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LINEAR TOPOLOGICAL INVARIANTS OF SPACES OF HOLOMORPHIC FUNCTIONS IN INFINITE DIMENSION

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Abstract

It is shown that if E is a Frechet space with the strong dual E^* then $H_b(E^*)$, the space of holomorphic functions on E^* which are bounded on every bounded set in E^*, has the property (DN) when $E \in (DN)$ and that $H_b(E^*) \in (\Omega)$ when $E \in (\Omega)$ and either E^* has an absolute basis or E is a Hilbert-Frechet-Montel space. Moreover the complementness of ideals $J(V)$ consisting of holomorphic functions on E^* which are equal to 0 on V in $H(E^*)$ for every nuclear Frechet space E with $E \in (DN) \cap (\Omega)$ is stablished when $J(V)$ is finitely generated by continuous polynomials on E^*.

1. Introduction

Let E be a Frechet space with a fundamental system of semi-norms $\{\|\cdot\|_k\}$. For each subset B of E we define a semi-norm $\|\cdot\|_B^*$ on E^*, the strongly dual space of E, with values in $[0, +\infty]$ by

$$\|u\|_B^* = \sup\{|u(x)| : x \in B\}.$$

We write

$$\|\cdot\|_k^* = \|\cdot\|_{U_k}^*, \text{ where } U_k = \{x \in E : \|x\|_k \leq 1\}.$$

Using the notation we say that E has the property

$$(\Omega) \qquad \forall p\, \exists q\, \forall k\, \exists d,\, C > 0 : \|\cdot\|_q^{*\ 1+d} \leq C\|\cdot\|_k^* \|\cdot\|_p^{*\ d}$$

$$(DN) \qquad \exists p\, \forall q\, \exists k,\, C > 0 : \|\cdot\|_q^2 \leq C\|\cdot\|_k \|\cdot\|_p.$$

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The above properties and many others properties were introduced and investigated by Vogt (see [**9-12**]). In [**9**] Vogt has proved that $E \in (\Omega)$ if and only if for every $p \geq 1$ there exists q such that for every k we have C, $d > 0$ with

$$U_q \subseteq CrU_p + \frac{1}{r^d} U_k$$

for all $r > 0$.

In [**3**] Meise and Vogt have investigated the structure of spaces of holomorphic functions in the relation with the above linear topological invariants in nuclear spaces of infinite dimension.

The aim of the present paper is to continue the study of Meise and Vogt in the non-nuclear case.

In Section 1 we prove that $H_b(E^*)$, the space of holomorphic functions on E^*, which are bounded on every bounded set in E^*, has the property (DN) if E has also the property (DN). We also prove that when E^* has an absolute basis, $H(E^*) \in (\Omega)$ if $E \in (\Omega)$. Moreover we prove also that $H(E^*) \in (\Omega)$ for every Hilbert-Frechet-Montel space E having the property (Ω).

Finally in Section 3 we consider completemented ideals of $H(E^*)$, the space of holomorphic functions on E^* equipped with the compact-open topology when E is a nuclear Frechet space. By applying the spliting Vogt's theorem we prove that if V is an algebraic hypersurface in E^* with $E \in (DN) \cap (\Omega)$, the space

$$J(V) = \{f \in H(E^*) : f|_V = 0\}$$

is complemented in $H(E^*)$. Moreover a more general result is also established. That is if V is an algebraic set of finite codimension in E^* such that $J(V)$ is generated by finite number of continuous polynomials, $J(V)$ is complemented in $H(E^*)$ when $E \in (DN) \cap (\Omega)$.

2. Properties (DN) and (Ω)

In this section we prove the following theorem which was proved in [**3**] by Meise and Vogt for the nuclear case.

2.1. Theorem. *Let E be a Frechet space. Then*

(i) $H_b(E^*) \in (DN)$ *if* $E \in (DN)$.

(ii) $H_b(E^*) \in (\Omega)$ *if* E^* *has an absolute basis.*

Proof: (i) Assume that $E \in (DN)$. By Vogt [**9**], E can be considered as a subspace of a space of the form $F \widehat{\otimes}_\pi s$, where F is some Banach space and s is the space of rapidly decreasing sequences. Since every bounded set in E^* can be extended to a bounded set in $(F\widehat{\otimes}_\pi s)^* \cong F^*\widehat{\otimes}_\pi s^*$, it follows that $H_b(E^*)$ can be considered as a subspace of $H_b(F^*\widehat{\otimes}_\pi s^*)$. Since $H_b(F^* \times s^*) \cong H_b(F^*, H(s^*))$ and $H_b(s^*) \in (DN)$ [**3**], (i) is an immediate consequence of the following two assertions.

Assertion 1. $H_b(F^*\widehat{\otimes}_\pi s^*)$ is isomorphic to a subspace of $H_b(F^* \times s^*)$.

Assertion 2. $H_b(F^*, H(s^*)) \in (DN)$.

Proof of Assertion 1: We check that the form $f \to g = f|_{F^* \times s^*}$ defines an isomorphic map from $H_b(F^*\widehat{\otimes}_\pi s^*)$ to a subspace of $H_b(F^* \times s^*)$.

Given B a bounded set in $F^*\widehat{\otimes}_\pi s^*$. Since F^* is a Banach space and s^* a (DFN)-space we can find a neighbourhood U of $0 \in s$ and $r > 0$ such that $B \subseteq \operatorname{conv}(D_r \times U^0)$ where $D_r = \{x \in F^* : \|x^*\| < r\}$. For each $f \in H_b(F^*\widehat{\otimes}_\pi s^*)$ consider its Taylor expansion at $0 \in F^*\widehat{\otimes}_\pi s^* : f(\omega) = \sum_{n\geq 0} P_n f(\omega)$ where $P_n f(\omega) = \frac{1}{2\pi i} \int\limits_{|t|=r} \frac{f(t\omega)}{t^{n+1}}\, dt$. Let $\pi : F^* \times s^* \to F^*\widehat{\otimes}_\pi s^*$ be the canonical map. We have

$$
\begin{aligned}
\|f\|_B &\leq \|f\|_{\operatorname{conv}(D_r \otimes U^0)} \\
&= \sup\{|f(\omega)| : \omega \in \operatorname{conv}(D_r \otimes U^0)\} \\
&\leq \sup\left\{\sum_{n\geq 0} |P_n f(\omega)| : \omega \in \operatorname{conv}(D_r \otimes U^0)\right\} \\
&= \sup\left\{\sum_{n\geq 0} \left| P_n f\left(\sum_{j\geq 1} \lambda_j (z_j \otimes u_j)\right)\right|\right. \\
&\qquad\qquad \left. : z_j \in D_r,\, u_j \in U^0,\, j \geq 1,\, \sum_{j\geq 1} |\lambda_j| \leq 1\right\} \\
&= \sup\left\{\sum_{n\geq 0} \sum_{j_1, j_2, \dots, j_n \geq 1} |\lambda_{j_1}||\lambda_{j_2}|\dots|\lambda_{j_n}| \widehat{P_n f}(z_{j_1} \otimes u_{j_1}, \dots, z_{j_n} \otimes u_{j_n})|\right. \\
&\qquad\qquad \left. : z_j \in D_r,\, u_j \in U^0\, \forall j \geq 1,\, \sum_{n\geq 1} |\lambda_j| \leq 1\right\} \\
&\leq \sum_{n\geq 0} \sup\{|\widehat{P_n f}(z_{j_1} \otimes u_{j_1}, \dots , z_{j_n} \otimes u_{j_n})| : z_j \in D_r,\, u_j \in U^0\, \forall j \geq 1\} \\
&\qquad \sup\left\{\sum_{j_1, \dots, j_n \geq 1} |\lambda_{j_1}| \dots |\lambda_{j_n}| : \sum_{j\geq 1} |\lambda_j| \leq 1\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{n\geq 0} \frac{n^{2n}}{(n!)^2} \sup\{|P_n f(z\otimes u)| : z\in D_r,\, u\in U^0\}\\
&\quad \sup\left\{\left(\sum_{j\geq 1}|\lambda_j|\right)^n : \sum_{j\geq 1}|\lambda_j|\leq 1\right\}\\
&= \sum_{n\geq 0} \frac{n^{2n}}{(n!)^2} \sup\left\{\left|\frac{1}{2\pi i}\int_{|\lambda|=\rho} \frac{f(\lambda(z\otimes u))}{\lambda^{n+1}}\,d\lambda\right| : z\in D_r,\, u\in U^0\right\}\\
&= \sum_{n\geq 0} \frac{n^{2n}}{(n!)^2} \sup\left\{\left|\frac{1}{2\pi i}\int_{|\lambda|=\rho} \frac{g((\lambda z, u))}{\lambda^{n+1}}\,d\lambda\right| : z\in D_r,\, u\in U^0\right\}\\
&\leq \sum_{n\geq 0} \frac{n^{2n}}{(n!)^2\rho^n} \sup\{|g((\lambda z,u))| : |\lambda|=\rho,\, z\in D_r,\, u\in U^0\}\\
&= \sum_{n\geq 0} \frac{n^{2n}}{(n!)^2\rho^n} \sup\{|g((z,u))| : z\in D_{\rho r},\, u\in U^0\}\\
&= \|g\|_{D_{\rho r}\times U^0} \sum_{n\geq 0} \frac{n^{2n}}{(n!)^2\rho^n} < \infty
\end{aligned}
$$

for ρ sufficiently large, where $\widehat{P_n f}$ denotes the continuous symetric n-linear map associated to $P_n f$. Hence the map $f \longmapsto g = f|_{F^*\times s^*}$ is isomorphic from $H_b(F^*\widehat{\otimes}_\pi s^*)$ to a subspace of $H_b(F^*\times s^*)$. The assertion is proved. ■

Proof of Assertion 2: It is easy to see that the topology of $H_b(F^*, H(s^*))$ can be defined by the system of seminorms given by

$$\|f\|_{(r,p)} = \sup\{\|P_n f(x)\|_p : \|x\| < r\}$$

where $\{\|\cdot\|_p\}$ is a fundamental sequence of seminorms on $H(s^*)$. Since $H(s^*)\in (DN)$ [**3**] there exists p_1 such that

$$\forall p\, \exists p_2,\, C>0 : \|\cdot\|_p^2 \leq C\|\cdot\|_{p_2}\|\cdot\|_{p_1}$$

on $H(s^*)$.

Then we have

$$
\begin{aligned}
\|f\|_{(r,p)}^2 &\leq \sup\{C\|P_n f(x)\|_{p_2}\|P_n f(x)\|_{p_1} : \|x\| < r\}\\
&= \sup\{Cr^{2n}\|P_n f(x)\|_{p_2}\|P_n f(x)\|_{p_1} : \|x\| < 1\}\\
&\leq C\|f\|_{(r^2,p_2)}\|f\|_{(1,p_1)}.
\end{aligned}
$$

The assertion 2 is proved. ■

Now we continue the proof of (ii) in Theorem 2.1. Let $\{e_j^*\}$ be an absolute basis of E^* and $\{e_j\} \in E^{**}$ be coefficient functionals. For each p put

$$N_p = \{j \in N : \|e_j^*\|_p^* < \infty\}$$

and

$$E^*(p) = \left\{ x^* \in E^* : \|x^*\|_p = \sum_{j \in N_p} |e_j(x^*)| \, \|e_j^*\|_p^* < \infty \right\}.$$

Obviously $N_p \subseteq N_{p+1}$ for every $p \geq 1$. Since E is a Frechet space it implies that every bounded set in E^* is contained and bounded in some $E^*(p)$. Indeed, given K a bounded set in E^*. Take p such that K is contained and bounded in E_p^*, where by E_p^* we denote the Banach space associated to p. Since for each bounded set B in E, the seminorm

$$\sum_{j \geq 1} |e_j(x^*)| \|e_j^*\|_B^*$$

is continuous on E_p^*, we have

$$C(B) = \sup \left\{ \sum_{j \geq 1} |e_j(x^*)| \, \|e_j^*\|_B^* : x^* \in K \right\} < \infty.$$

Assume that for every $q \geq p$ there exists $x_q^* \in K$ and $x_{q,j} \in U_q$ such that

$$\sum_{j \geq 1} |e_j(x_q^*)| \, |e_j^*(x_{q,j})| \geq 2q.$$

Now for each q we take m_q such that

$$\sum_{1 \leq j \leq m_q} |e_j(x_q^*)| \, |e_j^*(x_{q,j})| \geq q.$$

Then we have $C(B) = \infty$, where $B = \{x_{q,j} : 1 \leq j \leq m_q\}$. This is impossible, because B is bounded in E. By the hypothesis $E \in (\Omega)$ for every p there exists q such that for every k there exist $d, C > 0$ for which

$$\| \cdot \|_q^{* \, 1+d} \leq C \| \cdot \|_k^* \| \cdot \|_p^{*d}.$$

Put $a_{j,p} = \|e_j^*\|_p^*$ we have

$$a_{j,q}^{1+d} \le C a_{j,k} a_{j,p}^d \text{ for all } j \ge 1.$$

Let

$$M = \{m = (m_1, \dots, m_n, 0, \dots)\} \text{ and}$$
$$M_p = \{m \in M : m_j = 0 \text{ for } j \notin N_p\}.$$

Now for each $m \in M$ and each $f \in H_b(E^*)$ put

$$b_m(f) = \frac{1}{(2\pi i)^n} \int\limits_{|t_1|=1} \cdots \int\limits_{|t_n|=1} \frac{f(t_1 e_1^* + \cdots + t_n e_n^*)}{t^{m+1}} \, dt_1 \dots dt_n$$

where $t^{m+1} = t_1^{m_1+1} \dots t_n^{m_n+1}$.

By Ryan [**7**] the series $\sum\limits_{m \in M} b_m(f) z^m$ is absolutely convergent and uniformly on every bounded set in E^* to f. Moreover, since the map

$$\xi \longmapsto \sum_{j \in N_p} \frac{\xi_j e_j^*}{\|e_j^*\|_p}$$

is an isomorphism from l^1 onto $E^*(p)$, by Ryan [**7**], the topology of $H_b(E^*)$ can be defined by the system of seminorms $\|\cdot\|_{(\alpha,p)}$ given by

$$\|f\|_{(\alpha,p)} = \sup_{m \in M_p} \{\alpha^m |b_m(f)| m^m / a_{\cdot,p}^m |m|^{|m|}\}$$

where $\alpha \in R^+ = \{r \in \mathbf{R} : r > 0\}$ and $|m| = m_1 + m_2 + \cdots + m_n$. We check that for every (α, p) we can find (β, q) such that for every (γ, k) there exist d', $C' > 0$ such that

$$(*) \qquad W_{(\beta,q)} \subseteq C' r^{d'} W(\gamma, k) + \frac{1}{r} W_{(\alpha,p)} \text{ for all } r > 0$$

where $W_{(\alpha,p)}$ denotes the unit ball in $H_b(E^*)$ defined by the semi-norm $\|\cdot\|_{(\alpha,p)}$.

Given (α, p). Put $\beta = 2\alpha$. Take q such that for every k there exist d, $C > 0$ for wich

$$a_{j,q}^{1+d} \le C a_{j,k} a_{j,p}^d \text{ for all } j \ge 1.$$

Given (γ, k). We check that $(*)$ holds for $C' = 1$ and

$$d' = d_1 = d + \log_2 \gamma C/\beta.$$

Obviously $(*)$ holds for every $0 < r \le 1$ and every d', $C' > 0$. It remains to check that $(*)$ holds for every $r > 1$ and $d' = d_1$, $C' = 1$. Let $f \in W_{(\beta,q)}$. Put

$$M_p^1 = \{m \in M_p : |m| > \log_2 r\}$$
$$M_p^2 = \left\{m \in M_p : |m| > \log_2 r \text{ and } \frac{1}{a_{\cdot,p}^m} \le \frac{1}{r a_{\cdot,q}^m}\right\}$$
$$M_p^3 = M_p \backslash (M_p^1 \cup M_p^2).$$

We have

$$\begin{aligned}&\sup\{\alpha^m |b_m| m^m / a_{\cdot,p}^m |m|^{|m|} : m \in M_p^1\}\\ &\quad \le \sup\{(1/2)^{|m|} \beta^m |b_m| m^m / a_{\cdot,q}^m |m|^{|m|} : m \in M_p^1\}\\ &\quad \le (1/2)^{\log_2 r} \sup\{\beta^m |b_m| m^m / a_{\cdot,q}^m |m|^{|m|} : m \in M_q\} \le \frac{1}{r}\end{aligned}$$

and

$$\begin{aligned}&\sup\{\alpha^m |b_m| m^m / a_{\cdot,p}^m |m|^{|m|} : m \in M_p^2\}\\ &\quad \le (1/r) \sup\{\beta^m |b_m| m^m / a_{\cdot,q}^m |m|^{|m|} : m \in M_p^2\}\\ &\quad \le (1/r) \sup\{\beta^m |b_m| m^m / a_{\cdot,q}^m |m|^{|m|} : m \in M_q\} \le \frac{1}{r}.\end{aligned}$$

Thus

$$\sum_{m \in (M \backslash M_p) \cup (M_p^1 \cup M_p^2)} b_m(f) z^m \in (1/r) W_{(\alpha,p)} \text{ for all } r \ge 1.$$

It remains to show that

$$\sum_{m \in M_p^3} b_m(f) z^m \in r^{d_1} W_{(\gamma,k)}.$$

First we observe that

$$1/a_{\cdot,k}^m \le r^d C^{|m|} / a_{\cdot,q}^m \text{ for all } m \in M_p^3.$$

Indeed, in the converse case we have

$$1/a_{\cdot,k}^m > r^d C^{|m|} / a_{\cdot,q}^m \text{ for some } m \in M_p^3.$$

On the other hand, since $m \notin M_p^2$ we have

$$1/a_{\cdot,p}^m > 1/r a_{\cdot,q}^m.$$

Hence

$$a_{\cdot,q}^{(1+d)m} > C^{|m|} a_{\cdot,k}^{m} a_{\cdot,p}^{dm}.$$

This inequality shows that for $1 \leq j \leq n$ we have

$$(a_{j,q}^{(1+d)})^{m_j} > C^{m_j} (a_{j,k} a_{j,p}^{d})^{m_j}.$$

This is impossible, because

$$a_{j,q}^{1+d} \leq C a_{j,k} a_{j,p}^{d}.$$

Thus we have

$$\begin{aligned}
&\sup\{\gamma^m |b_m| m^m / a_{\cdot,k}^{m} |m|^{|m|} : m \in M_p^3\} \\
&\quad \leq r^d \sup\{(\gamma C/\beta)^{|m|} \beta^m |b_m| m^m / a_{\cdot,q}^{m} |m|^{|m|} : m \in M_p^3\} \\
&\quad \leq r^d (\gamma C/\beta)^{\log_2 r} \sup\{\beta^m |b_m| m^m / a_{\cdot,q}^{m} |m|^{|m|} : m \in M_p^3\} \leq r^{d_1}.
\end{aligned}$$

This means that

$$\sum_{m \in M_p^3} b_m(f) z^m \in r^{d_1} W_{\gamma,k)}$$

and, hence,

$$f \in r^{d_1} W_{(\gamma,k)} + \frac{1}{r} W_{(\alpha,p)}.$$

The theorem is proved. ■

2.2 Theorem. *Let E be a Hilbert-Frechet-Montel space with $E \in (\Omega)$. Then $H(E^*)$, the space of holomorphic functions on E^* equipped with the compact-open topology, has the property (Ω).*

Proof: Applying the spliting Vogt's theorem for exact sequences of Hilbert-Frechet spaces [**12**], it follows that E is isomorphic to a quotient space of $l^2 \widehat{\otimes}_\pi s$. This is possible, because for E we have a canonical exact sequence of the form

$$O \to E \to \prod_j E_j \to \prod_j E_j \to 0$$

where E_j are Hilbert spaces.

Since E is a Frechet-Montel space every bounded set in E is the image of a compact set in $l^2 \widehat{\otimes}_\pi s$, we infer that E^* is contained as a subspace of $(l^2 \widehat{\otimes}_\pi s)^* \cong l^2 \widehat{\otimes}_\pi s^*$. On the other hand, since $l^2 \widehat{\otimes}_\pi s^*$ has a fundamental system of Hilbert semi-norms by Colombeau and Mujica [**1**], $H(E^*)$ is a quotient space of $H_b(l^2 \widehat{\otimes}_\pi s^*)$. Thus it suffices to check that $H_b(l^2 \widehat{\otimes}_\pi s^*) \in (\Omega)$. The following lemma will solve that. ■

2.3. Lemma. *$H_b(l^2 \widehat{\otimes}_\pi s^*)$ is isomorphic to a quotient space of $H_b(l^2)\widehat{\otimes}_\pi H_b(s^*)$.*

Proof: Given
$$(f, g) \in H_b(l^2) \times H_b(s^*).$$
Consider the Taylor expansion of f and g at $0 \in l^2$ and $0 \in s^*$ respectively
$$f(u) = \sum_{n \geq 0} P_n f(u) \text{ for } u \in l^2$$
and
$$g(v) = \sum_{n \geq 0} P_n g(v) \text{ for } v \in s^*.$$
For each $n \geq 0$ by
$$\chi_n : \underbrace{(l^2 \widehat{\otimes}_\pi s^*) \times \cdots \times (l^2 \widehat{\otimes}_\pi s^*)}_{n} \to (l^2 \widehat{\otimes}_\pi s^*) \widehat{\otimes}_\pi (l^2 \widehat{\otimes}_\pi s^*) \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi (l^2 \widehat{\otimes}_\pi s^*)$$
we denote the canonical n-linear map and by
$$\begin{aligned} \mu_n : (l^2 \widehat{\otimes}_\pi s^*) \widehat{\otimes}_\pi (l^2 \widehat{\otimes}_\pi s^*) \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi (l^2 \widehat{\otimes}_\pi s^*) & \\ \to \underbrace{(l^2 \widehat{\otimes}_\pi l^2 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi l^2 \widehat{\otimes}_\pi l^2)}_{n} \widehat{\otimes}_\pi (s^* \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi s^*) & \end{aligned}$$
we denote the canonical isomorphism. Putting
$$\widehat{P_n}(f, g) = (\widehat{P_n f} \otimes \widehat{P_n g}) \mu_n \chi_n$$
we obtain a continuous symetric n-linear form on $(l^2 \widehat{\otimes}_\pi s^*) \times \cdots \times (l^2 \widehat{\otimes}_\pi s^*)$. Let $P_n f \otimes P_n g$ denote the n-homogeneous polynomial on $l^2 \widehat{\otimes}_\pi s^*$ induced by $\widehat{P_n}(f, g)$. We have
$$\begin{aligned} &(P_n f) \otimes (P_n g) \left(\sum_{j \geq 1} u_j \otimes v_j \right) \\ &\quad = \widehat{P_n}(f, g) \left(\sum_{j \geq 1} u_j \otimes v_j, \ldots, \sum_{j \geq 1} u_j \otimes v_j \right) \\ &\quad = \sum_{j_1, j_2, \ldots, j_n \geq 1} \widehat{P_n}(f, g)((u_{j_1} \otimes v_{j_1}, \ldots, u_{j_n} \otimes v_{j_n}) \\ &\quad = \sum_{j_1, j_2, \ldots, j_n \geq 1} \widehat{P_n f}(u_{j_1}, \ldots, u_{j_n}) \widehat{P_n g}(v_{j_1}, \ldots, v_{j_n}) \end{aligned}$$

Hence for every $r > 0$ and every neighbourhood V of $0 \in s$ we have

$$\begin{aligned}
&\Big\|\sum_{n\geq 0} P_n f \otimes P_n g\Big\|_B \leq \sum_{n\geq 0} \|P_n f \otimes P_n g\|_B \leq \sum_{n\geq 0} \|P_n f \otimes P_n g\|_{\mathrm{conv}(rD\times V)}\\
&= \sum_{n\geq 0} \sup\{|P_n f \otimes P_n g(\omega)| : \omega \in \mathrm{conv}(rD \times V^0)\}\\
&= \sum_{n\geq 0} \sup\Bigg\{\Bigg|P_n f \otimes P_n g\Bigg(\sum_{j\geq 1}\lambda_j u_j \otimes v_j\Bigg)\Bigg|\\
&\qquad : u_j \in rD,\ v_j \in V^o,\ \sum_{j\geq 1}|\lambda_j| \leq 1\Bigg\}\\
&\leq \sum_{n\geq 0} \sup\Bigg\{\sum_{j_1,j_2,\dots,j_n\geq 1} |\lambda_{j_1}|\dots|\lambda_{j_n}|\,|(P_n f \otimes P_n g)((u_{j_1}\otimes v_{j_1},\dots,u_{j_n}\otimes v_{j_n})|\\
&\qquad : u_j \in rD,\ v_j \in V^0,\ \sum_{j\geq 1}|\lambda_j| \leq 1\Bigg\}\\
&= \sum_{n\geq 0} \sup\Bigg\{\sum_{j_1,j_2,\dots,j_n\geq 1} |\lambda_{j_1}|\dots|\lambda_{j_n}|\,|(\widehat{P_n f}(u_{j_1},\dots,u_{j_n})|\,|\widehat{P_n g}(v_{j_1},\dots,v_{j_n})|\\
&\qquad : u_j \in rD,\ v_j \in V^0,\ \sum_{j\geq 1}|\lambda_j| \leq 1\Bigg\}\\
&\leq \sum_{n\geq 0} \frac{n^n}{n!}\frac{\|f\|_{\rho r D}}{\rho^n}\frac{n^n}{n!}\frac{\|g\|_{\rho V^0}}{\rho^n}\Bigg(\sum_{j_1,j_2,\dots,j_n\geq 1}|\lambda_{j_1}|\dots|\lambda_{j_n}|\Bigg)\\
&= \|f\|_{\rho r D}\|g\|_{\rho V^0}\sum_{n\geq 0}\left(\frac{n^n}{n!\rho^n}\right)^2\Bigg(\sum_{j\geq 1}|\lambda_j|\Bigg)^n\\
&\leq C_\rho\|f\|_{\rho r D}\|g\|_{\rho V^0}
\end{aligned}$$

for ρ sufficiently large. Thus the form

$$\theta(f,g) = \sum_{n\geq 0} P_n f \otimes P_n g$$

define a continuous bilinear map from $H_b(l^2) \times H_b(s^*)$ to $H_b(l^2\widehat{\otimes}_\pi s^*)$ which induces a continuous linear map

$$\widehat{\theta} : H_b(l^2)\widehat{\otimes}_\pi H_b(s^*) \to H_b(l^2\widehat{\otimes}_\pi s^*).$$

It remains to check that $\widehat{\theta}$ is surjective. Given $f \in H_b(l^2\widehat{\otimes}_\pi s^*)$. Let $\{e_j\}$ and $\{e_j^*\}$ be the canonical bases of s and s^* respectively. Formally we have, for every $\sum_{k\geq 1} u_k \otimes v_k \in l^2\widehat{\otimes}_\pi s^*$, the following equalities

$$
\begin{aligned}
&f\left(\sum_{k\geq 1} u_k \otimes v_k\right)\\
&= \sum_{n\geq 0} P_n f\left(\sum_{k\geq 1} u_k \otimes v_k\right)\\
&= \sum_{n\geq 0} P_n f\left(\sum_{k\geq 1}\left(u_k \otimes \sum_{j\geq 1} e_j(v_k)e_j^*\right)\right)\\
&= \sum_{n\geq 0} P_n f\left(\sum_{j\geq 1}\left(\sum_{k\geq 1} e_j(v_k)u_k\right)\otimes e_j^*\right)\\
&= \sum_{n\geq 0}\sum_{j_1,\dots,j_n\geq 1}\widehat{P_n f}\left(\left(\sum_{k\geq 1} e_{j_1}(v_k)u_k \otimes e_{j_1}^*\right),\dots,\left(\sum_{k\geq 1} e_{j_n}(v_k)u_k \otimes e_{j_n}^*\right)\right)\\
&= \sum_{n\geq 0}\sum_{j_1,\dots,j_n\geq 1}\sum_{k_1,\dots,k_n\geq 1}\widehat{P_n f}(u_{k_1}\otimes e_{j_1}^*,\dots,u_{k_n}\otimes e_{j_n}^*)e_{j_1}(v_{k_1})\dots e_{j_n}(v_{k_n})\\
&= \sum_{n\geq 0}\sum_{j_1,\dots,j_n\geq 1}\sum_{k_1,\dots,k_n\geq 1} P_n f(\cdot\otimes e_{j_1}^*,\dots,\cdot\otimes e_{j_n}^*)(u_{k_1},\dots,u_{k_n})\\
&\qquad\qquad\qquad\qquad (e_{j_1}(\cdot)\dots e_{j_n}(\cdot)(v_{k_1},\dots,v_{k_n})\\
&= \sum_{n\geq 0}\sum_{j_1,\dots,j_n\geq 1}([P_n f(\cdot\otimes e_{j_1}^*,\dots,\otimes e_{j_n}^*)\otimes(e_{j_1}(\cdot)\dots e_{j_n}(\cdot))]\left(\sum_{k\geq 1} u_k\otimes v_k\right)
\end{aligned}
$$

On the other hand if for $r > 0$ and V as above, choose a neighbourhood $\tilde{V}$ of $0 \in s$ such that $\sum \|e_j^*\|_{\tilde{V}}^*\|e_j\|_V < 1/2$ we get the following estimations

$$
\begin{aligned}
&\sum_{n\geq 0}\sum_{j_1,\dots,j_n\geq 1}\sup\left\{\sum_{k_1,\dots,k_n\geq 1}|\lambda_{k_1}|\dots|\lambda_{k_n}|\,|\widehat{P_n f}(u_{k_1}\otimes e_{j_1}^*,\dots,u_{k_n}\otimes e_{j_n}^*)|\right.\\
&\left.\times|e_{j_1}(v_{k_1})|\dots|e_{j_n}(v_{k_n})| : u_k \in rD,\ v_k \in V^0,\ \sum_{k\geq 1}|\lambda_k| \leq 1\right\}
\end{aligned}
$$

$$\leq \sum_{n\geq 0}\sum_{j_1,\dots,j_n\geq 1}\sup\Bigg\{\sum_{k_1,\dots,k_n\geq 1}|\lambda_{k_1}|\dots|\lambda_{k_n}|$$
$$\left|\widehat{P_n f}\left(u_{k_1}\otimes\frac{e^*_{j_1}}{\|e^*_{j_1}\|^*_{\bar V}},\dots,u_{k_n}\otimes\frac{e^*_{j_n}}{\|e^*_{j_n}\|^*_{\bar V}}\right)\right|$$
$$\|e^*_{j_1}\|^*_{\bar V}\dots\|e^*_{j_n}\|^*_{\bar V}|e_{j_1}(v_{k_1})|\dots|e_{j_n}(v_{k_n})| : u_k\in rD,\ v_k\in V^0,\ \sum_{k\geq 1}|\lambda_k|\leq 1\Bigg\}$$
$$\leq \sum_{n\geq 0}\sum_{j_1,\dots,j_n\geq 1}\sup\Bigg\{\sum_{k_1,\dots,k_n\geq 1}|\lambda_{k_1}|\dots|\lambda_{k_n}|\frac{n^n}{n!\rho^n}\|f\|_{\rho rD\otimes V^0}$$
$$\|e^*_{j_1}\|^*_{\bar V}\|e_{j_1}\|_V\dots\|e^*_{j_n}\|^*_{\bar V}\|e_{j_n}\|_V : u_k\in rD,\ v_k\in V^0,\ \sum_{k\geq 1}|\lambda_k|\leq 1\Bigg\}$$
$$= \|f\|_{\rho rD\otimes V^0}\sum_{n\geq 0}\frac{n^n}{n!\rho^n}\left(\sum_{j\geq 1}\|e^*_j\|^*_{\bar V}\|e_j\|_V\right)^n\left(\sum_{k\geq 1}|\lambda_k|\right)^n$$
$$\leq \|f\|_{\rho rD\otimes V^0}\sum_{n\geq 0}\frac{n^n}{n!2^n\rho^n}\leq C_{\rho,V}\|f\|_{\rho rD\otimes V^0}$$

for ρ sufficiently large.

Now for each $n\geq 0$ we define $\chi_n(f)\in H_b(l^2)\widehat{\otimes}_\pi H_b(s^*)\cong(l^2\times s^*)$ given by

$$\chi_n(f)(u,v)=\sum_{j_1,\dots,j_n\geq 1}P_n f(u\otimes e^*_{j_1},\dots,u\otimes e^*_{j_n})e_{j_1}(v)\dots e_{j_n}(v).$$

It follows that

$$\begin{aligned}\widehat{\theta}(\chi_n(f))&(u_1\otimes v_1,\dots,u_n\otimes v_n)\\ &=\sum_{j_1,\dots,j_n\geq 1}P_n f(u_1\otimes e^*_{j_1},\dots,u_n\otimes e^*_{j_n})e_{j_1}(v_1)\dots e_{j_n}(v_n)\\ &=P_n f(u_1\otimes v_1,\dots,u_n\otimes v_n).\end{aligned}$$

Thus for $g=\sum_{n\geq 0}\chi_n(f)\in H_b(l^2)\widehat{\otimes}_\pi H_b(s^*)$ we get $\widehat{\theta}g=f$. ■

To complete the proof of the theorem 2.2 it suffices prove the following.

2.4. Proposition. *Let B be a Banach space and E be a Frechet-Montel space such that E^* has an absolute basis. Then $H_b(B\times E^*)\in(\Omega)$ if $E\in(\Omega)$.*

Proof: Since E is Frechet-Montel, we have $H_b(B\times E^*)\cong H(E^*,H_b(B))$. By the hypothesis E^* has an absolute basis $\{e_j^*\}$ as in Theorem 2.1 (ii) the topology of $H_b(E^*,H_b(B))$ can be defined by the system of semi-norms given by

$$\|f\|_{(\alpha,\rho,r)}=\sup\{\alpha^m\rho^n\|P_nb_m(f)\|m^m/a^m_{\cdot,p}|m|^{|m|}\}$$

where

$$a_{j,p}=\|e_j^*\|_p^* \text{ for } j,\, p\geq 1.$$

As in the proof of Theorem 2.1 (ii) for each (α,ρ,p) take $\beta=2\alpha$, $\eta=2\rho$, and $d_1=d+\log_2\gamma\theta C/\beta$ we have

$$W_{(\beta,\eta,q)}\subseteq r^{d_1}W_{(\gamma,\theta,k)}+\frac{1}{r}W_{(\alpha,\rho,p)}$$

for all $r>0$, where for each p take q such that for each k there exist C, $d>0$ for which

$$a_{j,q}^{1+d}\leq Ca_{j,k}a_{j,p}^d \text{ for all } j\geq 1$$

and (γ,θ,k) is given.

Consequently $H(E^*,H_b(B))$ and, hence, $H_b(B\times E^*)$ has the property (Ω).

The proposition is proved. ■

3. Complemented ideals in $H_b(E^*)$

Let V be an analytic set in a locally convex space E and let

$$J_b(V)=\{f\in H_b(E):f|_V=0\}.$$

It is known [**2**] that $J_b(V)$ is a complemented subspace of $H_b(E)$ when $\dim E<\infty$ and V is an algebraic set in E. In this section we shall prove the above result for algebraic hypersurfaces in the space E^*, where E is a nuclear Frechet space with $E\in(DN)\cap(\Omega)$.

3.1. Theorem. *Let E be a nuclear Frechet space such that $E\in(DN)\cap(\Omega)$ and let V an algebraic hypersurface in E^*. Then $J(V)$ is a* *complemented subspace of $H(E^*)$, where $H(E^*)$ is the space of holomorphic functions on E^* and*

$$J(V) = \{f \in H(E^*) : f|_V = 0\}.$$

Proof: Considering the exact sequence of nuclear Frechet spaces

$$O \to J(V) \to H(E^*) \to H(E^*)/J(V) \to O$$

by the spliting Vogt's theorem [**9**] it suffices to check that $J(V) \in (\Omega)$ and $H(E^*)/J(V) \in (DN)$. ■

3.2. Proposition. *Let E be a Frechet space with the property (Ω) and let V be an algebraic hypersurface in E^*. Then $J_b(V) \in (\Omega)$ if one of the following two conditions holds*

(i) *E^* has an absolute basis*

(ii) *E is a Hilbert-Frechet-Montel space.*

Proof: By Theorem 2.1 and 2.2, $H_b(E^*) \in (\Omega)$. Let $P_1, \dots, P_m$ be irreducible polynomials on E^* such that

$$V = Z(P_1, \dots, P_m)$$

where by $Z(P_1, \dots, P_m)$ we denote the zero-set of $P_1, \dots, P_m$. Such polynomials exist by the factoriality of the ring $\mathbf{C}[E^*]$ of continuous polynomials on E^* (this can be proved as in [**6**] for the case where E is a Banach space). Take a decomposition of E^*, $E^* = F_1 \oplus \mathbf{C}e_1$ such that

$$P_1(x_1^* + z_1 e_1) = \sum_{0 \le j \le p_1} a_j^1(x_1^*) z_1^j \in C[F_1][z_1].$$

Let $D_1 \in \mathbf{C}[F_1]$ be the discriminant of P_1. Since P_1 is irreducible so $D_1 \neq 0$ and, hence,

$$G_1 = F_1 \backslash Z(D_1) = \{x_1^* \in F_1 : P_1(x_1^*) \text{ has different } p_1 \text{ solutions}\}$$

is dense in F_1.

As in [**6**] the map θ from $\mathbf{C}[E^*] \oplus \mathbf{C}[F_1][z_1]p_1$ to $\mathbf{C}[E^*]$ given by

$$(g, r) \to Pg + r$$

is an isomorphism, where

$$\mathbf{C}[F_1][z_1]p_1 = \{r \in \mathbf{C}[F_1][z_1] : \text{ degree (for } z_1) \ < p_1\}$$

Since $\mathbf{C}[E^*]$ and $\mathbf{C}[F_1]$ are dense in $H_b(E^*)$ and $H_b(F_1)$ respectively, the map θ is extended to an isomorphism $\widehat{\theta}$ from $H_b(E^*) \oplus H_b(F_1)[z_1]p_1$. Thus every $f \in H_b(E^*)$ is written uniquely in the form

$$(*) \qquad f = Pg(f) + r(f)$$

where $g(f) \in H_b(E^*)$ and $r(f) \in H_b(F_1)[z_1]p_1$. Now given $f \in J_b(V)$. Write f in the form $(*)$. Then

$$r(f)(x_1^*, \cdot) = 0 \text{ for } x_1^* \in G_1.$$

Since G_1 is dense in F_1, we have $r(f) = 0$ and, hence, $f = P_1 g_1$ with $g_1 = r(f)$.

It follows that $g_1 | Z(P_2, \dots, P_m) = 0$, because $Z(P_1), \dots, Z(P_m)$ are irreducible branchs of V. Applying the above argument to g_1 and P_2 we can write $g_1 = P_2 g_2$ with $g_2|_{Z(P_3,\dots,P_m)} = 0$. Continuing this progress we get

$$f = P_1 \cdot P_2 \dots P_m g_m \text{ for some } g_m \in H_b(E^*).$$

Consequently

$$J_b(V) = P_1 \dots P_m H_b(E^*) \cong H_b(E^*) \in (\Omega). \quad \blacksquare$$

3.3. Proposition. *Let E be a Frechet space with the property (DN) and V an algebraic set in E^* with* $\operatorname{codim} V < \infty$. *Then*

$$H_b(E^*)/J_b(V) \in (DN).$$

Proof: Let $P_1, P_2, \dots, P_m$ be polynomials such that $V = Z(P_1, \dots, P_m)$ where by $Z(P_1, \dots, P_m)$ we denote the common zero-set of $P_1, P_2, \dots, P_m$.

To simplify our reasoning we consider only the case $m = 2$. Take a decomposition of E^*, $E^* = F \oplus \mathbf{C}e_1 \oplus \mathbf{C}e_2$ such that

$$P = P_1 = a_p z_1^p + \sum_{0 \le j \le p-1} a_j z_1^j$$
$$Q = P_2 = b_q z_2^q + \sum_{0 \le k \le q-1} b_k z_2^k$$

where a_p, $b_q \in \mathbf{C}\backslash\{0\}$ and a_j and b_k are continuous polynomials on F and

$$\begin{aligned} f &= Pg + \sum_{0 \le j \le p-1} a_j(f) z_1^j \\ &= Pg + \sum_{0 \le j \le p-1} z_1^j \left[Q h_j + \sum_{0 \le k \le q-1} b_k^j z_2^k \right] \\ &= Pg + Q \sum_{0 \le j \le p-1} h_j z_1^j + \sum_{0 \le j \le p-1} \sum_{0 \le k \le q-1} b_k^j z_1^j z_2^k \end{aligned}$$

where $g \in \mathbf{C}[E^*]$, $a_j,\ h_j \in \mathbf{C}(F \oplus \mathbf{C}e_2)$ and $b_k^j \in \mathbf{C}[F]$. Then as in Proposition 3.2 every $f \in H_b(E^*)$ is written uniquely in the form $(*)$ with $g \in H_b(E^*)$, $a_j,\ h_j \in H_b(F \oplus \mathbf{C}e_1)$ and $b_k^j \in H_b(F)$. Let θ be the canonical map from $H_b(F)$ to $W = H_b(E^*)/_{J_b(V)}$. Obviously, θ is injective and defines W as a finitely generated $H_b(F)$-module. Hence there exists monic polynomials R and $S \in H_b(F)[X]$ such that

$$R(z_1) = S(z_2) = 0.$$

This implies that $(B \times (\mathbf{C}e_1 \oplus \mathbf{C}e_2)) \cap V$ is bounded for every bounded set in F. Then it is easy to see that θ is an embedding. Let us note that

$$W \cong Z = H_b(F)[z_1, z_2]p, q / J_b(V) \cap H_b(F)[z_1, z_2]p, q.$$

Thus b_k^j induces for each $0 \leq j \leq p-1$ and $0 \leq k \leq q-1$ a continuous linear map on Z and hence every $f \in Z$ can be written in the form

$$f = \sum_{0 \leq r \leq s} C_r(f) g_r$$

where C_r are continuous linear for $0 \leq r \leq s$ and $\{g_r\}$ is some finite system in Z. This yields that we can find u such that for every $f \in Z$ there exists a monic polynomial $Q \in H_b(F)[X]$

$$Q = X^u + d_{u-1}(f) X^{u-1} + \cdots + d_0(f)$$

where $Q(f) = 0$ and d_j are continuous polynomials on Z with values in $H_b(F)$.

To prove that $Z \in (DN)$ by Vogt [**11**] it suffices to shown that every continuous linear map from $\Lambda_1(\alpha)$ to Z is bounded on a neighbourhood of $0 \in \Lambda_1(\alpha)$ for every exponent sequence $\alpha = \{\alpha_n\}$ where

$$\Lambda_1(\alpha) = \left\{ \{\xi\} \in \mathbf{C}^N : \sum_{j \geq 1} |\xi_j| r^{\alpha_j} < \infty \text{ for } 0 < r < 1 \right\}.$$

Given such a map T. Since $d_j T$ are continuous polynomials on $\Lambda_1(\alpha)$ and $H_b(F) \in (DN)$ (Theorem 2.1) again by Vogt [**11**] these polynomials are bounded on some neighbourhood of U of $0 \in \Lambda_1(\alpha)$. Then from the relation

$$(Tf)^u + d_{u-1}(Tf)^{u-1} + \cdots + d_0(Tf) = 0, \text{ for } f \in U$$

it follows that T is bounded on U. The proposition is proved. ■

Since $IH_b(E^*) \in (\Omega)$ for every finitely generated ideal I in $\mathbf{C}[E^*]$ such that $IH_b(E^*)$ is closed using Proposition 3.3 we have

3.4. Theorem. *Let E be a nuclear Frechet space with $E \in (DN) \cap (\Omega)$ and let V be an algebraic set in E^* such that $J(V)$ is finitely generated by finite number of polynomials. Then (JV) is complemented in $H_b(E^*)$.*

Proof: Let $P_1, \dots, P_m \in \mathbf{C}[E^*]$ generate $J(V)$ and let I denote the ideal in $\mathbf{C}[E^*]$ generated by $P_1, \dots, P_m$. We consider only $m = 2$ because the other cases are similar. Given $f \in J(V)$. As in Proposition 3.3 we can write f as follows

$$f = Pg + Qh + \sum_{0 \le j \le p-1} \sum_{0 \le k \le q-1} b_k^j(x) z_1^j z_2^k$$

for some decomposition of E^*, $E^* = F \oplus \mathbf{C}e_1 \oplus \mathbf{C}e_2$. Then $r|_V := \sum_{0 \le j \le p-1} \sum_{0 \le k \le q-1} b_k^j(x) z_1^j z_2^k|_V = 0$. Considering the Taylor expansion of each b_k^j at $0 \in F$ we have

$$r = \sum_{n \ge 0} \sum_{0 \le j \le p-1} \sum_{0 \le k \le q-1} P_n b_k^j(x) z_1^j z_2^k.$$

Let $\pi : V \to F$ be the restriction of the canonical projection from E^* onto F to V. As in Proposition 3.3, π is a branched cover. By S we denote the branched locus of π. Let $x \in F \backslash S$. Then we can find a neighbourhood V of x in $F \backslash S$ such that

$$\pi^{-1}(V) = \sqcup_s U_s \text{ and } \pi : U_s \cong V \times u_s \text{ for every } s,$$

where $\pi^{-1}(X) = \{u_s\}$.

Since

$$\sum_{n \ge 0} \sum_{0 \le j \le p-1} \sum_{0 \le k \le q-1} P_n b_k^j(x) z_1^j z_2^k|_{U_s} = 0$$

for every s we infer that

$$\sum_{0 \le j \le p-1} \sum_{0 \le k \le q-1} P_n b_k^j(x) z_1^j z_2^k|_{\pi^{-1}(V)} = 0.$$

This implies that this function is equal to 0 on V because $V \backslash \pi^{-1}(S)$ is dense in V. Thus r can be approximated by elements of I. Consequently $IH(E^*) = \mathrm{Cl}(IH(E^*)) = J(V)$ and the theorem is proved. ■

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References

1. J. F. Colombeau and M. C. Mujica, "*Holomorphic and differentiable mappings of uniform type*," Functional Analysis, Holomorphy and Approximation Theory, J. A. Barroso, ed., North-Holland Math. Stud. **71**, 1982, pp. 179–200.
2. P. B. Dijakov and B. S. Mitiagin, The structure of polynomial ideals in the algebra of entire functions, *Studia Math.* **68** (1980), 85–104.
3. R. Meise and D. Vogt, Structure of spaces of holomorphic functions on infinite dimensional polydics, *Studia Math.* **75** (1983), 235–253.
4. R. Meise and D. Vogt, Holomorphic functions of uniformly bounded type on nuclear Frechet spaces, *Studia Math.* **83** (1986), 147–166.
5. A. Pietsh, "*Nuclear Locally Convex Spaces*," Ergeb. Math. Grenzgeb **66**, Springer-Verlag, 1972.
6. J. P. Ramis, "*Sous-ensembles Analytiques d'une variete Banachique complexe*," Springer-Verlag, 1970.
7. R. A. Ryan, Holomorphic mappings on l_1, *Trans. Amer. Math. Soc.* **302(2)** (1987), 797–811.
8. H. Shaefer, "*Topological Vector Spaces*," Springer-Verlag, 1971.
9. D. Vogt, On two classes of F-spaces, *Arch. Math.* **45** (1985), 255–266.
10. D. Vogt, Charakterisierung der Potenzreihenraume von endlichem Typ und ihre Folgerungen, *Manuscripta Math.* **37** (1982), 269–301.
11. D. Vogt, Frechtraume zwischen denen jede stelige linear Abbidung beschrankt ist., *J. Reine Angew. Math.* **345** (1983), 182–200.
12. D. Vogt, Interpolation of nuclear operators and a spliting theorem for exact sequence of Frechet spaces, Preprint.

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