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# WEIGHTED $L_p$ SPACES AND POINTWISE ERGODIC THEOREMS

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*Abstract*

In this paper we give an operator theoretic version of a recent result of F. J. Martín-Reyes and A. de la Torre concerning the problem of finding necessary and sufficient conditions for a nonsingular point transformation to satisfy the Pointwise Ergodic Theorem in $L_p$. We consider a positive conservative contraction $T$ on $L_1$ of a $\sigma$-finite measure space $(X, \mathcal{F}, \mu)$, a fixed function $e$ in $L_1$ with $e > 0$ on $X$, and two positive measurable functions $V$ and $W$ on $X$. We then characterize the pairs $(V, W)$ such that for any $f$ in $L_p(V\,d\mu)$ the averages

$$R_0^n(f, e) = \left(\sum_{k=0}^{n} T^k f\right) \Bigg/ \left(\sum_{k=0}^{n} T^k e\right)$$

converge almost everywhere to a function in $L_p(W\,d\mu)$. The characterizations are given for all $p$, $1 \le p < \infty$.

## 1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space and $T$ a positive linear contraction of $L_1(\mu)$. We assume $T$ to be a conservative operator. (For the usual notation we refer the reader to Krengel's book [**2**].) Thus the class

$$(1) \qquad \mathcal{I} = \mathcal{I}(T) = \{A \in \mathcal{F} : T^* 1_A = 1_A\}$$

of all invariant sets relative to $T$ forms a $\sigma$-field, where $1_A$ denotes the indicator function of $A$ and $T^*$ denotes the adjoint operator of $T$, acting on $L_\infty(\mu)$. Since $T$ is positive, we may extend by a canonical manner the domain of $T$ to the class $M^+(\mu)$ of all nonnegative extended real valued measurable functions on $X$. Similarly, this is done for $T^*$. Now let us fix an $e \in L_1(\mu)$ with $e > 0$ on $X$. Let $0 < V$, $W \le \infty$ be two measurable functions on $X$. Previously we observed in [**6**] that if $1 < p < \infty$ then for any $f \in L_p^+(V\,d\mu)$ the averages

$$R_0^n(f,e) = \left(\sum_{k=0}^{n} T^k f\right) \Big/ \left(\sum_{k=0}^{n} T^k e\right) \tag{2}$$

converge to a finite limit a.e. on $X$ if and only if

$$E\{e^{-1}V^{1-p'} | (X, \mathcal{I}, e\,d\mu)\} < \infty \text{ a.e. on } X, \tag{3}$$

where $1/p + 1/p' = 1$. In [**6**] we also observed implicitely (see especially p. 76–77 in [**6**]) that for any $f \in L_1^+(V\,d\mu)$ the averages $R_0^n(f,e)$ converge to a finite limit a.e. on $X$ if and only if there exists a function $U$, measurable with respect to $\mathcal{I}$, such that

$$V^{-1} \le U < \infty \text{ a.e. on } X. \tag{4}$$

In this paper we intend to study the problem of charactering the case where the limit function $R_0^\infty(f,e)$ belongs to $L_p^+(W\,d\mu)$ for every $f \in L_p^+(V\,d\mu)$. This study was inspired by the work [**3**] of Martín-Reyes and de la Torre. See also Assani and Wós [**1**]. As a result, this paper may be considered to be an operator theoretic version of Martín-Reyes and de la Torre's paper [**3**]. Using the result obtained we next consider multiparameter pointwise ergodic theorems for commuting positive linear contractions of $L_1(\mu)$ having a common strictly positive fixed point in $L_1(\mu)$.

## 2. The main result

**Theorem 1.** *Let $T$ be a conservative positive linear contraction of $L_1(\mu)$. Let $V$, $W$ be two positive real valued measurable functions on $X$. Fix an $e \in L_1(\mu)$ with $e > 0$ on $X$. If $1 < p < \infty$ and $1/p + 1/p' = 1$, then the following are equivalent:*

(a) *For any $f \in L_p^+(V\,d\mu)$ the averages $R_0^n(f,e)$ converge a.e. to a function belonging to $L_p^+(W\,d\mu)$.*
(b) $E\{e^{-1}W | (X, \mathcal{I}, e\,d\mu)\}^{1/p} \cdot E\{e^{-1}V^{1-p'} | (X, \mathcal{I}, e\,d\mu)\}^{1/p'} \le C$ *a.e. on $X$, where $C$ is a positive constant.*
(c) *For any $f \in L_{p'}^+(W^{1-p'}\,d\mu)$ the averages $R_0^n(f,e)$ converge a.e. to a function belonging to $L_{p'}^+(V^{1-p'}\,d\mu)$.*

*If $p = 1$, then (a) is equivalent to*

(d) $E\{e^{-1}W | (X, \mathcal{I}, e\,d\mu)\} \le CV$ *a.e. on $X$.*

*Proof:* Let $1 < p < \infty$.

(a) $\Rightarrow$ (b). By (a) the limit function

$$R_0^\infty(f,e) = \lim_n R_0^n(f,e) \tag{5}$$

is finite a.e. on $X$. Thus by (3) we have

$$E\{e^{-1}V^{1-p'}|(X,\mathcal{I},e\,d\mu)\} < \infty \text{ a.e. on } X.$$

Choose $X_n \in \mathcal{I}$, $n = 1,2,\dots$, so that

$$X_n \uparrow X \text{ and } \int_{X_n} V^{1-p'}\,d\mu < \infty. \tag{6}$$

Since $V^{(1-p')p} \cdot V = V^{1-p'}$, it follows that

$$V^{1-p'} \in L_p^+(X_n, V\,d\mu). \tag{7}$$

On the other hand, since $R_0^\infty(\cdot,e)$ is a positive linear operator from $L_p(V\,d\mu)$ into $L_p(W\,d\mu)$ by (a), it is bounded, i.e., there exists a constant $K > 0$ such that

$$\int |R_0^\infty(f,e)|^p W\,d\mu \le K^p \int |f|^p V\,d\mu \quad (f \in L_p(V\,d\mu)). \tag{8}$$

Therefore, for any $A \in \mathcal{I}$ with $A \subset X_n$, (7) yields

$$\int_A R_0^\infty(V^{1-p'},e)^p W\,d\mu \le K^p \int_A V^{1-p'}\,d\mu < \infty. \tag{9}$$

Since $R_0^\infty(V^{1-p'},e) = E\{e^{-1}V^{1-p'}|(X,\mathcal{I},e\,d\mu)\}$ a.e. on $X$ (cf. p. 73 in [**6**]), these inequalities imply

$$\begin{aligned} R_0^\infty(V^{1-p'},e)^p E\{e^{-1}W|(X,\mathcal{I},e\,d\mu)\} &\le K^p E\{e^{-1}V^{1-p'}|(X,\mathcal{I},e\,d\mu)\} \\ &< \infty \text{ a.e. on } X; \end{aligned}$$

and (b) follows.

(b) $\Rightarrow$ (a). Since $e^{-1}f = (e^{-1/p}fV^{1/p})(e^{-1/p'}V^{-1/p})$, the Hölder inequality for the conditional expectation operator and (b) imply that if $f \in L_p^+(V\,d\mu)$ then

$$\begin{aligned} R_0^\infty(f,e) &= E\{e^{-1}f|(X,\mathcal{I},e\,d\mu)\} \\ &\le E\{e^{-1}f^pV|(X,\mathcal{I},e\,d\mu)\}^{1/p} \cdot E\{e^{-1}V^{1-p'}|(X,\mathcal{I},e\,d\mu)\}^{1/p'} \\ &\le CE\{e^{-1}f^pV|(X,\mathcal{I},e\,d\mu)\}^{1/p} \cdot E\{e^{-1}W|(X,\mathcal{I},e\,d\mu)\}^{-1/p} \end{aligned}$$

a.e. on $X$; and thus

$$\begin{aligned}\int R_0^\infty(f,e)^p W\,d\mu &\le C^p \int \frac{E\{e^{-1}f^pV|(X,\mathcal{I},e\,d\mu)\}}{E\{e^{-1}W|(X,\mathcal{I},e\,d\mu)\}} W\,d\mu \\ &= C^p \int E\{e^{-1}f^pV|(X,\mathcal{I},e\,d\mu)\} e\,d\mu \\ &= C^p \int f^p V\,d\mu < \infty,\end{aligned}$$

which proves (a).

(b) $\Leftrightarrow$ (c). Direct from (a) $\Leftrightarrow$ ($b$).

Let $p = 1$.

(a) $\Leftrightarrow$ (d). For any $f \in L_1^+(V\,d\mu)$ we obtain

$$\begin{aligned}\int R_0^\infty(f,e) W\,d\mu &= \int E\{e^{-1}f|(X,\mathcal{I},e\,d\mu)\} W\,d\mu \\ &= \int E\{e^{-1}f|(X,\mathcal{I},e\,d\mu)\} E\{e^{-1}W|(X,\mathcal{I},e\,d\mu)\} e\,d\mu \\ &= \int f E\{e^{-1}W|(X,\mathcal{I},e\,d\mu)\}\,d\mu \\ &= \int fV(E\{e^{-1}W|(X,\mathcal{I},e\,d\mu)\}\cdot V^{-1})\,d\mu.\end{aligned}$$

Hence, by (8) with $p = 1$, (a) is equivalent to

(a$'$) $\int fV(E\{e^{-1}W|(X,\mathcal{I},e\,d\mu)\}\cdot V^{-1})\,d\mu \le K \int fV\,d\mu$ *for every* $f \in L_1^+(V\,d\mu)$;

and (a$'$) is clearly equivalent to (d). The proof is complete. ■

**Corollary 1.** *In addition to the hypotheses of Theorem 1, if we assume that $T$ is ergodic, i.e., that $\mathcal{I}$ is trivial, then the following are equivalent, for every $1 \le p < \infty$:*

(a) *For any $f \in L_p^+(V\,d\mu)$ the averages $R_0^n(f,e)$ converge a.e. to a function belonging to $L_p^+(W\,d\mu)$.*

(b) *$W \in L_1(\mu)$ and $V^{-1} \in L_{p'}(V\,d\mu)$, where $p' = \infty$ when $p = 1$.*

## 3. Applications

Let $d \ge 1$ be an integer and $T_1, \dots, T_d$ be commuting positive linear contractions of $L_1(\mu)$. In this section we assume that there exists an $e \in L_1(\mu)$ with $e > 0$ on $X$ such that

$$T_i e = e \quad (1 \le i \le d). \tag{10}$$

Thus each $T_i$ is a conservative operator and satisfies the mean ergodic theorem in $L_1(\mu)$. And by an induction argument we see that for any $f \in L_1(\mu)$ the averages

$$A_n(T_1, \dots, T_d)f = A_n(T_1)\dots A_n(T_d)f \tag{11}$$

converge in $L_1$-norm, where $A_n(T_i) = \frac{1}{n}\sum_{k=0}^{n-1} T_i^k$. By Theorem 1 of [**5**], for any $f \in L_1(\mu)$ the averages $A_n(T_1, \dots, T_d)f$ converge a.e. on $X$. Let us denote the limit function by $A(T_1, \dots, T_d)f$; thus

$$A(T_1, \dots, T_d)f = \lim_n A_n(T_1, \dots, T_d)f \text{ a.e. on } X. \tag{12}$$

If we let

$$T = \frac{1}{d}\sum_{i=1}^{d} T_i \tag{13}$$

then $T$ also satisfies the mean ergodic theorem in $L_1(\mu)$; and we get the direct decomposition

$$L_1(\mu) = \{f \in L_1(\mu) : Tf = f\} \oplus \{g - Tg : g \in L_1(\mu)\}^-.$$

Since $Tf = f$ if and only if $T_i f = f$ for each $1 \le i \le d$ by the Brunel-Falkowitz lemma (cf. p. 82 in [**2**]) and

$$\lim_n \|A_n(T_1, \dots, T_d)(g - Tg)\|_1 = 0$$

by the equation $g - Tg = \frac{1}{d}\sum_{i=1}^{d}(g - T_i g)$, it follows that for any $f \in L_1(\mu)$ the limit function $A(T_1, \dots, T_d)f$ coincides a.e. with the limit function

$$A(T)f = \lim_n A_n(T)f. \tag{14}$$

Further, since $\mathcal{I}(T) = \bigcap_{i=1}^{d} \mathcal{I}(T_i)$ (in the sequel $\mathcal{I}$ will denote this $\sigma$-field), it follows that for any $f \in L_1^+(\mu)$

$$\begin{aligned} A(T_1, \dots, T_d)f &= \lim_n A_n(T)f \\ &= \lim_n e\left(\sum_{k=0}^{n} T^k f\right)\Big/\left(\sum_{k=0}^{n} T^k e\right) \\ &= eE\{e^{-1}f|(X, \mathcal{I}, e\,d\mu)\} \text{ a.e. on } X. \end{aligned} \tag{15}$$

Hence, by an approximation argument, for any $f \in M^+(\mu)$ the limit $A(T_1, \dots, T_d)f = \lim_n A_n(T_1, \dots, T_d)f$ exists a.e. on $X$ and satisfies (15).

We are now in position to state the first application of Theorem 1.

**Theorem 2.** *Let $T_1, \dots, T_d$ be commuting positive linear contractions of $L_1(\mu)$ such that $T_i e = e$ $(1 \le i \le d)$ for some $e \in L_1(\mu)$ with $e > 0$ on $X$. Let $0 < V, W < \infty$ be two measurable functions on $X$. If $1 < p < \infty$ and $1/p + 1/p' = 1$, then the following are equivalent:*

(a) *For any $f \in L_p^+(V\,d\mu)$ the limit function $A(T_1, \dots, T_d)f$ belongs to $L_p^+(W\,d\mu)$.*

(b) $E\{e^{p-1}W|(X, \mathcal{I}, e\,d\mu)\}^{1/p} E\{e^{-1}V^{1-p'}|(X, \mathcal{I}, e\,d\mu)\}^{1/p'} \le C$ *a.e. on $X$.*

(c) *For any $f \in L_{p'}^+(e^{-p'}W^{1-p'}\,d\mu)$ the limit function $A(T_1, \dots, T_d)f$ belongs to $L_{p'}^+(e^{-p'}V^{1-p'}\,d\mu)$.*

*If $p = 1$, then (a) is equivalent to*

(d) $E\{W|(X, \mathcal{I}, e\,d\mu)\} \le CV$ *a.e. on $X$.*

*Consequently, in case $\mathcal{I}$ is trivial, (a) is equivalent, for every $1 \le p < \infty$, to*

(e) *$e^p W \in L_1(\mu)$ and $V^{-1} \in L_{p'}(V\,d\mu)$, where $p' = \infty$ when $p = 1$.*

*Proof:* For any $f \in L_p^+(V\,d\mu)$ we have, by (15), $A(T_1, \dots, T_d)f = eR_0^\infty(f, e)$. Thus (a) is equivalent to

(a$'$) *For any $f \in L_p^+(V\,d\mu)$ the limit function $R_0^\infty(f, e)$ (relative to $T$) belongs to $L_p^+(e^p W\,d\mu)$.*

Therefore, by Theorem 1, we see (a) $\Leftrightarrow$ (b) when $1 < p < \infty$, and (a) $\Leftrightarrow$ (d) when $p = 1$. When $1 < p < \infty$, (b) $\Leftrightarrow$ (c) follows from the equivalence (a) $\Leftrightarrow$ (b). This completes the proof. ∎

**Corollary 2.** *Let $T_1, \dots, T_d$ and $e$ be the same as in Theorem 2. If $1 < p < \infty$ and $1/p + 1/p' = 1$, then the following are equivalent:*

(a) *For any $f \in L_p^+(\mu)$ the limit function $A(T_1, \dots, T_d)f$ belongs to $L_p^+(\mu)$.*

(b) $E\{e^{p-1}|(X, \mathcal{I}, e\,d\mu)\}^{1/p} E\{e^{-1}|(X, \mathcal{I}, e\,d\mu)\}^{1/p'} \le C$ *a.e. on $X$.*

*Consequently, in case $\mathcal{I}$ is trivial, (a) is equivalent, for every $1 < p < \infty$, to*

(c) *$\mu(X) < \infty$ and $e \in L_p(\mu)$.*

**Remark.** We note that (a) of Corollary 2 always holds when $p = 1$.

We next consider the adjoint operators $T_1^*, \dots, T_d^*$. Since $\int (T_i^* f)e\,d\mu = \int f(T_i e)\,d\mu = \int fe\,d\mu$ for $f \in L_\infty^+(\mu)$, $T_1^*, \dots, T_d^*$ can be regarded as commuting positive linear contractions of $L_1(e\,d\mu)$. Since

$$T_i^* 1 = 1 \in L_1(e\,d\mu) \quad (1 \le i \le d), \tag{16}$$

if we replace the measure $\mu$ and the function $e$ by $e\,d\mu$ and 1, respectively, then the above-given argument shows that for any $f \in M^+(\mu) = M^+(e\,d\mu)$ the limit

$$A(T_1^*, \dots, T_d^*)f = \lim_n A_n(T_1^*, \dots, T_d^*)f \tag{17}$$

exists a.e on $X$; further, since $\mathcal{I} = \bigcap_{i=1}^{d} \mathcal{I}(T_i) = \bigcap_{i=1}^{d} \mathcal{I}(T_i^*)$, it follows that

$$A(T_1^*, \dots, T_d^*)f = \lim_n A_n(T^*)f = E\{f|(X, \mathcal{I}, e\,d\mu)\} \text{ a.e. on } X, \tag{18}$$

where $T^* = \dfrac{1}{d}\displaystyle\sum_{i=1}^{d} T_i^*$.

**Theorem 3.** *Let $T_1, \dots, T_d$ and $e$ be the same as in Theorem 2. Let $0 < V$, $W < \infty$ be two measurable functions on $X$. If $1 < p < \infty$ and $1/p + 1/p' = 1$, then the following are equivalent:*

(a) *For any $f \in L_p^+(V\,d\mu)$ the limit function $A(T_1^*, \dots, T_d^*)f$ belongs to $L_p^+(W\,d\mu)$.*

(b) $E\{e^{-1}W|(X, \mathcal{I}, e\,d\mu)\}^{1/p} E\{(e^{-1}V)^{1-p'}|(X, \mathcal{I}, e\,d\mu)\}^{1/p'} \le C$ *a.e. on $X$.*

(c) *For any $f \in L_{p'}^+(e^{p'}W^{1-p'}\,d\mu)$ the limit function $A(T_1^*, \dots, T_d^*)f$ belongs to $L_{p'}^+(e^{p'}V^{1-p'}\,d\mu)$.*

(d) *For any $f \in L_{p'}^+(W^{1-p'}\,d\mu)$ the limit function $A(T_1, \dots, T_d)f$ belongs to $L_{p'}^+(V^{1-p'}\,d\mu)$.*

(e) *For any $f \in L_p^+(e^{-p}V\,d\mu)$ the limit function $A(T_1, \dots, T_d)f$ belongs to $L_p^+(e^{-p}W\,d\mu)$.*

*If $p = 1$, then (a) is equivalent to*

(f) $E\{e^{-1}W|(X, \mathcal{I}, e\,d\mu)\} \le C(e^{-1}V)$ *a.e. on $X$.*

*Consequently, in case $\mathcal{I}$ is trivial, (a) is equivalent, for every $1 \le p < \infty$, to*

(g) *$W \in L_1(\mu)$ and $eV^{-1} \in L_{p'}(V\,d\mu)$, where $p' = \infty$ when $p = 1$.*

*Proof:* Since $L_p^+(V\,d\mu) = L_p^+(e^{-1}V e\,d\mu)$ and $L_p^+(W\,d\mu) = L_p^+(e^{-1}W e\,d\mu)$, if we apply Theorem 2 to commuting positive linear contractions $T_1^*, \dots, T_d^*$ of $L_1(e\,d\mu)$, then (16) yields (a) $\Leftrightarrow$ (f) when $p = 1$, and (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) when $1 < p < \infty$. If we write (b) as

$$E\{e^{p-1}(e^{-p}W)|(X, \mathcal{I}, e\,d\mu)\}^{1/p} E\{e^{-1}(e^{-p}V)^{1-p'}|(X, \mathcal{I}, e\,d\mu)\}^{1/p'} \le C \text{ a.e. on } X,$$

and apply Theorem 2 to commuting positive linear contractions $T_1, \dots, T_d$ of $L_1(\mu)$, then we obtain (b) $\Leftrightarrow$ (e) $\Leftrightarrow$ (d) when $1 < p < \infty$. The proof is complete. ■

**Corollary 3 (cf. [3] and [4]).** *Let $T_1, \dots, T_d$ and $e$ be the same as in Theorem 2. If $1 < p < \infty$ and $1/p + 1/p' = 1$, then the following are equivalent:*

(a) *For any $f \in L_p^+(\mu)$ the limit function $A(T_1^*, \dots, T_d^*)f$ belongs to $L_p^+(\mu)$.*

(b) $E\{e^{-1}|(X, \mathcal{I}, e\,d\mu)\}^{1/p} E\{e^{p'-1}|(X, \mathcal{I}, e\,d\mu)\}^{1/p'} \le C$ *a.e. on $X$.*

(c) *For any $f \in L_{p'}^+(\mu)$ the limit function $A(T_1, \dots, T_d)f$ belongs to $L_{p'}^+(\mu)$.*

*If $p = 1$, then (a) is equivalent to*

(d) $E\{e^{-1}|(X, \mathcal{I}, e\,d\mu)\} \le Ce^{-1}$ *a.e. on $X$.*

*Consequently, in case $\mathcal{I}$ is trivial, (a) is equivalent, for every $1 \le p < \infty$, to*

(e) $\mu(X) < \infty$ *and $e \in L_{p'}(\mu)$, where $p' = \infty$ when $p = 1$.*

**Corollary 4.** *Suppose $(X, \mathcal{F}, \mu)$ is a finite measure space. Let $T_1, \dots, T_d$ be commuting positive linear contractions of $L_1(\mu)$, and assume that $\mu$ is invariant under $T_1, \dots, T_d$, i.e., that $T_i 1 = 1 \in L_1(\mu)$ $(1 \le i \le d)$. Let $0 < V$, $W < \infty$ be two measurable functions on $X$. If $1 < p < \infty$ and $1/p + 1/p' = 1$, then the following are equivalent:*

(a) *For any $f \in L_p^+(V\,d\mu)$ the limit function $A(T_1^*, \dots, T_d^*)f$ belongs to $L_p^+(W\,d\mu)$.*

(b) $E\{W|(X, \mathcal{I}, \mu)\}^{1/p} E\{V^{1-p'}|(X, \mathcal{I}, \mu)\}^{1/p'} \le C$ *a.e. on $X$.*

(c) *For any $f \in L_{p'}^+(W^{1-p'}\,d\mu)$ the limit function $A(T_1^*, \dots, T_d^*)f$ belongs to $L_{p'}^+(V^{1-p'}\,d\mu)$.*

*If $p = 1$, then (a) is equivalent to*

(d) $E\{W|(X, \mathcal{I}, \mu)\} \le CV$ *a.e. on $X$.*

*Consequently, in case $\mathcal{I}$ is trivial, (a) is equivalent, for every $1 \le p < \infty$, to*

(e) $W \in L_1(\mu)$ *and $V^{-1} \in L_{p'}(V\,d\mu)$, where $p' = \infty$ when $p = 1$.*

**Remark.** Under the hypotheses of Corollary 4, it follows (see (15) and (18)) that for any $f \in M^+(\mu)$

$$A(T_1, \dots, T_d)f = A(T_1^*, \dots, T_d^*)f = E\{f|(X, \mathcal{I}, \mu)\} \text{ a.e. on } X,$$

so that the function $A(T_1^*, \dots, T_d^*)f$ can be replaced by the function $A(T_1, \dots, T_d)f$ in Corollary 4, without any influence.

## 4. Concluding remarks

Throughout this section, $(X, \mathcal{F}, \mu)$ is a $\sigma$-finite measure space, and $T_1, \ldots, T_d$ are commuting positive linear contractions of $L_1(\mu)$ such that $T_i e = e$ $(1 \le i \le d)$ for some $e \in L_1(\mu)$ with $e > 0$ on $X$. Here we briefly discuss the problem of characterizing a positive measurable function $V$ on $X$ such that if $f \in L_p^+(V\,d\mu)$ then the limit function $A(T_1, \ldots, T_d)f$ (or $A(T_1^*, \ldots, T_d^*)f$) is finite a.e. on $X$. As in the preceding section, we will denote $\mathcal{I} = \bigcap_{i=1}^{d} \mathcal{I}(T_i)$. The results may be stated as follows. (For a related result we refer the reader to [**7**].)

**Theorem 4.** *Let $0 < V \le \infty$ be a measurable function on $X$. If $1 < p < \infty$ and $1/p + 1/p' = 1$, then the following are equivalent:*

(a) *For any $f \in L_p^+(V\,d\mu)$ the limit function $A(T_1, \ldots, T_d)f$ is finite a.e. on $X$.*

(b) *$E\{e^{-1}V^{1-p'} | (X, \mathcal{I}, e\,d\mu)\} < \infty$ a.e. on $X$.*

*If $p = 1$, then (a) is equivalent to*

(c) *$V^{-1} \le U < \infty$ a.e. on $X$ for some $U$, measurable with respect to $\mathcal{I}$.*

*Proof:* By virtue of (15) and the result mentioned in Introduction (see especially (3) and (4)), Theorem 4 follows immediately. ■

**Corollary 5.** *If $1 < p < \infty$, then the following are equivalent:*

(a) *For any $f \in L_p^+(\mu)$ the limit function $A(T_1, \ldots, T_d)f$ is fintie a.e. on $X$.*

(b) *There exist $X_n \in \mathcal{I}$, $n = 1, 2, \ldots$, such that $X_n \uparrow X$ and $\mu(X_n) < \infty$.*

(c) *For any $f \in \bigcup_{1 \le r \le \infty} L_r^+(\mu)$ the limit function $A(T_1, \ldots, T_d)f$ is finite a.e. on $X$.*

*Proof:* Since the implications (b) $\Rightarrow$ (c) $\Rightarrow$ (a) are obvious, we only prove (a) $\Leftrightarrow$ $(b)$. To do this we apply Theorem 4 with $V = 1$ on $X$ and see that (a) is equivalent to

$$E\{e^{-1} | (X, \mathcal{I}, e\,d\mu)\} < \infty \text{ a.e. on } X,$$

which is clearly equivalent to (b). The proof is complete. ■

**Theorem 5.** *Let $0 < V \leq \infty$ be a measurable function on $X$. If $1 < p < \infty$ and $1/p + 1/p' = 1$, then the following are equivalent:*

(a) *For any $f \in L_p^+(V\,d\mu)$ the limit function $A(T_1^*, \dots, T_d^*)f$ is finite a.e. on $X$.*

(b) *$E\{(e^{-1}V)^{1-p'}|(X, \mathcal{I}, e\,d\mu)\} < \infty$ a.e. on $X$.*

*If $p = 1$, then (a) is equivalent to*

(c) *$eV^{-1} \leq U < \infty$ a.e. on $X$ for some $U$, measurable with respect to $\mathcal{I}$.*

*Proof:* By the proof of Theorem 4, we see that in this case it is enough to use (18) instead of (15), completing the proof. ■

**Corollary 6.** *If $1 \leq p < \infty$ and $1/p + 1/p' = 1$, then the following are equivalent:*

(a) *For any $f \in L_p^+(\mu)$ the limit function $A(T_1^*, \dots, T_d^*)f$ is finite a.e. on $X$.*

(b) *There exists an $\tilde{e} \in L_1(\mu) \cap L_{p'}(\mu)$ such that $\tilde{e} > 0$ on $X$ and $T_i\tilde{e} = \tilde{e}$ $(1 \leq i \leq d)$.*

(c) *For any $f \in \bigcup_{p \leq r \leq \infty} L_r^+(\mu)$ the limit function $A(T_1^*, \dots, T_d^*)f$ is finite a.e. on $X$.*

*Proof:* (a) $\Rightarrow$ (b). By Theorem 5 with $V = 1$ on $X$, (a) implies the existence of $X_n \in \mathcal{I}$, $n = 1, 2, \dots$, such that

$$X_n \uparrow X \text{ and } e \cdot 1_{X_n} \in L_{p'}(\mu).$$

Thus choosing a suitable sequence $d_n$, $n = 1, 2, \dots$, of positive real numbers we have

$$\tilde{e} = \sum_{n=1}^{\infty} d_n(e \cdot 1_{X_n}) \in L_1(\mu) \cap L_{p'}(\mu)$$

and

$$T_i\tilde{e} = \tilde{e} \text{ for all } 1 \leq i \leq d.$$

(b) $\Rightarrow$ (c). Since $T_1^*, \dots, T_d^*$ are commuting positive linear contractions of $L_1(\tilde{e}\,d\mu)$ with $T_i^*1 = 1 \in L_1(\tilde{e}\,d\mu)$ $(1 \leq i \leq d)$, it is enough to show that

$$f \in L_1(\tilde{e}\,d\mu) \text{ for every } f \in \bigcup_{p \leq r \leq \infty} L_r^+(\mu).$$

And this follows, as $\tilde{e} \in L_1(\mu) \cap L_{p'}(\mu)$ implies

$$\tilde{e} \in \bigcap_{1 \leq r' \leq p'} L_{r'}(\mu)$$

by the Hölder inequality.

(c) $\Rightarrow$ (a). Trivial. The proof is complete. ■

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