Publicacions Matemàtiques, Vol **39** (1995), 355–366.

ABELIAN INTEGRALS OF QUADRATIC HAMILTONIAN VECTOR FIELDS WITH AN INVARIANT STRAIGHT LINE*

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Abstract

We prove that the lowest upper bound for the number of isolated zeros of the Abelian integrals associated to quadratic Hamiltonian vector fields having a center and an invariant straight line after quadratic perturbations is one.

1. Introduction

Let $H(x, y)$ be a real polynomial of degree $n + 1$, and let $P(x, y)$ and $Q(x, y)$ be real polynomials of degree at most m. The problem of finding an upper bound $N(n, m)$ for the number of isolated zeros of the Abelian integrals

$$I(h) = \int_{\Gamma_h} Q(x, y)\, dx - P(x, y)\, dy, \tag{1.1}$$

where Γ_h varies in the compact components of $H^{-1}(h)$ is called the *weakened 16th Hilbert problem.* It was posed by Arnold in [**1**].

The weakened 16th Hilbert problem is closely related to the problem of determinating an upper bound for the number of limit cycles of the perturbed Hamiltonian system

$$(1.2)_\varepsilon \qquad \begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial y} + \varepsilon P(x, y), \\ \frac{dy}{dt} &= -\frac{\partial H}{\partial x} + \varepsilon Q(x, y), \end{aligned}$$

*The authors are partially supported by DGICYT grants, the first and third authors are also partically supported by NSF of China, the second one is also partially supported by a CIRIT grant.

where $0 < |\varepsilon| << 1$. The relationship between both problems comes from the following two facts:

(1) If $I(h^*) = 0$ and $I'(h^*) \neq 0$, then there exists a hyperbolic limit cycle L_{h^*} of system $(1.2)_\varepsilon$ such that $L_{h^*} \to \Gamma_{h^*}$ as $\varepsilon \to 0$; and conversely, if there exists a hyperbolic limit cycle L_{h^*} of system $(1.2)_\varepsilon$ such that $L_{h^*} \to \Gamma_{h^*}$ as $\varepsilon \to 0$, then $I(h^*) = 0$.

(2) The total number of isolated zeros of (1.1) (taking into account their multiplicity) is an upper bound for the number of limit cycles of system $(1.2)_\varepsilon$ tending to some periodic orbit Γ_h of system $(1.2)_{\varepsilon=0}$ when $\varepsilon \to 0$.

Khovansky [**16**] and Varchenko [**24**] proved independently that $N(n, m)$ is finite, but an explicit expression for $N(n, m)$ is unknown. Many authors have contributed to estimate the number $N(n, m)$ for some values of n and m, and for some classes of polynomial functions $H(x, y)$, see for instance Bogdanov [**3**] and [**4**], Petrov [**20**] and [**21**], Cushman and Sanders [**10**], Dumortier, Roussarie and Sotomayor [**13**], Drachman, van Gils and Zhang [**12**], Li and Rousseau [**18**], Gavrilov and Horozov [**14**], Li, Llibre and Zhang [**17**], ...

A *quadratic Hamiltonian vector field* is a vector field of the form $(\partial H/\partial y, -\partial H/\partial x)$ where $H = H(x, y)$ is a real polynomial of degree 3.

Our main result is to show that $N(2, 2) = 1$ for the quadratic perturbations of the quadratic Hamiltonian vector fields having an invariant straight line under the flow defined by the vector field. This result is proved in Section 2. In Section 3 we characterize the phase portraits of the quadratic Hamiltonian vector fields having an invariant straight line, we will see that there are six of such phase portaits.

We want to thank to Armengol Gasull his comments on a preliminary version of this paper.

2. Statement of the main results

Let $(\partial H/\partial y, -\partial H/\partial x)$ be a quadratic Hamiltonian vector field, as usual we say that

$$\frac{dx}{dy} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x},$$

is its associated *quadratic Hamiltonian system*, and vice versa. For abbreviation we denote by QH the class of quadratic Hamiltonian systems having a center and an invariant straight line.

Lemma 1. *Every system in QH can be reduced to another system in QH of the form*

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial y} = 2xy, \\ \frac{dy}{dt} &= -\frac{\partial H}{\partial x} = -y^2 + f(x), \end{aligned} \tag{2.1}$$

where $f(x)$ is a polynomial of degree 2.

Proof: Let

$$\frac{dx}{dt} = \frac{\partial H_1}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H_1}{\partial x}, \tag{2.2}$$

be a system in QH. By a translation we can put the origin of coordinates on the invariant straight line of system (2.2). Then by a rotation we can transform the invariant straight line of system (2.2) in the y-axis. Since both transformations are canonical, they preserve the Hamiltonian structure of the system, and clearly they do not increase the degree of the Hamiltonian function. Thus system (2.2) can be written in the form

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H_2}{\partial y} = x(ax + by + c), \\ \frac{dy}{dt} &= -\frac{\partial H_2}{\partial x}. \end{aligned} \tag{2.3}$$

Notice that $b \neq 0$; otherwise $\dot{x} = x(ax+c)$, and consequently system (2.3) would not have periodic orbits.

Now consider the change of variables

$$\overline{x} = x, \quad b\overline{y} = ax + by + c. \tag{2.4}$$

Clearly

$$\det\left(\frac{\partial(\overline{x},\overline{y})}{\partial(x,y)}\right) = 1,$$

and the transformation (2.4) is canonical. For simplicity rewriting (x, y) instead of $(\overline{x}, \overline{y})$ in system (2.3) after doing the change of variables (2.4), we get

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H_3}{\partial y} = bxy, \\ \frac{dy}{dt} &= -\frac{\partial H_3}{\partial x}. \end{aligned} \tag{2.5}$$

Since system (2.5) belongs to QH, it follows that

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H_3}{\partial y} = bxy, \\ \frac{dy}{dt} &= -\frac{\partial H_3}{\partial x} = -\frac{b}{2}y^2 + g(x), \end{aligned}$$

where $g(x)$ is a polynomial of degree 2. Now rescaling the variable y from y to $2y/b$, the lemma follows. ■

Notice that system (2.1) is invariant under the symmetry $(x, y, t) \mapsto (x, -y, -t)$ and that it has $x = 0$ as an invariant straight line.

Now we present two preliminary results on the Abelian integrals $I(h)$ (see Section 1). These results will allow us to simplify the quadratic perturbation of a system in QH. They are well-known but since we cannot find any reference for them, we prove them here.

Proposition 2. *The following equality holds*

$$I(h) = \int_{\Gamma_h} \left(Q(x, y) + \int \frac{\partial P(x, y)}{\partial x}\, dy \right) dx.$$

Proof: From Stokes theorem we get

$$\begin{aligned} &\int_{\Gamma_h} \left[P(x, y)\, dy + \left(\int \frac{\partial P(x, y)}{\partial x}\, dy \right) dx \right] \\ &= \int_{\mathrm{Int}(\Gamma_h)} \left[\frac{\partial P(x, y)}{\partial x}\, dx \wedge dy + \frac{\partial}{\partial y} \left(\int \frac{\partial P(x, y)}{\partial x}\, dy \right) dy \wedge dx \right] \\ &= \int_{\mathrm{Int}(\Gamma_h)} \left(\frac{\partial P(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial x} \right) dx \wedge dy \\ &= 0. \end{aligned}$$

Then from (1.1) the propesition follows. ■

From Proposition 2 it follows immediately.

Corollary 3. *The Abelian integral $I(h)$ associated to the system*

$$\frac{dx}{dt} = \frac{\partial H}{\partial y},$$
$$\frac{dy}{dt} = -\frac{\partial H}{\partial x} + \varepsilon \left(Q(x,y) + \int \frac{\partial P(x,y)}{\partial x}\, dy \right),$$

is identical to the Abelian integral associated to system (1.2).

From Proposition 2 and Corollary 3 we get easily the following result.

Corollary 4. *Assume that the perturbation (P,Q) of system (1.2)$_{\varepsilon=0}$ is of degree 2. Then the Abelian integral $I(h)$ associated to system (1.2)$_\varepsilon$ is identical to the Abelian integral associated to the following system*

$$\frac{dx}{dt} = \frac{\partial H}{\partial y},$$
$$\frac{dy}{dt} = -\frac{\partial H}{\partial x} + \varepsilon(\mu_1 y + \mu_2 xy + \mu_3 y^2).$$

Now we will apply Corollary 4 to a system in QH.

Lemma 5. *Every system in QH after a quadratic perturbation has the same Abelian integral that the following perturbed system of QH:*

$$\frac{dx}{dt} = \frac{\partial H}{\partial y} = 2xy,$$
$$\frac{dy}{dt} = -\frac{\partial H}{\partial x} = -y^2 + f(x) + \mu_1 y + \mu_2 xy, \tag{2.6}$$

where $f(x)$ is a polynomial of degree 2.

Proof: By Lemma 1 every system in QH can be transformed in the form (2.1) with Hamiltonian function

$$H(x,y) = x(y^2 + F(x)), \tag{2.7}$$

where $F(x)$ is a polynomial of degree 2 such that minus the derivative of $xF(x)$ is equal to $f(x)$. From Corollary 4 the Abelian integral of a Hamiltonian system (2.1) with an arbitrary quadratic perturbation is the same that the Abelian integral of the system:

$$\frac{dx}{dt} = 2xy,$$
$$\frac{dy}{dt} = -y^2 + f(x) + \mu_1 y + \mu_2 xy + \mu_3 y^2. \tag{2.8}$$

Since for a periodic orbit Γ_h of the Hamiltonian system we have $\Gamma_h \cap \{x = 0\} = \emptyset$, it follows from (2.7) that

$$\int_{\Gamma_h} y^2\, dx = \int_{\Gamma_h} \left(\frac{h}{x} - F(x)\right) dx = 0.$$

So the Abelian integral of system (2.8) is the same that the Abelian integral of system (2.6). ∎

Notice that system (2.6) only depends on two parameters.

Now we state our main result.

Theorem 6. *We have that $N(2,2) = 1$ for the quadratic perturbation of quadratic Hamiltonian vector fields having an invariant straight line.*

Proof: We denote by

$$I_i(h) = \int_{\Gamma_h} x^i y\, dx$$

for $i = 0, 1$. Then, from Lemma 5, the Abelian integral for the quadratic perturbation of a system in QH is

$$I(h) = \mu_1 I_0(h) + \mu_2 I_1(h),$$

where $h \in (h_0, h_1)$, and h_0 and h_1 corespond to some equilibrium point and some homoclinic or heteroclinic loop of the phase partrait of the Hamiltonian system

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial y} = 2xy, \\ \frac{dx}{dt} &= -\frac{\partial H}{\partial x} = -y^2 + f(x), \end{aligned}$$

in the Poincaré disc.

Since (2.6) is a quadratic system having an invariant straight line, by the properties of quadratic systems (see [**5**], [**6**], [**15**], [**25**], [**21**], [**8**] and [**9**]), system (2.6) has at most one limit cycle, and if it exists then it is hyperbolic. So $I(h)$ can have at most one zero. Hence, $N(2,2) = 1$. ∎

The number of isolated zeros of the Abelian integral (1.1) associated to quadratic perturbations of a system in QH, and the number of limit cycles of the perturbed system in general is not the same. It may exist limit cycles of the perturbed system such that when the quadratic perturbation goes to zero they tend to the equilibrium point (the center for the system in QH) or to the homoclinic or heteroclinic loop forming the boundary of the center in the Poincaré disc. Both possibilities has been confirmed by Zoladek in [**26**] and [**27**]. Hence, in order to obtain all the limit cycles which can be bifurcated from a center it is necessary to study simultaneously three cases. The first one is to control the number of limit cycles which can be bifurcated from the equilibrium point, secondly those which can be bifurcated from periodic orbits of the center (for instance, through Abelian integrals), and finally those which can be bifurcated from the homoclinic or heteroclinic loop at the boundary of the center. For more details see [**4**], [**19**], [**18**] and [**22**].

3. Classification of the System in QH

We shall use the classification of all the phase portraits of quadratic Hamiltonian vector fields given by Artes and Llibre in [**2**] to characterize the phase portraits of the systems in QH, i.e. the phase portraits of the quadratic Hamiltonian vector fields having an invariant straight line and at least one center.

By Lemma 1 every system in QH can be transformed to a system

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}, \tag{3.1}$$

with $H(x, y) = x(y^2 + ax^2 + bx + c)$. In order to classify the phase portrarits of system (3.1) in QH we distinguish three main cases.

Case 1. $a = 0$. If $b = 0$ then system (3.1) has no center. So we assume that $b \neq 0$. After rescaling the variable x the parameter b can be reduced to 1. Therefore $H(x, y) = x(y^2+x+c) = 0$ has two branches, the straight line $x = 0$ and the parabola $y^2+x+c = 0$. If $c \geq 0$ then system (3.1) has no centers. So we assume $c < 0$. Then the two branches of $H(x, y) = 0$ intersect at two saddle points of system (3.1). The other singular point $(-c/2, 0)$ of system (3.1) is a center. Hence, from [**2**], system (3.1) has the phase portrait of type Vulpe 5 (see Figure 1).

Case 2: $a > 0$. After rescaling the variable x the parameter a can be taken equal to 1. So $H(x, y) = x[y^2 + (x + b/2)^2 + c - b^2/4]$.

Subcase 1. $c - b^2/4 < 0$. Then $H(x, y) = 0$ has two branches, the straight line $x = 0$ and the circle centered at the point $(-b/2, 0)$ with radius equal to $\sqrt{b^2/4 - c}$. If $c < 0$ then both branches intersect at two saddles of system (3.1). Therefore, from [**2**], system (3.1) has the phase portrait of type Valpe 3 (see Figure 1). If $c = 0$ the straight line $x = 0$ is tangent to the cercle $y^2 + (x + b/2)^2 = b^2/4$, and from [**2**] system (3.1) has the phare partrait of type Vulpe 2 (see Figure 1). If $c > 0$ the two branches of $H(x, y) = 0$ do not intersect, then from [**2**] system (3.1) has the phase portrait of type Vulpe 2 (see Figure 1).

Subcase 2. $c - b^2/4 \geq 0$. Then $H(x, y) = 0$ has a unique branch, the straight line $x = 0$. If $b^2 - 3c \leq 0$ then system (3.1) has no centers. Therefore we assume $b^2 - 3c > 0$. Now system (3.1) has exactly two singular points of coordinates $((-b \pm \sqrt{b^2 - 3c})/3, 0)$, a center and a saddle. Therefore, from [**2**] the phase portrait of system (3.1) is of type Vulpe 2 (see Figure 1).

Case 3. $a < 0$. After rescaling the variable x the parameter a can be taken equal to -1. So $H(x, y) = x[y^2 - (x - b/2)^2 + c + b^2/4]$.

Subcase 1. $c + b^2/4 = 0$. Then $H(x, y) = 0$ has three branches, the straight lines $x = 0, y = x - b/2$ and $y = x + b/2$. Each two of them intersects at a saddle. Therefore, from [**2**], system (3.1) has the phase portrait of type Vulpe 10 (see Figure 1).

Subcase 2. $c + b^2/4 \neq 0$. Then $H(x, y) = 0$ has three branches, the straight line $x = 0$ and the two branches of the hyperbola $y^2 - (x - b/2)^2 = -(c + b^2/4)$. If $b^2 + 3c \leq 0$ then system (3.1) has no centers. So we assume that $b^2 + 3c > 0$. If $c + b^2/4 < 0$ then $x = 0$ intersects the two branches of the hyperbola, at two saddles. Therefore, from [**2**] it follows that system (3.1) has the phase portrait of type Vulpe 9 (see Figure 1). If $c + b^2/4 > 0$ then $x = 0$ intersects only one branch of the hyperbola, at two saddles. Hence, from [**2**], system (3.1) has the phase portrait of type Vulpe 8 (see Figure 1).

Thus we have obtained six different topological phase portraits for the systems in QH. We remark that the unique of these six phase portraits which can be realized for a quadratic Hamiltonian system without having an invariant straight line is the phase portrait of Vulpe 2. This is due to the property that if a quadratic system has an orbit with α-limit one saddle and ω-limit another saddle, then this orbit is contained in an invariant straight line. This result is due to Dong [**11**], see also Ye Yanqian and others [**25**]. An improvement of this result is due to Chicone and Shafer, see Theorem 2.8 of [**7**].

We also remark that Vulpe 2 without invariant straight line is related with the Bogdanov-Takens bifurcation restricted to quadratic vector fields.

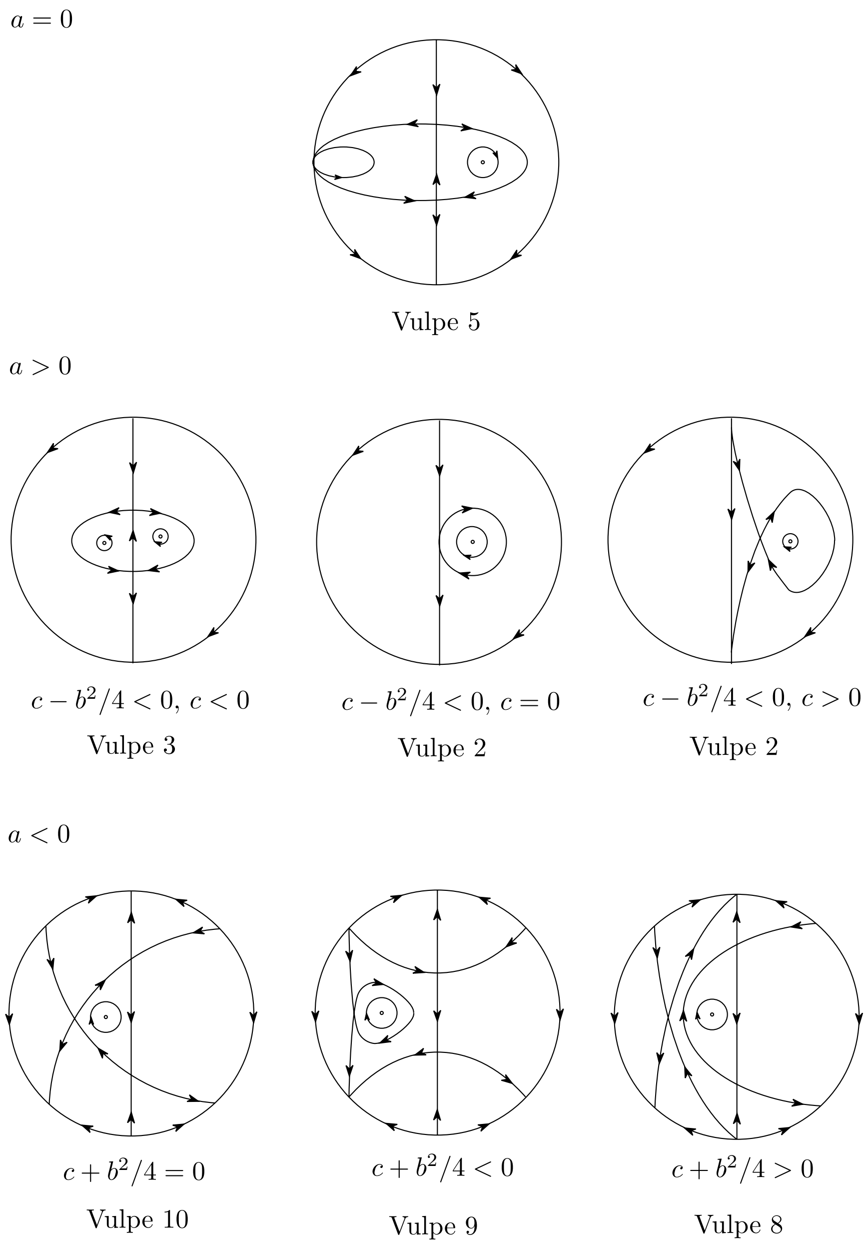

Figure 1: The six different topological phase portraits of the quadratic Hamiltonian vector fields having an invariant straight line and at least one center.

References

1. V. I. Arnold, Loss of stability of self-oscillation close to resonance and versal deformations of equivariant vector fields, *Funct. Anal. Appl.* **11** (1977), 1–10.
2. J. C. Artés and J. Llibre, Quadratic Hamiltonian vector fields, *J. Diff. Equations* **107** (1994), 80–95.
3. R. I. Bogdanov, Bifurcation of the limit cycle of a family of a planar vector fields, *Selecta Math. Soviet* **1** (1981), 373–387.
4. R. I. Bogdanov, Versal deformation of a singularity of a vector field on the plane in the case of zero eigenvalues, *Selecta Math. Soviet* **1** (1981), 389–421.
5. L. A. Cherkas and L. I. Zhilevich, Some criteria for the absence of limit cycles and for the existence of a single limit cycle, *Diff. Equations* **6** (1970), 891–897.
6. L. A. Cherkas and L. I. Zhilevich, Limit cycles of some differential equations, *Diff. Equations* **8** (1972), 924–929.
7. C. Chicone and D. S. Shafer, Separatrix and limit cycles of quadratic systems and Dulac's theorem, *Trans. Amer. Math. Soc.* **278** (1983), 585–612.
8. B. Coll and J. Llibre, Limit cycles for a quadratic system with an invariant straight line and some evolution of phase portraits, *Colloquia Mathematica Societatis János Bolyai* **53**, Qualitative Theory of Differential Equations, Bolyai Institut, Szeged, Hungary (1988), 111–123.
9. W. A. Coppel, Some quadratic systems with at most one limit cycle, *Dynamics Reported* **2** (1989), 61–88.
10. R. Cushman and J. A. Sanders, A codimension two bifurcation with a third order Picard-Fuchs equation, *J. Diff. Equations* **59** (1985), 243–256.
11. D. Jinzhu, The structure of the separatrix cycles of the system $\dot{x} = \sum_{0\le i+k\le 2} a_{ik}x^i\, y^k$, $\dot{y} = \sum_{0\le i+k\le 2} b_{ik}x^i y^k$ (in Chinese), *Acta Math. Sinica* **12** (1962), 251–257.
12. B. Drachman, S. A. van Gils and Z. F. Zhang, Abelian integrals for quadratic vector fields, *J. Reine Angew. Math.* **382** (1987), 165–180.
13. F. Dumortier, R. Roussarie and J. Sotomayor, Generic 3-parameter families of vector fields on the plane, unfolding a singularity

with nilpotent linear part, The cusp case, *Ergod. Th. & Dyn. Sys.* **7** (1987), 375–413.

14. L. GAVRILOV AND E. HOROZOV, "*Limit cycles and zero of Abelian integrals satisfying third order Picand-Fuchs equations*," Lect. Notes in Math. **1455**, Springer-Verlag, 1990, pp. 160–196.

15. G. NAIDAN AND C. LANSUN, "*Theory of limit cycles*," by Ye Yanqian and others, Transl. of Math. Monographs **66**, Amer. Math. Soc., 1984, pp. 336, 359–360.

16. A. G. KHOVANSKY, Real analytic manifolds with finiteness properties and complex Abelian integrals, *Funct. Anal. Appl.* **18** (1984), 119–128.

17. C. LI, J. LLIBRE AND Z. F. ZHANG, Weak focus, limit cycles and bifurcations for bounded quadratic systems, *J. Diff. Equations* **115** (1995), 193–223.

18. C. LI AND C. ROUSSEAU, A system with three limit cycles appearing in a Hopf bifurcation and dying in a homoclinic bifurcation: The cusp of order 4, *J. Diff. Equations* **79** (1989), 132–161.

19. P. MARDESIC, The number of limit cycles of polynomial deformations of a Hamiltonian vector field, *Ergod. Th. & Dynam. Sys.* **10** (1990), 523–529.

20. G. S. PETROV, Number of zeros of complete elliptic integrals, *Funct. Anal. Appl.* **18** (1984), 148–150.

21. G. S. PETROV, Elliptic integrals and their nonoscillations, *Funct. Anal. Appl.* **20** (1986), 37–40.

22. G. S. PETROV, The Chebyshev property of elliptic integrals, *Funct. Anal. Appl.* **22** (1988), 72–73.

23. G. S. RYCHKOV, Limit cycles of the equation $u(x+1)\,du = (-x+ax^2+bxu+cu+du^2)\,dx$, *Diff. Equations* **8** (1972), 1748–1750.

24. A. N. VARCHENKO, Estimate of the number of zeros of an Abelian integral depending on a parameter and limit cycles, *Funct. Anal. Appl.* **18** (1984), 98–108.

25. Y. YANQIAN AND OTHERS, "*Theory of Limit cycle*," Tranl. of Math. Monographs **66**, Amer. Math. Soc., 1984.

26. H. Zoladek, Quadratic systems with center and their perturbations, preprint (1992).
27. H. Zoladek, The cyclicity of triangles in quadratic systems, preprint (1992).

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Rebut el 24 de Març de 1995