generated by such elements will be conjugate to each other, and thus isomorphic, as long as they have the same value for the constant $\tau = \mathrm{tr}(AB) - 2$. Most of the work done to date on this problem has put the two generators in the form

$$A_m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \; B_m = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}, \tag{1}$$

or

$$A_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \; B_\lambda = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \tag{2}$$

so that $\tau = m^2 = 2\lambda$. In this paper, we use

$$A = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \tag{3}$$

and ask: for which rational values of τ is $G_\tau = \langle A, B \rangle$ nonfree? Henceforth, τ will be regarded as an indeterminate, so that $G_\tau \subseteq \mathrm{SL}_2(\mathbb{C}(\tau))$.

We write α when considering a specific complex value for τ; the subgroup of $\mathrm{SL}_2(\mathbb{C})$ obtained when τ is specialized to $\alpha \in \mathbb{C}$ will be denoted G_α. We still write A to denote $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \in G_\alpha$. There should be no confusion.

Following [**7**] and [**10**], we say a complex number α is *τ-nonfree* if G_α is not free of rank 2 with free generators A, B. (By this definition, zero is τ-nonfree, even though $G_0 \cong \mathbb{Z}$ is a free group.) We define τ-free, m-nonfree, λ-free, etc. analogously. Thus α is τ-nonfree if and only if $\sqrt{\alpha}$ is m-nonfree if and only if $\alpha/2$ is λ-nonfree. It is known that if $|\alpha| \geq 2$, then α is m-free (see, for example [**8**, p. 167–168]) so that if $|\alpha| \geq 4$, α is τ-free. It was shown in [**10**] that λ-nonfree values are dense on the interval $(-2, 2) \subseteq \mathbb{R}$, i.e. that τ-nonfree values are dense on $(-4, 4)$. It seems reasonable, especially in light of the results in [**3**], to conjecture that all rational numbers in this latter interval are τ-nonfree. We obtain below some evidence supporting this hypothesis.

2. Good Numerators

Let G be a group generated by two elements x, y. In general, G is free of rank 2 if and only if there is no nontrivial word $y^{h_{2n}} x^{h_{2n-1}} \cdots x^{h_1}$ which gives the value 1 in G. Up to conjugacy, we may assume that such a word is x^n, y^n, or that it does, in fact, begin with a nonzero power of y and end with a nonzero power of x. In G_α,

$$A^n = \begin{pmatrix} 1 & n\alpha \\ 0 & 1 \end{pmatrix}, \; B^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}. \tag{4}$$

Certainly, then, $A^n \neq 1 \neq B^n$ if $n \neq 0$, $\alpha \neq 0$. Therefore we have

Lemma 1. *For $\alpha \neq 0$, G_α is nonfree if and only if there is some sequence of nonzero integers $h_1, \ldots, h_{2n}$, $(n > 0)$, such that $B^{h_{2n}} \cdots A^{h_1} \in G_\alpha$ gives the value* 1.

From (4), it also follows that

Lemma 2. *If α is τ-nonfree, then so is α/n for any nonzero integer n.*

We now shift our attention to the rationals. Unless stated otherwise, let $|\alpha| = |a/b| < 4$, a, b nonzero integers. We call a a *good* numerator if a/b is τ-nonfree for every b with $|a/b| < 4$.

A typical element of G_τ is

$$g = B^{h_{2n}} A^{h_{2n-1}} \cdots B^{h_2} A^{h_1} = \begin{pmatrix} 1 + \tau p_{11}(\tau) & \tau p_{12}(\tau) \\ p_{21}(\tau) & 1 + \tau p_{22}(\tau) \end{pmatrix} \tag{5}$$

where the p_{jk}'s are elements of $\mathbb{Z}[\tau]$ dependent on the exponents h_i. We assume in the sequel, unless otherwise stated, that $h_i \neq 0$ for all $i \leq 2n$.

One method we might use in order to prove G_α is nonfree is to find an element of finite order. The next theorem, which generalizes the result in [**5**], says that this usually will be a waste of time. We do, though, get three good numerators.

Theorem 1. *If $\alpha = a/b$, with a, b relatively prime integers, and $a > 0$, then G_α has torsion if and only if $a = 1, 2,$ or 3. In particular, $1, 2,$ and 3 are good numerators.*

Proof: By Lemma 2 and the fact that

$$\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\right)^4 = 1 = \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}\right)^3$$

we immediately see that if $a = 1, 2, 3$, then G_α has nontrivial elements of finite order, and is therefore nonfree. Conversely, suppose G_α has torsion. Then there is an element $g = B^{h_{2n}} A^{h_{2n-1}} \cdots B^{h_2} A^{h_1} \neq 1$ with

$$g^p = I_2$$

for some prime p. As a 2×2 matrix, g satisfies the quadratic polynomial $g^2 - \mathrm{tr}(g)g + \det(g)I_2$. Therefore,

$$g^2 - \mathrm{tr}(g)g + I_2 = 0.$$

These two relations in $\mathbb{Q}[x]$ imply that $p = 2$ or 3, and hence, that either $g = -I_2$ or $\mathrm{tr}(g) = -1$. In the former case, referring back to (5) , we see that $1 + \alpha p_{11}(\alpha) = -1$; α is a rational root of a polynomial with integral coefficients whose constant term is 2, and so α has numerator 1 or 2. If, on the other hand, $\mathrm{tr}(g) = -1$, then, again from (5), we have $2 + \alpha(p_{11}(\alpha) + p_{22}(\alpha)) = -1$ and $a = 1$ or 3. ■

Having reached this dead end, we revert to looking for sequences as in Lemma 1. Fortunately, it is possible to simplify this process greatly. We spend most of the rest of this paper doing this and defining an algorithm that (we hope) in most cases will find a sequence which will prove nonfreeness. We begin by noting (as in [**9**]) that it suffices to concentrate on only one of the matrix entries.

Lemma 3. *Let $\alpha \in \mathbb{C}$, $g = B^{h_{2n}} \cdots A^{h_1}$, $n \geq 1$, $h_i \neq 0$ for $i < 2n$. If $g = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$, then α is τ-nonfree.*

Proof: $[B, g] = BgB^{-1}g^{-1} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$, and so $[B, [B, g]] = 1$. ■

In order to use this lemma, we need to relate a sequence of nonzero integer exponents $h_1 \ldots h_l$ (hereafter called an h-sequence) to the entries of its associated matrix. Thus, given $\alpha \in \mathbb{C}$ and an h-sequence $h_1, \ldots, h_\ell$ we define (following [**9**] and [**3**]) a new sequence recursively:

$$\begin{aligned} x_0 &= 0 \\ x_1 &= 1 \\ x_{2n} &= x_{2n-2} + \alpha h_{2n-1} x_{2n-1} \\ x_{2n+1} &= x_{2n-1} + h_{2n} x_{2n}. \end{aligned}$$

Then

$$g_{2n-1} = A^{h_{2n-1}} \cdots B^{h_2} A^{h_1} = \begin{pmatrix} * & x_{2n} \\ * & x_{2n-1} \end{pmatrix} \in G_\alpha \tag{6}$$

$$g_{2n} = B^{h_{2n}} A^{h_{2n-1}} \cdots B^{h_2} A^{h_1} = \begin{pmatrix} * & x_{2n} \\ * & x_{2n+1} \end{pmatrix} \in G_\alpha. \tag{7}$$

In particular, G_α is nonfree if and only if there is an h-sequence of nonzero $h_1, \ldots, h_{2n-1}$ with $x_{2n} = 0$.

When $\alpha = a/b$, we define

$$z_{2n} = b^n x_{2n}, \quad z_{2n+1} = b^n x_{2n+1}$$

and see that these will satisfy

$$\begin{aligned} z_0 &= 0 \\ z_1 &= 1 \\ z_{2n} &= b z_{2n-2} + a h_{2n-1} z_{2n-1} \\ z_{2n+1} &= b z_{2n-1} + h_{2n} z_{2n}. \end{aligned}$$

Since $x_\ell = 0$ when $z_\ell = 0$, we still have nonfreeness if and only if there is some h-sequence with some $z_{2n} = 0$. The advantage here is that we are now dealing exclusively with integers.

A few things are worth noting at this point. Using induction, it is easy to see that no matter what the h_i's are, we have

$$a | z_{2n};$$

also, if we assume that a, b are relatively prime,

$$(a, z_{2n+1}) = 1$$

for any $n \geq 0$. Thus, if $a \neq \pm 1$, it is never the case that $z_{2n+1} = 0$. Second, if some h-sequence yields $z_{2n-1} = \pm 1$ $(n > 1)$, then choosing $h_{2n-1} = \mp b z_{2n-2}/a$ (which is an integer) results in $z_{2n} = 0$. Note that if this particular $h_{2n-1} = 0$, then $z_{2n-2} = 0$, so a/b is τ-nonfree anyhow. This leads to

Theorem 2. *If $\alpha = a/(ar \pm 1)$, $r \neq 0$, then α is τ-nonfree.*

Proof: $h_1 = 1$, $h_2 = -r$ yields $z_2 = a$, $z_3 = \pm 1$. ■

Note that since $h_2 = -r$, we can do this only when $r \neq 0$. Indeed, if we allowed $r = 0$, we would contradict the fact that unless $|\alpha| < 4$, G_α is free.

This theorem gives another way of showing that $1, 2$, and 3 are good numerators. Now consider $a = 4$. $4/(4r \pm 1)$ is τ-nonfree (when $r \neq 0$) by the above theorem. $4/4r$ and $4/(4r + 2)$ are τ-nonfree because (respectively) 1 and 2 are good numerators. Therefore, 4 is a good numerator. In a similar way, we see that 6 is a good numerator.

It is also worth noting that if $(z_i, z_{i+1}) = d \neq 1$, then $d | z_j$ for all $j \geq i$. Hence if all we care about is finding some $z_\ell = 0$, we can use z_i/d and z_{i+1}/d for calculating higher values of z. For example, $z_{2n+1} = d(b\frac{z_{2n-1}}{d} + h_{2n}\frac{z_{2n}}{d})$. In order to take advantage of this situation, we define

$$\tilde{z}_i = z_i/(z_i, z_{i-1}) \text{ (for } i > 1).$$

These will be called *effective*, rather than actual, values. If $(z_{i-1}, z_i) = 1$, then we will get $(z_i, z_{i+1}) = d$ when and only when $(h_{i-1}, b) = d$.

In general, given a rational number $\alpha = a/b$, how can we construct a satisfactory h-sequence? An obvious strategy is, at each step, to choose an h_{i-1} that will minimize the absolute value of z_i. This method has the advantage that, in order to calculate z_{i+1}, we are really finding some

number (namely bz_{i-1}) either modulo z_i or modulo az_i. Thus small values of z_i beget small values of z_{i+1}. It is also easier to tell a computer to find an h-sequence if we simply use this 'greedy' algorithm.

One reason we might *not* want to take the smallest value is that a larger value for z_{i+1} might have the property that $(z_{i+1}, z_i) = d > 1$. Then $\tilde{z}_{i+1} = z_{i+1}/d$ might be smaller than the greedy algorithm value, and we use $z_i/d, z_{i+1}/d$ to calculate z_{i+2}. The author has done some calculations to try to determine whether rationals are τ-nonfree using a method that takes advantage of this possibility. In this modified greedy algorithm, we first find the h provided by the 'pure' greedy algorithm. (That is, we find the h_{i-1} that would minimize z_i). Then we test the resulting $\tilde{z}_i$ against the two z-values obtained from the h-values $h \pm 1$ to find the smallest value of $z_i/(h_{i-1}, b))$, making sure, of course, that we never use $h_{i-1} = 0$. Practical experimentation shows that, in general, this modified greedy algorithm gives much better results than the pure greedy algorithm. Indeed, in [**9**], where only the pure greedy algorithm seems to have been followed, it was not determined whether $7/4$ is m-nonfree; that is, whether $49/16$ is τ-nonfree. If we let $\alpha = 49/16$ and use this modified greedy algorithm, then we find that $z_{148} = 0$, and therefore, that $49/16$ is τ-nonfree.

There are cases where a greedy algorithm does not yield $z_i = 0$, and yet does provide enough information to prove τ-nonfreeness with the help of the following theorem:

Theorem 3. *Given $\alpha = a/b \in \mathbb{Q}$, if there is an infinite sequence of nonzero integers $h_1, h_2, \ldots$ and some $N \in \mathbb{Z}^+$ such that for all i, $|\tilde{z}_i| < N$, then α is τ-nonfree.*

Proof: Because the $\tilde{z}_i$'s are bounded, so are the entries of the ordered pairs $(z_i/(\gcd(z_i,z_{i+1})), z_{i+1}/(\gcd(z_i,z_{i+1})))$. Therefore, there must eventually be a repetition among such pairs, i.e. there is a pair (z_{2n}, z_{2n+1}), an integer d, and some $j \neq 0$ so that $dz_{2n} = z_{2n+2j}$, $dz_{2n+1} = z_{2n+2j+1}$. Then, if we take g_{2n} as in (7), and $g' = B^{h_{2n+2j}} A^{h_{2n+2j-1}} \cdots A^{h_{2n+1}}$,

$$g_{2n}\begin{pmatrix}0\\1\end{pmatrix} = b^{-n}\begin{pmatrix}z_{2n}\\z_{2n+1}\end{pmatrix},$$
$$g'g_{2n}\begin{pmatrix}0\\1\end{pmatrix} = db^{-n-j}\begin{pmatrix}z_{2n}\\z_{2n+1}\end{pmatrix},$$

hence

$$g_{2n}^{-1}g'g_{2n}\begin{pmatrix}0\\1\end{pmatrix} = db^{-j}\begin{pmatrix}0\\1\end{pmatrix}.$$

Certainly $g_{2n}^{-1}g'g_{2n}$ is neither of the form B^k, nor the word 1 in the free group on $\{A, B\}$; but the only matrices that have $\binom{0}{1}$ as an eigenvector are of the form $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$. Therefore, by Lemma 3, G_α will be nonfree. ■

The situation treated in this theorem comes up in two ways: on the one hand, we shall give general arguments below that for certain kinds of α, h-sequences with the stated properties can be found. On the other hand, in particular calculations, we occasionally come across sequences that do not eventually yield $z_i = 0$, but do have the property that there is some $\ell > i$ and some d so that $dz_i = z_\ell$, $dz_{i+1} = z_{\ell+1}$. In that case, to show τ-nonfree ness, we can either use the proof of the above theorem; or we note that we can force $|\tilde{z}_{i+k}| = |\tilde{z}_{\ell+k}|$ for all $k > 0$, and thus $N = \max\{|\tilde{z}_j| : j \le \ell\}$ satisfies the hypotheses of the theorem.

We now have some ways to prove that a particular rational number a/b is τ-nonfree. The next theorem allows us to use this fact to show that other rationals, related to a/b, are τ-nonfree, and brings us closer to finding good numerators.

Theorem 4 (cf. [3, Theorem 2.6]).
Suppose that $\alpha = a/b$ is τ-nonfree, and that $h_1, \dots, h_\ell$ is an h-sequence such that for some $j < \ell$, $z_{\ell+1} = 0$ or $dz_j = z_\ell$, $dz_{j+1} = z_{\ell+1}$. Let $M = \mathrm{lcm}\{z_i | 1 \le i \le \ell\}$. If $r \in \mathbb{Z}$, and $(Mr + b) \nmid M$, then $a/(Mr + b)$ is τ-nonfree.

Proof: First, let $\alpha' = a/(Mr+b)$. We will show, using induction, that there is a sequence $h'_1, \dots, h'_\ell$ so that $z_i = z'_i$ for all $i \le \ell + 1$. Then α' will be τ-nonfree for the same reason that α is.

By definition, $z_0 = z'_0 = 0$ and $z_1 = z'_1 = 1$, so suppose that $z_i = z'_i$ for all $i < 2n$. Let $h'_i = h_i + s_i$ (s_i is an integer variable and will be given a value below). Then

$$\begin{aligned} z'_{2n} &= (Mr + b)z'_{2n-2} + a(h_{2n-1} + s_{2n-1})z'_{2n-1} \\ &= (bz_{2n-2} + ah_{2n-1}z_{2n-1}) + Mrz_{2n-2} + as_{2n-1}z_{2n-1} \\ &= z_{2n} + Mrz_{2n-2} + as_{2n-1}z_{2n-1}, \\ z'_{2n+1} &= \cdots \\ &= (bz_{2n-1} + h_{2n}z'_{2n}) + Mrz_{2n-1} + s_{2n}z'_{2n}. \end{aligned}$$

If we choose

$$s_{2n-1} = -\frac{Mrz_{2n-2}}{z_{2n-1}a}, \quad s_{2n} = -\frac{Mrz_{2n-1}}{z_{2n}},$$

we get $z'_{2n} = z_{2n}$ and $z'_{2n+1} = z_{2n+1}$.

It remains to show that h'_i is a nonzero integer. We have already seen that $a|z_{2n-2}$ for any n, so $z_i|M$ implies that s_i (and hence h'_i) is an integer. If $h'_{2n-1} = 0$, then $z_{2n} = z'_{2n} = (Mr+b)z'_{2n-2} = (Mr+b)z_{2n-2}$. We can assume that $z_{2n-2} \neq 0$, and thus, since $z_{2n}|M$, we have $(Mr + b)|M$, which contradicts the hypothesis. We reach a similar contradiction if we assume that $h'_{2n} = 0$. ■

In general, the condition $(Mr + b) \nmid M$ does not seem to eliminate any interesting cases. Also, at no point do we restrict r to positive integers. Thus as long as a/b is τ-nonfree and $(M(-r) + b) \nmid M$, we have that $\alpha'' = a/(M(-r) + b)$ is τ-nonfree. Lemma 2 then tells us that $-\alpha'' = a/(Mr - b)$ is also τ-nonfree.

3. Some Calculations

Consider the case $\alpha = 5/3$. The h-sequence $(h_1 =)1, -1, 1, 1, 3(= h_5)$ yields the z-sequence $(z_0 =)0, 1, 5, -2, 5, -1, 0(= z_6)$. Here, $M = \mathrm{lcm}\{1, 5, -2\} = 10$. We conclude that all rationals of the form $5/(10r \pm 3)$ are τ-nonfree. We have already shown that the groups G_α for $\alpha = 5/10r$ and $5/(10r + 5)$ have torsion; $5/(10r \pm 1)$ (for $r \neq 0$) and $5/(10r \pm 4) = 5/(5(2r \pm 1) \mp 1)$ are τ-nonfree by Theorem 2; and if $r \neq 0$, then $5/(10r \pm 2) = 5/2(5r \pm 1)$ is τ-nonfree by Theorem 2 and Lemma 2. $5/2$ may be checked separately: the greedy algorithm yields a satisfactory h-sequence. Therefore 5 is a good numerator.

To show that $\alpha = 5/3$ is τ-nonfree, we found a specific h-sequence; but it happens that whenever $a = 5$ and b is odd, the h_i's can be chosen so that $z_{2n} = \pm 5$ (or 0), and so that $z_{2n+1} \in \{\pm 1, \pm 2\}$. Thus, even if we could never get $z_i = 0$, we would have $|z_i| \leq 5$ for all i. Then, by Theorem 3, a/b is τ-nonfree. This phenomenon is examined in the following theorem.

Theorem 5. *Let $\alpha = a/b$ $(b > 1)$. Suppose there exists a finite set of integers $\mathcal{I}$ such that for all positive integers k, $b^k \equiv i \bmod a$ for some $i \in \mathcal{I}$, and that for all $i \in \mathcal{I}$, $i|(b-1)$ or $i|(b+1)$. Then α is τ-nonfree. Furthermore, if we choose M so that $i|M$ for all $i \in \mathcal{I}$, and let $b' = aM \pm b$, then a/b' is also τ-nonfree (provided that $b' \nmid M$).*

Proof: We will find an infinite (nonzero) sequence $h_1, h_2, \ldots$, so that, if $n > 0$,

1) $z_{2n} = \pm a$,
2) $z_{2n+1} \equiv b^k \bmod a$ for some k, and
3) $z_{2n+1} \in \mathcal{I}$.

Then Theorem 3, with $N = \max\{|a|, |i| \ (i \in \mathcal{I})\}$ will imply that α is τ-nonfree.

We may assume that $1 \in \mathcal{I}$. Choose $h_1 = 1$, then $z_1 = 1 \in \mathcal{I}$, $z_2 = a$. Now suppose, inductively, that z_{2n-1}, z_{2n} satisfy the above three conditions. Note that

$$z_{2n+1} = z_{2n-1}b + z_{2n}h_{2n} = z_{2n-1}b \pm ah_{2n}$$

by hypothesis. Also, $z_{2n-1} \equiv b^k \bmod a$, so $z_{2n-1}b \equiv b^{k+1} \equiv i_{2n+1} \bmod a$ for some $i_{2n+1} \in \mathcal{I}$. Thus, we can choose h_{2n} so that $z_{2n+1} = i_{2n+1} \in \mathcal{I}$. Because $b \neq 1$, and $i_{2n+1}|(b \pm 1)$, it cannot be that $z_{2n-1}b = i_{2n+1}$, so $h_{2n} \neq 0$.

For the even terms, we have

$$z_{2n+2} = z_{2n}b + az_{2n+1}h_{2n+1} = a(\pm b + i_{2n+1}h_{2n+1}).$$

Again, i_{2n+1} divides one of $(b \pm 1)$, hence there is some h_{2n+1} $(\neq 0)$ with $i_{2n+1}h_{2n+1} = \mp(b \pm 1) \neq 0$, and then $z_{2n+2} = \pm a$.

The last statement of the theorem follows immediately from Theorem 4. ■

Note that Theorem 2 is a special case of the above result.

What this theorem says is that G_α is τ-nonfree whenever the subgroup of $U(\mathbb{Z}/a\mathbb{Z})$ generated by b is well-behaved. From this, we easily prove that the next few numerators are good.

Corollary 1. *If $|a| \leq 11$, and $b = 6m \pm 1$, with $|\alpha| = |a/b| < 4$, $(a, b) = 1$, then α is τ-nonfree.*

Proof: Assume, via Lemma 2, that $a \geq 0$, $b > 0$. We have already shown that 0 through 6 are good numerators. For $7 \leq a \leq 11$, let $\mathcal{I} = \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6\}$. We have assumed that $(a, b) = 1$, so when $a = 10$ it is never the case that $b^k \equiv 5 \bmod a$. For all other a's under consideration, this is a complete set of residues modulo a. Since b is odd, one of $b + 1$, $b - 1$ will be divisible by 4, and certainly $\pm 1, \pm 2, \pm 3, \pm 6$ divide $6m = b \mp 1$. The conditions of Theorem 5 are therefore satisfied, as long as $m \neq 0$, which is automatic if $|a/b| < 4$, $a > 6$. ■

Lemma 4. *If $|a| \leq 11$, then a is a good numerator.*

Proof (sketch): Again, assume that a, $b > 0$. Suppose some such a is not a good numerator, and b is the lowest denominator with $|a/b| < 4$ and $\alpha = a/b$ τ-free. Then $|a| > 6$ and, from the above corollary, b is divisible by 2 or 3. If b is composite, then, in view of Lemma 2 and our minimality assumption, $\alpha = 9/4$ or $11/4$. Otherwise, α is one of $7/2$, $7/3$, $8/3$, $10/3$, $11/3$. Theorem 5 takes care of $7/3$ ($\mathcal{I} = \{\pm 1, \pm 2, \pm 4\}$), and the rest may be proved τ-nonfree by using the modified greedy algorithm (see appendix). ■

For higher numerators a, it is usually impossible to restrict the values of z_{2n} to $\pm a$. The modified greedy algorithm, together with Theorem 4, still works most of the time, but some tinkering is necessary to keep the value of M within reason.

Lemma 5. *The integers* $12, 13, 14, 15, 16$ *are good numerators.*

Proof (sketch): Let $\alpha = 12/b$. Because of lower good numerators and Theorem 2, we need only consider the case $b \equiv \pm 5 \bmod 12$. $12/5$ may be proved τ-nonfree using the modified greedy algorithm (see appendix), and so if $5|b$, we're done by Lemma 2. Taking $\mathcal{I} = \{\pm 1, \pm 5\}$, we are done if $b \equiv \pm 1 \bmod 5$. If $b \equiv \pm 2 \bmod 5$ and $\pm 5 \bmod 24$ (respectively $\pm 5 \bmod 36$), we choose $h_1 = 2$ so $z_2 = 24$, (respectively $h_1 = 3, z_2 = 36$), and then $z_3 = \pm 5$, $z_4 = \pm 12$, $z_5 = \pm 1$. If $b \equiv \pm 2 \bmod 5$ and $\pm 7 \bmod 36$, then we can force $z_2 = 12$, $z_3 = \pm 5$, $z_4 = \pm 36$, $z_5 = \pm 1$. (The $\pm$ signs in the preceding few sentences are independent.) Note that all values of z_i used thus far divide 360. We now have, using Theorem 4 that $12/(360r \pm b)$ is τ-nonfree for all $b < 180$ except for 17 and 127. We take care of these final two cases either by raising the M (of Theorem 4) to 2520, or equivalently, by considering the possible residues modulo 7 of r (in the denominator of $12/(360r \pm b)$).

13 through 16 can be shown to be good numerators in a similar manner. The details are tedious, and we rely on a computer (see appendix) for the first time because the M's (from Theorem 4) are larger. Results are displayed in the appendix. ■

It may or may not be a coincidence that sticking to the 'pure' greedy algorithm, both $12/17$ and $12/127$ —the two numbers that give us trouble in the above proof— quickly settle into repeating series rather than reaching zero. $12/17$ seems to be the simplest rational number for which this happens. This repetition phenomenon is discussed in, and is the reason for, Theorem 3.

In order to show that numbers of the form $16/(16r \pm 7)$ are free, the modified greedy algorithm (together with some common sense), required $M = 10,080$. This seems bad, in that it is a large M for a relatively small a. However, in [**3**] when it was shown, using a result analogous to Theorem 4, that $4/b$ is m-nonfree (i.e. that $16/b^2$ is τ-nonfree, which follows from the above), $M = 50,400$ was needed.

To continue in this manner —inching our way up the number line— is probably futile. It may not even be possible. As might be expected, for rationals closer to 4, the number of z's it is necessary to calculate, as well as the values of the z's (and the $\tilde{z}$'s), grows large. The author was unable to prove that $27/7$ (or any rational value greater than $27/7$) is τ-nonfree.

4. Appendix

Table I gives some of the details of a proof that the integers 13 through 16 are good numerators. For the most part, the entries in this table were obtained by finding z-sequences for all denominators of the indicated form (with the provisos $(a, b) = 1$, $|a/b| < 4$, and $|b| < \frac{1}{2}M$; the latter is used, and is sufficient in the cases considered, because it assures that $Mr \pm b \nmid M$ for $r \neq 0$, and because every denominator satisfying the first two provisos can be written as $Mr \pm b$ for some b subject to all the provisos). In some cases, though, to obtain the value of M for an entry, instead of calculating a z-sequence, it was convenient to apply the knowledge that the entries above the entry in question are τ-nonfree; for example, once we know that $13/(13r \pm 4)$ is τ-nonfree, we can deduce that if $b = 13r \pm 6$ and $5|b$, then $b/5 \equiv \mp 4 \bmod 13$, and so $13/b$ is τ-nonfree by Lemma 2.

Table II lists some of the values of α that have been discussed above (as well as a few which are interesting because they are close to 4), and a reason we can infer that α is τ-nonfree. The modified greedy algorithm is used for all listed values; the only exception is that in some cases, h_1 is taken to be 2 rather than 1. Note that because we are not using the 'pure' greedy algorithm, we do not get a repeating sequence for $\alpha = 12/17$, as was mentioned above.

A Fortran program run on a Sun 3/50 was used to obtain all numerical calculations included herein.

Table I

Numerator	Form of Denominator	M used
13	$13r \pm 3, \pm 4$	$312 = 13 \cdot 24$
	$13r \pm 5$	130
	$13r \pm 2, \pm 6$	780
14	$420r \pm 137$	$1260 = 14 \cdot 90$
	all others of the form $14r \pm 3$	420
	$420r \pm 187$	1260
	all others of the form $14r \pm 5$	420
15	$120r \pm 13$	$2520 = 15 \cdot 8 \cdot 3 \cdot 7$
	$120r \pm 43, \pm 77$	840
	all others of the form $15r \pm 2$	120
	$120r \pm 37$	2520
	$120r \pm 53$	840
	all others of the form $15r \pm 8$	120
16	$16r \pm 7$	$10080 = 16 \cdot 2 \cdot 9 \cdot 7 \cdot 5$
	$10080r \pm 3229$	20160
	$10080r \pm 61, \pm 2147, \pm 2851$	110880
	all others of the form $16r \pm 3$	10080
	$10080r \pm 1163$	110880
	$10080r \pm 3659$	20160
	all others of the form $16r \pm 5$	10080

Table II

α	Reason α is τ-nonfree	(Partial) h-sequences
5/2	$z_6 = 0$	$(h_1 =)1, -1, 1, -1, -2$
7/2	$z_{10} = 0$	$2, -1, 1, -1, 1, -1, -1, 1, 2$
8/3	$z_6 = 0$	$1, -1, 1, -1, 6$
10/3	$z_{14} = 0$	$1, -1, 1, -1, 1, -1, 2, \ldots$
11/3	$z_{24} = 0$	$1, \ldots$
9/4	$z_6 = 0$	$1, -1, 1, -2, -2$
11/4	$z_8 = 0$	$1, -1, 1, -1, 2, 1, -4$
12/5	$z_8 = 0$	$1, -1, 1, -1, -1, 5, -1$
12/17	$z_8 = 0$	$1, -1, -3, -4, 3, 17, -1$
12/127	$z_2 = z_6 = 12,\ z_3 = z_7 = -5$	$1, -11, 25, 26, 23, 116$
49/16	$z_{148} = 0$	$1, -1, 1, -1, 1, -3, -18, \ldots$
15/4	$z_{18} = 0$	$2, \ldots$
19/5	$z_{38} = 0$	$2, \ldots$
23/6	$z_{28} = 0$	$2, \ldots$

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