RINGS WHOSE MODULES HAVE MAXIMAL SUBMODULES

CARL FAITH

Dedicated to Laci Fuchs on his 70th birthday

Abstract

A ring \( R \) is a right max ring if every right module \( M \neq 0 \) has at least one maximal submodule. It suffices to check for maximal submodules of a single module and its submodules in order to test for a max ring; namely, any cogenerating module \( E \) of \( \text{mod-}R \); also it suffices to check the submodules of the injective hull \( E(V) \) of each simple module \( V \) (Theorem 1). Another test is transfinite nilpotence of the radical of \( E \) in the sense that \( \text{rad}^\alpha E = 0 \); equivalently, there is an ordinal \( \alpha \) such that \( \text{rad}^\alpha(E(V)) = 0 \) for each simple module \( V \). This holds if each \( \text{rad}^\beta(E(V)) \) has a maximal submodule, or is zero (Theorem 2). If follows that \( R \) is right max iff every nonzero (subdirectly irreducible) quasi-injective right \( R \)-module has a maximal submodule (Theorem 3.3). We characterize a right max ring \( R \) via the endomorphism ring \( \Lambda \) of any injective cogenerator \( E \) of \( \text{mod-}R \); namely, \( \Lambda/L \) has a minimal submodule for any left ideal \( L = \text{ann}_\Lambda M \) for a submodule (or subset) \( M \neq 0 \) of \( E \) (Theorem 8.8). Then \( \Lambda/L_0 \) has socle \( \neq 0 \) for:

1. any finitely generated left ideal \( L_0 \neq \Lambda \); (2) each annihilator left ideal \( L \neq \Lambda \); and (3) each proper left ideal \( L_0 = L + L' \), where \( L = \text{ann}_\Lambda M \) as above (e.g. as in (2)) and \( L' \) finitely generated (Corollary 8.9A).

HAMSHER MODULES

A module \( M \) is a Hamsher module provided each submodule \( S \neq 0 \) has a maximal submodule.\(^1\)

\(^1\)Hamsher modules are called max modules by Shock [S].
1. **One-Module Theorem.** A ring \( R \) is a right max ring iff \( R \) has a cogenerating right Hamsher module \( E \). A n.a.s.c. for this is that the injective hull \( E(V) \) of each simple right \( R \)-module \( V \) is a Hamsher module.

**Proof:** A module \( E \) cogenerates the category \( \text{mod-} R \) of all right \( R \)-modules iff for every module \( M \neq 0 \), there is a nonzero map \( h : M \rightarrow E \) ([F1, pp. 91, 148 & 165]). Then \( h(M) = M' \) is a nonzero submodule of \( E \). Thus, when \( E \) is a Hamsher module, then \( M' \) has a maximal submodule \( M'' \), so \( h^{-1}(M'') \) is a maximal submodule of \( M \).

This proves the first statement in Theorem 1. Next let \( E = \oplus E(V) \), as \( V \) range over all simple \( R \)-modules. Then \( E \) is a cogenerator module for \( \text{mod-} R \) ([F1, p. 167, prop. 3.55]). Let \( P_V \) be the projection \( E \rightarrow E(V) \). Then, in the above, \( 0 \neq M' = h(M) \subseteq E \) implies \( 0 \neq P_V h(M) = M_V \subseteq E(V) \) is a nonzero submodule of \( E(V) \) for some \( V \), and so \( M \) has a maximal submodule, as before, whenever \( E(V) \) is a Hamsher module for all \( V \).

**Note.** \( E(V) \) is direct summand of any cogenerator \( E \) of \( \text{mod-} R \), hence the Hamsher condition on \( E(V) \) is a consequence of that on \( E \) in Theorem 1. Moreover, this is sufficient for \( E \) to be Hamsher.

1.1. **Corollary.** If \( R \) is a ring such that each simple module \( V \) has Noetherian injective hull \( E(V) \), then \( R \) is a right max ring.

To illustrate when \( E(V) \) is not only Noetherian, but simple we will cite a theorem of Kaplansky, but first we recall some terminology:

\( R \) is right V-ring in case \( R \) has the equivalent properties. (See [F1, p. 356, 7.32A].)

(V1) Every simple right \( R \)-module \( V \) is injective, that is, \( E(V) \) is simple.
(V2) \( \text{rad} M = 0 \) for each right \( R \)-module \( M \).
(V3) Every right ideal \( I \neq R \) is the intersection of maximal right ideals, that is, \( \text{rad}(R/I)_R = 0 \).

**Note.** A right V-ring is a right max ring since \( \text{rad} M \neq M \) for every \( M \neq 0 \).

**Kaplansky’s Theorem.** A commutative ring \( R \) is a V-ring iff \( R \) is Von Neuman regular (\( = \text{VNR} \)).

\(^2\)According to my inquiry of Professor Kaplansky, “It worked its way into the public domain” (Letter of October 12, 1994).
Let $J = \text{rad } R$. Then $J$ is left vanishing (= $T$-nilpotent in [B], [H]) if for every sequence \( \{a_n\}_{n=1}^{\infty} \) of elements of $A$, there is an $n \geq 1$ so that $a_n \cdots a_1 = 0$, that is the left-hand partial product $a_n \cdots a_1$ vanishes.

**First Max Theorem** ([H], [K]). A commutative ring $R$ is a max ring iff $R/J$ is VNR and $J = \text{rad } R$ is vanishing.

Expressed otherwise: $R$ is a max ring iff $R/J$ is a V-ring, and $J$ is vanishing. The radical series $\text{rad}^\alpha(M)$ is defined inductively for each ordinal $\alpha$ in the usual way, where $\text{rad}(M)$ is the intersection of all maximal submodules of $M$, $\text{rad}^{\alpha+1}(M) = \text{rad}(\text{rad}^\alpha(M))$ for any ordinal, and

$$\text{rad}^\beta(M) = \bigcap_{\alpha \in \beta} \text{rad}^\alpha(M)$$

for each limit ordinal $\beta$.

**Second Max Theorem** ([H], [K]). A ring $R$ is right max iff $R/J$ is right max and $J$ is left vanishing.

We next show that the modules in the radical series are test submodules for a Hamsher module.

2. Theorem. 3 The f.a.e.c.’s on a right $R$-module $M$.

1. $M$ is Hamsher.
2. $\text{rad}^\beta(M)$ has a maximal submodule, or is 0, for every ordinal $\beta$.
3. $\text{rad}^\alpha(M) = 0$ for some $\alpha$.

Proof: (1) ⇒ (2) is obvious, and (2) ⇒ (3) follows by cardinal number theory for any $\alpha$ of cardinal greater than that of $R$. (3) ⇒ (1). If $S \neq 0$ is a submodule of $M$, then $S \nsubseteq \text{rad}^\lambda(M)$ for least ordinal $\lambda < \alpha$, and obviously $\lambda$ is not a limit ordinal, so $S \subseteq \text{rad}^{\lambda-1}(M)$. If $S = \text{rad}^{\lambda-1}M$, then $S$ has a maximal submodule since $\text{rad}S = \text{rad}^{\lambda}(M)$ $\neq S$. And if $S \neq \text{rad}^{\lambda-1}(M)$, then $S$ is not contained in a maximal submodule $M'$ of $\text{rad}^{\lambda-1}(M)$, hence $S \cap M'$ is a maximal submodule of $S$. This proves that $M$ is a Hamsher module. ■

3.1. Corollary. Let $E$ be a right cogenerator module for $R$. The $R$ is right max iff $E$ has transfinite nilpotent radical. A n.a.s.c. for

\footnote{The equivalence (1) ⇔ (3) is a theorem of Shock [S] who also proved that every semi-Artinian Hamsher module is Noetherian.}
this is that $E(V)$ have transfinite nilpotent radical for each simple right $R$-module $V$.

3.2. Lemma. If $M$ is a quasi-injective right $R$-module, then so is every fully invariant submodule, in particular, so is $\text{rad}^\alpha(M)$, for each ordinal $\alpha$.

Proof: A theorem of Wong-Johnson ([W-J]) characterizes a quasi-injective module as the fully invariant submodules of their injective hulls (see, e.g. [F2, p. 63, Prop. 19.2]). For example, if $E = E(M)$ has endomorphism $A$, then $M$ is quasi-injective iff $\lambda(M) \subseteq M$ $\forall \lambda \in A$. Now let $M_0$ be a fully invariant submodule of $M$. Since $E_0 = E(M) \subseteq E$, and since $E$ is injective, then every element $\lambda_0 \in \Lambda_0 = \text{End} E_0$ is induced by an element $\lambda \in \Lambda$. Since $\lambda$ induces an endomorphism $\lambda$ in $M$, and since $\lambda(M_0) \subseteq M_0$ by the hypothesis that $M_0$ is fully invariant in $M$, then $\lambda_0(M_0) \subseteq M_0$ for each $\lambda_0 \in \Lambda_0$, that is, $M_0$ is fully invariant in $E(M_0)$, hence is quasi-injective.

It follows that $\text{rad}^{\alpha+1}(M)$ is quasi-injective for all $\alpha$, since $\text{rad}^{\alpha+1}(M)$ is fully invariant in $\text{rad}^\alpha(M)$ which by an inductive hypothesis may be assumed to be quasi-injective. Furthermore, $\text{rad}^\beta(M)$ is fully invariant hence quasi-injective for each limit ordinal $\beta$, since it is the intersection of fully invariant submodules of $M$.

3.3. Theorem. For a ring $R$, the f.a.e.c.’s:

1. $R$ is right max.
2. Every nonzero quasi-injective module has a maximal submodule.
3. Every nonzero subdirectly irreducible quasi-injective module has a maximal submodule.

Proof: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is trivial, and (3) $\Rightarrow$ (1) is an immediate consequence of Theorem 2, Corollary 3.1 and Lemma 3.2.

4. Corollary. If a right $R$ module $M$ is faithful and has transfinite nilpotent radical, then $R$ has transfinite nilpotent radical $J$.

Proof: One shows inductively that $\text{rad}^\alpha(M) \supseteq MJ^\alpha$, where $J = \text{rad} R$. 

Note. Let $R$ be a commutative Noetherian ring. Then $J^\omega = 0$ by the Krull intersection Theorem and if $R$ is a domain, then $I^\omega = 0$ for any ideal $I \neq R$ ([Z-S, p. 216, Theorem 12 and Corollary]). Thus, $J$ is transfinite but not $T$-nilpotent when $R$ is e.g., a Noetherian local domain not a field.
Rings whose modules have maximal submodules

LOEWY SERIES
AND TRANSFINITE SEMISIMPLE MODULES

A descending or dual Loewy series for a module $M$ is descending chain \( \{ M_\alpha \} \alpha \in \Lambda \) of submodules indexed by an ordinal $\Lambda$ such that $M_0 = M$, and $M_\alpha / M_{\alpha + 1}$ is semisimple

\[ M_\beta = \cap_{\alpha \in \beta} M_\alpha \]

for any limit ordinal $\beta \in \Lambda$. We say that $M$ is transfinely semisimple if there is a descending Loewy series \( \{ M_\alpha \} \) with $M_\alpha = 0$ for some $\alpha \in \Lambda$.

5. Theorem. Any transfinely semisimple module $M$ is a Hamsher module.

Proof: By transfinite induction,

\[ M_\alpha \supseteq \text{rad}^\alpha(M) \]

for each $M_\alpha$ as defined above, hence $\text{rad}^\alpha(M) = 0$ for some ordinal $\alpha$, and Theorem 1 applies: $M$ is Hamsher module.

By Theorem 1, we also have the following:

5.1. Corollary. If $E(V)$ is transfinite semisimple for each simple right $R$-module $V$, then $R$ is right max.

BASS MODULES

Recall that a module $M$ is a Bass module ([F2]) if every submodule $M' \neq M$ is contained in a maximal submodule of $M$.

6. Theorem. Let $E$ be an quasi-injective right $R$-module that contains a copy of each simple image of $E$ and $\Lambda = \text{End}_E R$. If $E$ is a Bass module, then $\Lambda$ has essential left socle, $\text{soc}_L \Lambda$.

Proof: By the Harada-Ishii ([H-I]) double annihilator condition (= DAC) for a quasi-injective modules,

\[ \text{ann}_\Lambda \text{ann}_E I = I \]

for finitely generated left ideals of $\Lambda$, one can show that each such $I \neq 0$ contains a minimal left ideal $L$. For if $E'$ is a maximal submodule, containing $\text{ann}_E I$ the fact $V = E/E' \hookrightarrow E$ yields $\lambda \in \Lambda$ such that $\lambda E \approx V$, hence $L = \lambda \Lambda$ is a minimal left ideal contained in $I$. Thus, $\text{soc}_L \Lambda$ is an essential left ideal of $\Lambda$.

In the next corollary, we see what happens to $\Lambda$ when $E$ is Noetherian.
6.1. **Corollary.** If $E$ is a Noetherian quasi-injective right module over $R$, then $\Lambda = \text{End} E_R$ is a right perfect ring, hence a right max ring.

Proof: By the Harada-Ishii DAC cited in the proof of Theorem 6, $E_R$ Noetherian implies that $\Lambda$ satisfies the DAC on finitely generated left ideals, hence $\Lambda$ is right perfect ([B]). □

**DOUBLE ANNIHILATOR CONDITIONS FOR COGENERATORS**

It is known that any cogenerator $F$ satisfies the double annihilator conditions (DAC)

$$I = \text{ann}_R \text{ann}_F I$$

(see, e.g. [F1]). We next prove another DAC for $F$.

7. **Dac Theorem.** If $F$ is any right cogenerator of $R$, and $I$ and $M$ are submodules of $R_R$ and $F_R$ respectively, then they satisfy the DAC's:

(a) \hspace{1cm} I = \text{ann}_R \text{ann}_F I

(b) \hspace{1cm} M = \text{ann}_F \text{ann}_\Omega M

where $\Omega = \text{End} F_R$.

Proof:

(1) Since $F$ is a cogenerator then $R/I \hookrightarrow F^\alpha$ for some cardinal $\alpha$, and if $(x_i)$ is the image in $F$ of the coset $1 + I$ in $R/I$, one sees that

$$I = \text{ann}_R \{x_i\},$$

so (a) follows.

(2) $F/M$ embeds in a direct product $F^\alpha$ of copies of $F$, and hence there is a map $h : F \to F^\alpha$ that has $\ker h = M$. Then, if $p_\alpha : F^\alpha \to F$ is the $\alpha$-th projection, it follows that $\omega_\alpha = p_\alpha \circ h \in \Omega$ and that

$$M = \cap_\alpha \ker \omega_\alpha.$$

Then,

(3) \hspace{1cm} M = \cap_\alpha \ker \omega_\alpha.

(4) \hspace{1cm} M = \text{ann}_F L,

where $L = \Sigma_\alpha \omega_\alpha$.

Since (4) $\implies$ (b), the proof is complete. □

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4After this was written, I found Kurata’s report [Ku] where (b) is stated without proof in greater generality.
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INJECTIVE COGENERATORS

If any cogenerator of mod-R is a Hamsher module, then R is a right max ring. In this section we list two conditions on a minimal injective cogenerator E that are each necessary and sufficient in order that R be a right V-ring: (1) \( \text{rad } E = 0 \) (Theorem 8.1) and (2) \( E_R \) is a Bass module, and \( \Lambda = \text{End } E_R \) has zero Jacobson radical (Theorem 8.2).

8.1. Theorem. Let \( E \) be a minimal injective cogenerator of \( R \), and \( W \) the direct sum of a complete set of non-isomorphic simple right \( R \)-modules. (Thus, \( E \) is the injective hull of \( W \), and \( W \) is the socle of \( E \).) Then, the f.a.e.c.'s:

(1) \( R \) is a right \( V \)-ring.

(2) \( \text{rad } E = 0 \).

Proof: (1) \( \Rightarrow \) (2). As stated, (1) \( \Leftrightarrow \) \( \text{rad } M = 0 \) for every right \( R \)-module \( M \).

(2) \( \Rightarrow \) (1). If \( V \) is a simple submodule of \( E \), then (2) implies that there exists a maximal submodule \( M \) of \( E \) not containing \( V \). Then since \( V \cap M = 0 \), and \( V + M \supset M \), we see that \( E = V \oplus M \), so \( V \) is injective. Since every simple right \( R \)-module embeds in \( E \), then \( R \) is a right \( V \)-ring.

8.2. Theorem. If the right minimal injective cogenerator \( E \) of a ring \( R \) is a Bass Module, and if \( \Lambda = \text{End } E_R \) has zero Jacobson radical, then \( R \) is a right \( V \)-ring (and \( E \) is semisimple).

Proof: Let \( W = \text{soc } E \), the sum of all simple module, one for each isomorphism class. If \( W = E \), then every submodule of \( E \) is a direct summand, hence is injective, so \( R \) is right \( V \)-ring. We may therefore assume that \( E \neq W \), and hence by our Bass module assumption that there is a maximal submodule \( M \) of \( E \) that contains \( W \). Since \( V = E/M \hookrightarrow W \), there is an endomorphism \( \lambda \) of \( E \) such that \( \text{ker } \lambda = M \). Since \( M \) is an essential submodule of \( E \), then \( \lambda \in J = J(\Lambda) \) by a theorem of Utumi (e.g. [F2, p. 76, Theorem 19.27(a)]) contradicting the \( J = 0 \) assumption, and completing the proof.

8.3. Proposition. If \( S \) is any semisimple right \( R \)-module with injective hull \( E = E(S) \), then the endomorphism ring \( \Lambda \) has radical

(1) \[ J(\Lambda) = \{ \lambda \in \Lambda : \ker \lambda \supseteq S \} \],
and moreover,

\[ J(\Lambda) = \text{ann}_A S. \]

Furthermore,

\[ \overline{\Lambda} = \Lambda/J(\Lambda) = \text{End} S_R \]

is a full product \( = \prod_{i \in A} L_i \) of full linear rings, where \( L_i = \text{End} W_{D_i} \), and \( W_i \) is a vector space over a field \( D_i, \forall i \in A \).

**Proof:** By Utumi’s theorem cited above (proof of 8.2), (2) has the description (1) above. Since a submodule \( M \) of \( E = E(S) \) is essential iff \( M \supseteq S \), this shows that (2) holds. Furthermore since \( E \) is injective, any element of \( \text{End} S_R \) is induced by some \( \lambda \in \Lambda \), so (2) \( \Rightarrow \) (3). Finally, \( \overline{\Lambda} \) is a product as described by classical ring theory. \( \blacksquare \)

**8.4. Corollary.** If \( E \) is a minimal injective cogenerator of \( \text{mod}-R \), and \( \Lambda = \text{End} E_R \), then \( \overline{\Lambda} = \Lambda/J(\Lambda) \) is product \( \prod_{i \in A} D_i \) of fields \( D_i = \text{End} V_i \), one for each isomorphy class \( [V_i] \) of simple modules. Consequently, \( \overline{\Lambda} \) is a \( V \)-ring.

**Proof:** Follows from 8.3. \( \overline{\Lambda} \) is thus abelian \( VNR \) (=strongly regular), hence is a right and left \( V \)-ring. \( \blacksquare \)

**8.5. Corollary.** If (in Theorem 8.3) \( E \) is a minimal injective cogenerator, then \( E = E(S), \) where \( S = \bigoplus V_i, \) exactly one simple module \( V_i \) of each isomorphy class, and

\[ \overline{\Lambda} = \Lambda/J(\Lambda) = \prod_{i \in A} D_i \]

where \( D_i = \text{End} V_i, \) one for each \( V_i \).

Furthermore, \( \overline{\Lambda} \) is a right and left \( V \)-ring. Finally, \( \Lambda \) is a right (left) max ring iff \( J(\Lambda) \) is left (right) vanishing. Moreover, \( \Lambda \) is right max iff \( E_R \) satisfies the acc on kernels of finite products \( \{j_n \cdots j_2 j_1\} \) of elements of \( J(\Lambda) \).

**Proof:** Follows from Corollary 8.4, the Harada-Ishii theorem, and the Second Max Theorem. \( \blacksquare \)
8.6. Corollary. If the minimal injective cogenerator \( E \) of \( \text{mod-} R \) satisfies the acc on essential submodules (equivalently, \( E/\text{soc} E \) is Noetherian), then \( \Lambda = \text{End} E_R \) is a right max ring.

Proof: Since \( \Lambda/J(\Lambda) \) is a \( V \)-ring (both sides) hence a max ring, then by Hamsher’s theorem, \( \Lambda \) is right max iff \( J(\Lambda) \) is left vanishing. But this follows from Corollary 8.5 and the Harada-Ishi Theorem as in the proof of Theorem 6. (Since \( \text{soc} E \) is the intersection of all essential submodule by a theorem of Kasch-Sandomierski, the parenthetical equivalence holds.) \( \square \)

Remark 8.6A. The condition of Corollary 8.6 implies that \( E(V) \) is Noetherian for any simple module \( V \), and by Corollary 1.1, this is also a sufficient condition for \( R \) to be right max.

8.7. Theorem (Partial Converse of Theorem 6). If \( E \) is an injective cogenerator for \( \text{mod-} R \), and if \( \Lambda = \text{End} E_R \) has essential left socle then \( E \) is a Bass module.

Proof: The proof is a straightforward application of the Harada-Ishii theorem. For if \( M \) is a proper submodule of \( E \), the fact that \( E \) is an injective cogenerator yields \( \text{hom}(E/M, E) \neq 0 \), hence some \( \lambda \in \Lambda \) with \( \ker \lambda \supseteq M \). Then, if \( \Lambda \lambda_0 \) is a minimal left ideal of \( \Lambda \) contained in \( \Lambda \lambda \), by the Harada-Ishii theorem, \( E_0 = \ker \lambda_0 \) is a maximal submodule containing \( \ker \lambda \), hence \( M \). \( \square \)

In the proof of the next theorem, we let \( \ker L = \cap_{\lambda \in L} \ker \lambda \).

8.8. Theorem. For a ring \( R \), right injective cogenerator \( E \), and \( \Lambda = \text{End} E_R \) the f.a.e.c.’s:

1. \( R \) is right max.
2. \( E \) is a Hamsher module.
3. \( \Lambda/L \) has nonzero socle for any left ideal \( L = \text{ann} \Lambda M \), where \( M \) is a nonzero submodule of \( E \).

Proof: (1) \( \Leftrightarrow \) (2) by Theorem 1. (2) \( \Rightarrow \) (3). By the DAC Theorem 6.2, if \( L = \text{ann} \Lambda M \), then \( M = \ker L \), hence, since \( E \) is Hamsher module, \( M \) has a maximal submodule \( M_0 \). Since \( \text{hom}_R(M/M_0, E) \neq 0 \) and \( E \) is injective, then there exists \( \lambda_0 \in \Lambda \) such that \( \lambda_0 M_0 = 0 \) and \( \lambda_0 M \neq 0 \). Moreover, if \( L_0 = \text{ann}_\Lambda M_0 \), then by the DAC Theorem 7, \( \text{ann}_E L_0 = M_0 \), and since \( M \cap (\ker \lambda_0) = M_0 \), then:

\[
\text{ann}_E(L + \Lambda \lambda_0) = (\ker L) \cap (\ker \lambda_0) = M \cap (\ker \lambda_0) = M_0 = \text{ann}_E L_0. \quad \square
\]
By the Harada-Ishii theorem, \( L + \Lambda \lambda_0 \) satisfies the DAC, hence
\[
L + \Lambda \lambda_0 = \text{ann}_\Lambda (L + \Lambda \lambda_0) = \text{ann}_\Lambda M_0 = L_0.
\]
Moreover, the same argument shows that
\[
L_0 = L + \Lambda \lambda' \quad \text{for all } \Lambda' \in L_0 \setminus (L)
\]
that is, necessarily \( \text{ann}_E (L + \Lambda \lambda') = M_0 \) so \( L + \Lambda \lambda' = \text{ann}_\Lambda M_0 = L \). Thus \( L_0 \setminus L \) is a minimal submodule of \( \Lambda \setminus L \), so \((2) \Rightarrow (3)\).

\((3) \Rightarrow (2)\). Let \( L = \text{ann}_\Lambda M \). Then, by the DAC Theorem 6.2, \( M = \text{ann}_E L \). Let \( L_0/L \) be a minimal submodule of \( \Lambda/L \), and let \( M_0 = \text{ann}_E L_0 \). Since \( L_0 = L + \Lambda \lambda \) for any \( \lambda \in L_0 \setminus L \), then by the Harada-Ishii DAC, necessarily \( L_0 = \text{ann}_\Lambda M_0 \). If \( M' \neq M \) is a submodule of \( M \) containing \( M_0 \), then by simplicity of \( L_0/L \), necessarily \( \text{ann}_\Lambda M' = L_0 \) whence by the DAC Theorem 6.2,
\[
M' = \text{ann}_E \text{ann}_\Lambda M' = \text{ann}_E L_0 = M_0
\]
so \( M_0 \) is a maximal submodule of \( M \). Thus, \((3) \Rightarrow (2)\). \( \square \)

**8.9A. Corollary.** If \( R \) is right max, \( E \) an injective cogenerator, and \( \Lambda = \text{End}_R E \), then \( \Lambda/I \) has nonzero socle for each proper left ideal \( I \) of the (3) types:

1. \( L_0 \) finitely generated left ideal of \( \Lambda \).
2. \( L_1 \) an annihilator left ideal of \( \Lambda \).
3. \( L_2 = L + L_0 \), where \( L_0 \) is finitely generated and \( L = \text{ann}_\Lambda M \) for a submodule \( M \) of \( E \).

In particular, \( L_1 = \text{ann}_\Lambda M_1 \), where \( M_1 = L_1^1 E \), so \( L \) can have the form \( L_1 \) in (2).

**Proof:** By the Harada-Ishii DAC, any left ideal \( L_2 \) of the form (2) satisfies the DAC, hence \( L_2 = \text{ann}_\Lambda M_2 \), where \( M_2 = \text{ann}_\Lambda L_2 = \ker L_2 \), so Theorem 8.8 applies.

Furthermore, if \( L_1 \) is the left annihilator \( ^+X \) in \( \Lambda \) of a subset \( X \) of \( \Lambda \), then \( L_1 = ^+ (L_1^1) \) so
\[
L_1 = \text{ann}_\Lambda (^+ L_1 E)
\]
is the annihilator of an \( R \)-submodule of \( E \). \( \square \)
8.9B. Corollary. If \( E \) is an injective cogenerator of \( \text{mod-} R \) with left Loewy (equivalently, left semiartinian) endomorphism ring \( \Lambda \), then \( R \) is right max and \( \Lambda \) is right perfect. Moreover, \( R \) has just finitely many simple right modules.

Proof: If \( \Lambda \) is left Loewy, then \( \Lambda/L \) has nonzero socle for all left ideals \( L \neq \Lambda \), so Theorem 8.8 applies to establish that \( R \) is right max. Since \( \overline{\Lambda} = \Lambda/J(\Lambda) \) is also left Loewy and right self-injective (see, e.g. (3) of Prop. 8.3), then \( \overline{\Lambda} \) is semisimple Artinian and \( J = J(\Lambda) \) is left vanishing, hence \( \Lambda \) is right perfect. (See, for example, the discussion in [C-P, esp. Lemma 1 and the proof of Proposition 2].) Furthermore, since \( \overline{\Lambda} \) is semisimple and isomorphic to the endomorphism ring of the socle \( S \) of \( E \) (see the proof of 8.3), then \( S \) has finite length. This shows that the isomorphy set of simple right \( R \)-modules is finite.

8.10. Corollary. If \( E \) is an injective cogenerator of \( \text{mod-} R \), and \( \Lambda = \text{End}_{E_R} \), then \( R \) is right max iff \( J = \text{rad} R \) left vanishing, and \( \Lambda/L \) has nonzero left socle for any left ideal \( L = \text{ann}_M \), where \( M \) is a nonzero \( R \)-submodule of \( E \) annihilated by \( J \).

Proof: One knows that \( F = \text{ann}_E J \) is an injective cogenerator of \( \text{mod-} R/J \) (\( F \) is injective as an \( R/J \)-module and contains a copy of each simple \( R \)-module). Moreover, \( F \) is a fully invariant \( R \)-submodule of \( E \), hence, by injectivity of \( E \),

\[
\overline{\Lambda} = \Lambda/\text{ann}_F \approx \text{End}_F R.
\]

The corollary now follows from Hamsher’s Second Theorem and Theorem 8.8.

8.11. Example. Let \( M \) be any bimodule over a right max ring \( A \). Then the split-null or trivial extension \( R = (A, M) \) is a right max ring.

Proof: Let \( J(A) \) be the (left vanishing) radical of \( A \). Then \( J(R) = J(A) \) and

\[
R/J(R) \approx A/J(A)
\]

is a right max ring, so \( R \) is right max iff \( J(R) \) is left vanishing. But

\[
J(R)/(0, M) \approx J(A)
\]

is left vanishing and \((0, M)^2 = 0\), and then an easy computation shows that \( J(R) \) is left vanishing.
REMARKS ON THE LITERATURE

A module $M$ is quotient finite dimensional (=q.f.d.) provided that all factor modules have finite Goldie dimension, i.e., contain no infinite direct sums. Generalizing a theorem of Shock [S], Camillo [C1] proved that an $R$-module $M$ is q.f.d. iff every submodule $N$ contains a finitely generated submodule $K$ with $N/K$ having no maximal submodules. This implies that a q.f.d. module $M$ is Noetherian iff every factor module $M/K$ is Hamsher. Since linearly compact modules are q.f.d., then by duality theory [M] one shows that a Morita ring $R(=R$ has a Morita duality) is right max iff left Loewy (= semi-Artinian and iff $R$ is right and left Artinian.

Results of Camillo and Fuller [C-F1], [C-F2] and Nastasescu and Popescu [N-P] are germane here: A left Loewy ring $R$ of finite Loewy length is right max ([C-F1], [N-P]). More generally, any left Loewy ring with acc on primitive ideals is right max ([C-F2]). The example of a right but not left $V$-ring $R$ of the author’s in [F4] is a $VNR$ of left Loewy length 2 hence left max.

As an application of Theorem 1, we prove in [F3] that for a commutative ring $R$ that the f.e.c.’s : (1) $R$ is locally a perfect ring (= $R_m$ is perfect at each maximal ideal $m$); (2) $R_m$ is a max ring for each maximal ideal $m$; (3) $R$ is a max ring.

QUESTIONS

(1) If $Λ=\mathrm{End} E_R$ is a right max ring, for a minimal injective cogen- erator $E$ of $\text{mod-}R$, is $R$ right max?

(2) If $R$ is right max, is $Λ$?

In [C2], Camillo proves that a right max right and left $PIDR$ is simple, and that given two maximal right ideals, $pR$ and $qR$, either $R/pqR$ or $R/qpR$ is semisimple.

(3) Characterize when a $PID$ ring $R$ is right (or left) max. It is of course if $R/aR$ (or $R/Ra$) is semisimple for any $0 ≠ a ∈ R$. (See [C2].)

(4) (Hamsher [H]) When is a full linear ring right or left max? (Re- garding the corresponding question for $V$-rings, see Osofsky [0].)
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References


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Rutgers University
Department of Mathematics
New Brunswick, New Jersey 08903
U.S.A.

Author’s Permanent address:
199 Longview Drive
Princeton, NJ 08540
U.S.A.

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