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ATTRACTING DOMAINS FOR SEMI-ATTRACTIVE TRANSFORMATIONS OF $\mathbf{C}^p$

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Abstract

Let F be a germ of analytic transformation of $(\mathbf{C}^p, 0)$. We say that F is semi-attractive at the origin, if $F'_{(0)}$ has one eigenvalue equal to 1 and if the other ones are of modulus strictly less than 1. The main result is: either there exists a curve of fixed points, or $F - \mathrm{Id}$ has multiplicity k and there exists a domain of attraction with $k-1$ petals. We study also the case where F is a global isomorphism of $\mathbf{C}^2$ and $F - \mathrm{Id}$ has multiplicity k at the origin. This work has been inspired by two papers: one of P. Fatou (1924) and the other one of T. Ueda (1986).

1. Introduction

Let F be a germ of analytic transformation from $(\mathbf{C}^p, 0)$ to $(\mathbf{C}^p, 0)$, i.e., a holomorphic map defined in a neighborhood of the origin in $\mathbf{C}^p$ which leaves the origin $0 = (0,0)$ of $\mathbf{C}^p$ fixed. We are interested in the behavior of the iterates $(x_n, y_n) = F^{(n)}(x, y)$ of points (x, y) near the origin. We implicitly assume that they are defined. We study the situation where $F'_{(0)}$ has one eigenvalue equal to 1 and the others are $\{\lambda_j\}_{2\le j\le p}$, with $0 \le |\lambda_j| < 1$. We call semi-attractive such trasnformations. We want to investigate the existence of attracting domains at 0 in a neighbourhood of 0. As the partial derivative $\frac{\partial}{\partial x_1} F_1^{(n)}(0) = 1$ in some coordinate system, the family $\{F^{(n)}\}$ cannot converge to 0 in a neighborhood of 0. So by attracting domains in a neighborhood of 0, we mean open domains D with $0 \in \partial D$ such that $x_n = F^{(n)}(x)$ converge to 0 for $x \in D$.

When $p = 1$, the dynamics of analytic transformations from $(\mathbf{C}, 0)$ to $(\mathbf{C}, 0)$ with eigenvalue 1, i.e., transformations which can be written with convergent power series in x as

$$F(x) = x_1 = x(1 + a_1 x + a_2 x^2 + \dots)$$

have been studied by Fatou and Leau. Their theory is quite complete (see for instance [**B**]).

In his paper [**F**] on transformations of $(\mathbf{C}^2, 0)$ Fatou investigates the case of transformations with eigenvalues 1 and b, with $0 < |b| < 1$. He proves the existence of a coordinate system (x, y) where F can be written

$$\text{(1.1)} \qquad \begin{cases} x_1 = f(x,y) = a_1(y)x + a_2(y)x^2 + \dots \\ y_1 = g(x,y) = by + b_1(y)x + b_2(y)x^2 + \dots \end{cases}$$

where the $a_j(y)$, $b_j(y)$ are holomorphic functions in a neighborhood of $0 \in \mathbf{C}$ such that $a_1(0) = 1$, $b_1(0) = 0$, the x-coordinate being chosen in such a way that $\{x = 0\}$ is the invariant curve of Poincaré. Then Fatou shows that, if $a_2(0) \neq 0$, there exists an attracting domain at 0. The projection on the x-plane of the dynamics is of the same type as the one we get in $\mathbf{C}$ with the a_j's constant. This case has been studied again by Ueda [**U**] with a better reduced form for F, which allows him to give a simple and more complete description of the domain of convergence.

When $y = 0$ is a curve of fixed points, the transformation F is of type

$$\begin{cases} x_1 = x + xyf(x,y) \\ y_1 = y(b + b_1(y)x + b_2(y)x^2 + \dots) \end{cases},$$

a case considered by Lattès [**L**]. Fatou shows that the coordinates can be chosen in such a way that we get the reduced form

$$\text{(1.2)} \qquad \begin{cases} x_1 = x \\ y_1 = y\left(b + \displaystyle\sum_{i+j \geq 1} b_{ij} x^i y^j\right) \end{cases}.$$

So a neighborhood of 0 is attracted by the curve of fixed points along the trajectories $x =$ constant. Then Fatou asks if there exis other cases for which there is no attracting domain at 0. He asks what happens for instance with a transformation like

$$\text{(1.3)} \qquad \begin{cases} x_1 = \dfrac{x}{1 + xy} \ ? \\ y_1 = by + x^2 \end{cases}$$

In this paper, we will see that there is an attracting domain of 0. We prove the following result.

Theorem 1.1. *Let F be an analytic germ of transformation from $(\mathbf{C}^p, 0)$ to $(\mathbf{C}^p, 0)$, with eigenvalues in 0 $\{1, \{\lambda_j\}_{2\leq j\leq p}\}$, such that $0 \leq |\lambda_j| < 1$, for $2 \leq j \leq p$, then either there exists a curve of fixed points or there exists an attracting domain of 0.*

More precisely, let Id be the identity of $(\mathbf{C}^p, 0)$. Either there exists a curve of fixed points or $F - \mathrm{Id}$ has a finite multiplicity $k \geq 2$. We show that in the case of multiplicity $k \geq 2$, there exists an attracting domain D of 0 made of $k-1$ attracting petals, i.e. $k-1$ disjoint open sets $\{D_j\}$ positively invariant by F such that $0 \in \partial D_j$ and that every $x \in D_j$ is attracted by 0. Conversely, if a point x is attracted by 0, for n big enough, $x_n = F^{(n)}(x)$ is in one of the D_j.

Let us recall (see for instance [**C**]) the definition of the multiplicity of a holomorphic transformation Φ from $(\mathbf{C}^p, 0)$ to $(\mathbf{C}^p, 0)$ such that 0 is isolated in the fiber $\Phi^{-1}(\{0\})$. Let V be a compact neighborhood of 0 such that the restriction Φ_V of Φ to V is proper from V to $W = \Phi(V)$; let $\mathrm{Br}(\Phi)$ be the branched locus of Φ, i.e. the set of $z \in V$ where $\mathrm{Det}(J(\Phi)(z)) = 0$; here $J(\Phi)$ is the matrix $\left(\frac{\partial\Phi_i}{\partial z_j}\right)_{1\leq i\leq p,\, 1\leq j\leq p}$. The multiplicity of Φ at 0 is then the number of points in a fiber $\Phi^{-1}(\{\zeta\})$ for ζ a point in W which does not belong to $\Phi(\mathrm{Br}(\Phi))$. We use then the lemma

Lemma 1.2 [**C**, p. 102]. *Let Φ be a holomorphic transformation Φ from $(\mathbf{C}^p, 0)$ to $(\mathbf{C}^p, 0)$ such that 0 is isolated in its fiber $\Phi^{-1}(\{0\})$. Suppose that the matrix $\left(\frac{\partial\Phi_i}{\partial z_j}\right)_{2\leq i\leq p,\, 2\leq j\leq p}$ has rank $p-1$. Let C be the curve $\{z_j = \varphi_j(z_1)\}_{2\leq j\leq p}$ defined via the implicit function theorem by $\Phi_2 = \Phi_3 = \cdots = \Phi_p = 0$, then the multiplicity of Φ at 0 is equal to the multiplicity at 0 of the function of one variable*

$$\Phi_1(z_1, \varphi_2(z_1), \ldots, \varphi_p(z_1)).$$

Proof of Lemma 1.2: From the relations

$$\frac{\partial\Phi_i}{\partial z_1} + \sum_{j=2}^{p} \varphi_j'(z_1)\frac{\partial\Phi_i}{\partial z_j} = 0, \text{ for } 2 \leq i \leq p$$

on C, we get

$$\mathrm{Det}(J(\Phi)_{|C}) = \left(\frac{\partial\Phi_1}{\partial z_1} + \sum_{j=2}^{p} \varphi_j'(z_1)\frac{\partial\Phi_1}{\partial z_j}\right) \mathrm{Det}\left(\frac{\partial\Phi_i}{\partial z_j}\right)_{2\leq i\leq p,\, 2\leq j\leq p}.$$

As 0 is isolated in its fiber $\Phi^{-1}(\{0\})$,

$$d\Phi_{1|C} = \frac{\partial \Phi_1}{\partial z_1} + \sum_{j=2}^{p} \varphi_j'(z_1)\frac{\partial \Phi_1}{\partial z_j},$$

is not identically zero. Hence C is not in $\mathrm{Br}(\Phi)$. So to count the multiplicity, we can restrict ζ to $\Phi(C)$, and we have just to count the zeros of $\phi_1(z_1) = \Phi_1(z_1, \varphi_2(z_1), \dots, \varphi_p(z_1)) = \zeta_1$ for ζ_1 in a neighborhood of 0 in $\mathbf{C}$. By Rouché's theorem, this multiplicity is given by the order in 0 of ϕ_1.

Let us for instance, compute the multiplicity of $\phi = F - \mathrm{Id}$ in the example (1.3). According to lemma 1.2, we solve the equation

$$y = by + x^2$$

and replace y in the first relation. We get

$$x_1 = \frac{x}{1 + \frac{x^3}{1-b}} = x\left(1 - \frac{x^3}{1-b} + \dots\right).$$

The multiplicity of $F - \mathrm{Id}$ at 0 is 4. By theorem 1 we see that there exists an attracting domain at 0 with three petals.

The result is also true if $F'_{(0)}$ is not invertible. For instance the transformation

$$(1.4) \qquad \begin{cases} x_1 = x(1 + xy - x^3) \\ y_1 = y^2 + x^2 \end{cases},$$

where $F'_{(0)}$ has for eigenvalues 1 et 0, $F - \mathrm{Id}$ has multiplicity 6 in 0. Indeed, solve

$$y = y^2 + x^2,$$

we get

$$y = x^2 + x^4 + \dots,$$

replacing y in the first relation

$$x_1 - x = x^2 y - x^4 = x^6 + o(x^6).$$

So there is an attracting domain at 0 with 5 petals. ■

In [**U**], Ueda studies the analytic transformations of $(\mathbf{C}^2, 0)$ with eigenvalues $\{1, b\}$ at 0, such that $0 < |b| < 1$. He calls them transformations of type $(1, b)$. Then he defines a classification $\{(1, b)_k\}$, for k integer,

$1 \le k \le +\infty$, on these transformations. In fact, the integer $k+1$ for type $(1,b)_k$ of Ueda is precisely the multiplicity of $F - \mathrm{Id}$. Ueda concentrates his work on the case $(1,b)_1$. This is the case considered by Fatou when in the expression (1.1), we have $a_2(0) \neq 0$. Ueda introduces transformations of the coordinates which give simple reduced forms to study the attracting domain, in the case $a_2(0) \neq 0$. Similar transformations will be used here.

Ueda studies also the case of a global automorphism F in $\mathbf{C}^2$. Let F be a global automorphism in $\mathbf{C}^2$ with a fixed point of type $(1,b)_1$, he proves that, like in the examples given by Fatou and Bieberbach with eigenvalues of modulus strictly less than 1, the attracting domain is isomorphic to $\mathbf{C}^2$. This is, for instance, the case for the attraction domain of the Hénon transformation

$$\begin{cases} x_1 = x(1+b) - by + x^2, \\ y_1 = x \end{cases},$$

for $0 < |b| < 1$. This statement has the following generalization.

Theorem 1.3. *Let F be a global automorphism in $\mathbf{C}^2$ with a fixed point p, such that $F'_{(p)}$ has eigenvalues $\{1, \lambda\}$, with $|\lambda| < 1$, and that $F - \mathrm{Id}$ has a multiplicity $k+1$ in p. The attracting domain of p has then k components and each component is isomorphic to $\mathbf{C}^2$.*

Theorem 1.3 applies for instance to Hénon transformations

$$\begin{cases} x_1 = x(1+b) - by + P(x) \\ y_1 = x \end{cases},$$

where P is a polynomial with a zero of order $k+1$ at the origin.

2. Reduced forms of semi-attractive transformations

Proposition 2.1. *Let F be a semi-attractive germ of transformation of $(\mathbf{C}^p, 0)$, with eigenvalues $\{1, \{\lambda_j\}_{2 \le j \le p}\}$, $0 \le |\lambda_j| < 1$, for $2 \le j \le p$. There exist coordinates (x,y), $x \in \mathbf{C}$, $y \in \mathbf{C}^{p-1}$ in which F has the form*

$$(2.1) \qquad \begin{cases} x_1 = u(x,y) = a_1(y)x + a_2(y)x^2 + \dots \\ y_1 = v(x,y) = g(y) + xh(x,y) \end{cases},$$

where $\{a_j(.)\}$, $j = 1, 2, \dots, g(.)$ and $h(.,.)$ are respectively germs of holomorphic functions from $(\mathbf{C}^{p-1}, O)$ to $\mathbf{C}$, from $(\mathbf{C}^{p-1}, O)$ to $\mathbf{C}^{p-1}$ and *from* $(\mathbf{C}^p, O)$ *to* $\mathbf{C}^{p-1}$, *with* $a_1(0) = 1$, $g(0) = 0$, $h(0,0) = 0$, *and* $g'_{(0)}$ *is triangular with eigenvalues* $\{\lambda_j\}_{2\leq j\leq p}$.

Proof: Let $E_1 \oplus E_2$ be the Jordan decomposition of $\mathbf{C}^p$ in characteristic subspaces. Here E_1 is associated to the eigenvalue 1 and E_2 to the set of the eigenvalues $\{\lambda_j\}_{2\leq j\leq p}$. There exists an analytic stable submanifold X attracted by 0 and tangent to E_2 (see [**R**] for a sketch of the proof and a complete bibliography). We then just choose coordinates (x, y), $x \in \mathbf{C}$, $y \in \mathbf{C}^{p-1}$, in such a way that X is $\{x = 0\}$ and that matrix $F'_{(0)}$ is triangular. ■

Proposition 2.2. *Let F be a semi-attractive germ of transformation of $(\mathbf{C}^p, 0)$. For every integer m there exists coordinates (x, y), $x \in \mathbf{C}$, $y \in \mathbf{C}^{p-1}$, in which the transformation has the form*

$$(2.2) \qquad \begin{cases} x_1 = x + a_2x^2 + \dots a_m x^m + a_{m+1}(y)x^{m+1} + \dots \\ y_1 = g(y) + xh(x,y) \end{cases},$$

i.e. like in (2.1), but with $a_1 = 1$ and $a_2, \dots, a_m$ are constants.

Remark. This proposition is in [**U**] in the case of a semi-attractive invertible germ of $(\mathbf{C}^2, 0)$. The following proof is just a generalization of it.

Proof: We start with

$$(2.1) \qquad \begin{cases} x_1 = a_1(y)x + a_2(y)x^2 + \dots \\ y_1 = g(y) + xh(x,y) \end{cases},$$

and we proceed inductively on h.

1) Reduction to $a_1(y) = 1$. We use the coordinate transformation

$$\begin{cases} X = u(y)x \\ Y = y \end{cases} \quad \text{or} \quad \begin{cases} s = X/u(Y) \\ y = Y \end{cases},$$

with $u(.)$ a germ of analytic function from $(\mathbf{C}^{p-1}, 0)$ to $\mathbf{C}$ such that $u(0) = 1$, to be chosen. We want

$$\begin{aligned} X_1 &= u(y_1)x_1 = u(g(y) + xh(x,y))[a_1(y)x + a_2(y)x^2 + \dots] \\ &= u(g(Y) + \dots)[a_1(Y)X/u(Y) + \dots] \\ &= \frac{a_1(Y)u(g(Y))}{u(Y)}X + O(X^2) = X + O(X^2). \end{aligned}$$

So we must choose u such that

$$\begin{aligned}
&u(Y) = a_1(Y)u(g(Y))\\
&u(g(Y)) = a_1(g(Y)u(g^{(2)}(Y))\\
&\dots\\
&u(g^{(n)}(Y)) = a_1(g^{(n)}(Y))u(g^{(n+1)}(Y)).
\end{aligned}$$

This gives for u the unique solution

$$u(Y) = \prod_{n=0}^{\infty} a_1(g^{(n)}(Y)).$$

Since $a_1(0) = 1$ and since there exists α, $0 < \alpha < 1$, such that for $\|y\|$ small enough, one has $\|g(y)\| \leq \alpha\|y\|$, so $\|g^{(n)}(y)\| \leq \alpha^n\|y\|$, the infinity product above is convergent in a neighborhood of 0.

2) Suppose that for $m \geq 2$, with some coordinates (x, y), F takes the form

$$\begin{cases} x_1 = x + a_2x^2 + \dots a_{m-1}x^{m-1} + a_m(y)x^m + \dots \\ y_1 = g(y) + xh(x,y) \end{cases},$$

with the a_j's constant for $1 \leq j \leq m-1$. We then use a coordinate transformation

$$\begin{cases} X = x + v(y)x^m \\ Y = y \end{cases} \quad \text{or} \quad \begin{cases} x = X - v(Y)X^m + \dots \\ y = Y \end{cases},$$

with $v(y)$ a holomorphic function in a neighborhood of 0 in $\mathbf{C}^{p-1}$ such that $v(0) = 0$, v to be chosen. We get

$$\begin{aligned}
X_1 &= x_1 + v(y_1)x_1^m\\
&= x + a_2x^2 + \dots a_{m-1}x^{m-1} + a_m(y)x^m + v(g(y))x^m + O(x^{m+1})\\
&= X - v(Y)X^m + a_2X^2 + \dots a_{m-1}X^{m-1} + a_m(y)X^m\\
&+ v(g(y))X^m + O(X^{m+1}).
\end{aligned}$$

So we need that

$$\begin{aligned}
&v(Y) - v(g(Y)) = a_m(y) - a_m(0),\\
&v(g(Y)) - v(g^2(Y)) = a_m(g(y)) - a_m(0)\\
&\dots\\
&v(g^n(Y)) - v(g^{n+1}(Y)) = a_m(g^n(y)) - a_m(0).
\end{aligned}$$

The unique solution is then

$$v(y) = \sum_{n=0}^{\infty}\{a_m(g^n(y)) - a_m(0)\}.$$

The series converges in a neighborhood of 0 because g is contraction and because $a_m(y) - a_m(0) = 0$ for $y = 0$. ∎

Proposition 2.3. *Let F be a semi-attractive germ of transformation of $(\mathbf{C}^p, 0)$, such that $F - \mathrm{Id}$ is of multiplicity $k+1$ in 0. Then the transformation can be written in some coordinates (x, y), $x \in \mathbf{C}$, $y \in \mathbf{C}^{p-1}$*

$$\left\{ \begin{array}{l} x_1 = x(1 + x^k + Cx^{2k} + a_{2k+1}(y)x^{2k+1} + \dots) \\ y_1 = g(y) + xh(x,y) \end{array} \right. , \tag{2.3}$$

with C a constant.

Proof: Assume then that the transformation is written in the form (2.2) with $m > k+1$. We want to evaluate the multiplicity of $F - \mathrm{Id}$ at 0. Since $(I_{p-1} - g')(0)$ is invertible, we can use lemma 1.2. Using the implicit function theorem, we can solve locally in $y = y(x)$ the equation $y = y_1$. The multiplicity is then given by the order at the origin of

$$x_1 - x = a_2 x^2 + \dots a_m x^m + a_{m+1}(y) x^{m+1} + \dots .$$

Since $F - \mathrm{Id}$ is of multiplicity $k+1$ in 0, we have $a_1 = \dots = a_k = 0$ and $a_{k+1} \neq 0$. If we use proposition (2.2) with $m = 2k+1$, we get

$$x_1 = x(1 + a_{k+1}x^k + \dots + a_{2k+1}x^{2k} + a_{2k+2}(y)x^{2k+1} + \dots).$$

As in the case of one variable, a polynomial transformation in the single variable x leads to the required form. The coefficient of x^{k+1} can be an arbitrary constant not equal to 0, the coefficient of x^{2k+1} is then fixed (see for instance [**B**, theorem 6.5.7, page 122]. ∎

3. Existence of attracting domains and curve of fixed points

We will now prove the theorem 1.1 stated in the introduction. Let F be a semi-attractive germ of transformation of $(\mathbf{C}^p, 0)$, given as before in the form

$$\left\{ \begin{array}{l} x_1 = u(x,y) = x(1 + a_2(y)x + \dots) \\ y_1 = v(x,y) = g(y) + xh(x,y) \end{array} \right. ,$$

with $x \in \mathbf{C}$, $y \in \mathbf{C}^{p-1}$ and g, h like in proposition (2.1). To find the fixed points of F, one can first solve locally in $y = y(x)$ the relation $y = v(x,y)$, thus obtaining an analytic curve $y = \varphi(x)$. There exists a curve of fixed points if and only if the relation

$$x = u(x, \varphi(x))$$

is also satisfied. If not, $F - \mathrm{Id}$ is of finite multiplicity, and the multiplicity is given by the order at 0 of $x - u(x, \varphi(x))$.

The following corollary is an answer to a question of Fatou [**F**, page 131]) who asked if for such F, there could exist a curve of fixed points through 0 for some iterate $F^{(n)}$ but not for F.

Corollary. *Let F be a semi-attractive germ of transformation of $(\mathbf{C}^p, 0)$. There exists a curve of fixed points through 0 for an iterate $F^{(n)}$ of F if and only if there is one for F.*

Proof: It is an immediate consequence of proposition (2.3), for if F is in the form (2.3), we have

$$x_n = x(1 + nx^k + \dots),$$

so $F^{(n)} - \mathrm{Id}$ has the same multiplicity as $F - \mathrm{Id}$. ■

The proof of theorem 1.1 is then a consequence of the following proposition.

Proposition 3.1. *Let F be a semi-attractive germ of transformation of $(\mathbf{C}^p, 0)$, of multiplicity $k+1$ in 0, then there exists an attracting domain with k petals.*

Proof: We can suppose that F is in the form given by proposition (2.3)

$$\begin{cases} x_1 = x(1 + a_k x^k + a_{2k} x^{2k} + a_{2k+1}(y) x^{2k+1} + \dots) \\ y_1 = g(y) + x h(x, y) \end{cases},$$

with $a_k \neq 0$. Changing x in some λx one can assume $a_k = -\frac{1}{k}$. We can then imitate a Fatou's method simplified here by using the reduced form for F which gives easily the Abel-Fatou invariant functions.

Let R and ρ be positive constants to be adjusted later. The half complex-plane P_R and the subset $V_{R,\rho}$ of $\mathbf{C} \times \mathbf{C}^{p-1}$ are defined by

$$\begin{aligned} P_R &= \{X \in \mathbf{C}; \operatorname{Re} X \geq R\}, \\ V_{R,\rho} &= \{(X, y) \in \mathbf{C} \times \mathbf{C}^{p-1};\ X \in P_R, \|y\| < \rho\}. \end{aligned} \tag{3.1}$$

Let D_R and $U_{R,\rho}$ be the images of P_R and $V_{R,\rho}$ by the inversion $z = \frac{1}{X}$, so we have

$$\begin{aligned} D_R &= \left\{ z \in \mathbf{C}; \left| z - \frac{1}{2R} \right| < \frac{1}{2R} \right\}, \\ U_{R,\rho} &= \{(z, y) \in \mathbf{C} \times \mathbf{C}^{p-1};\ z \in D_R, \|y\| < \rho\}. \end{aligned}$$

There are k branches of $z^{1/k}$ in D_R. Let $\{\Delta_{R_j}\}_{0 \leq j \leq k-1}$ be the images of D_R by these determinations. We will show that, for R big enough and ρ small enough, the domains

$$W_{R,\rho,j} = \{(x, y) \in \mathbf{C} \times \mathbf{C}^{p-1};\ x \in \Delta_{Rj}, \|y\| < \rho\}, \quad 0 \leq j \leq k-1, \tag{3.2}$$

are attracting domains.

Raising the relation

$$x_1 = x\left(1 - \frac{1}{k}x^k + a_{2k}x^{2k} + \dots\right)$$

to the power k, we get

$$\begin{aligned} x_1^k &= x^k\left(1 - \frac{1}{k}x^k + a_{2k}x^{2k} + \dots\right)^k \\ &= x^k(1 - x^k + c_{2k}x^{2k} + c_{2k+1}(y)x^{2k+1} + \dots) \\ y_1 &= g(y) + xh(x,y). \end{aligned}$$

We then restrict (x, y) to a $W_{R,\rho,j}$ for fixed R, ρ, j, and we make the transformations

$$(z = x^k, y = 1) \text{ from } W_{R,\rho,j} \text{ to } U_{R,\rho},$$

and

$$\left(X = \frac{1}{z}, y = y\right) \text{ from } U_{R,\rho} \text{ to } V_{R,\rho}.$$

For R big enough and ρ small enough, the transformation F is defined in $V_{R,\rho}$, where we get

$$\begin{aligned} X_1 &= \frac{X}{1 - x^k + c_{2k}x^{2k} + c_{2k+1}(y)x^{2k+1} + \dots} \\ &= X\left(1 + \frac{1}{X} + c\frac{1}{X^2} + \alpha(y)\frac{x}{X^2} + \dots\right). \end{aligned}$$

So F becomes

$$\text{(3.3)} \qquad \begin{cases} X_1 = X + 1 + c\dfrac{1}{X} + O_y\left(\dfrac{1}{|X|^{1+1/k}}\right) \\ y_1 = g(y) + xh(x,y) = g(y) + O_y\left(\dfrac{1}{|X|^{1/k}}\right) \end{cases}.$$

Here the notation $O_y\left(\frac{1}{|X|^\alpha}\right)$ represents a holomorphic function of (X, y) in $V_{R,\rho}$ which is bounded by $\frac{K}{|X|^\alpha}$ for some constant K.

Let K be a constant such that

$$\text{(3.4)} \qquad \begin{cases} |X_1 - X - 1| \le \dfrac{K}{|X|} \le \dfrac{K}{R} \\ \|y_1 - g(y)\| \le \dfrac{K}{|X|^{1/k}} \le \dfrac{K}{R^{1/k}} \end{cases},$$

in $V_{R,\rho}$. Since g is a contraction, there exists b, $0 < b < 1$, such that for ρ small enough we get in $\|y\| \le \rho$

$$\|g(y)\| \le b\|y\|.$$

The condition $\frac{K}{R} < \frac{1}{2}$ implies $\operatorname{Re} X_1 \ge \operatorname{Re} X + \frac{1}{2}$ and the condition

$$\frac{K}{R^{1/k}} < (1-b)\rho$$

implies $\|y_1\| \le \|y\| \le \rho$. So for R big enough, $V_{R,\rho}$ is mapped to itself.

Then to prove that $W_{R,\rho,j}$ is attracted by 0, we have to show that $V_{R,\rho}$ is attracted by $(\infty, 0)$. We see inductively that

$$\operatorname{Re} X_n \ge R + \frac{n}{2}.$$

Let C be a constant big enough to have $C \ge \frac{2K}{1-b}$, and $\rho \le \frac{C}{(R)^{1/k}}$, we prove by induction that, if R is big enough, we have

$$\|y_n\| \le \frac{C}{\left(R + \frac{n}{2}\right)^{1/k}}.$$

Indeed, since

$$\|y_{n+1}\| \le b\|y_n\| + \frac{K}{|X_n|^{1/k}} \le \frac{bC + K}{\left(R + \frac{n}{2}\right)^{1/k}}.$$

The inequality

$$\|y_{n+1}\| \le \frac{C}{\left(R + \frac{n+1}{2}\right)^{1/k}}$$

will be satisfied if we have

$$\left(\frac{R + \frac{1}{2}}{R}\right)^{1/k} \le \frac{C}{bC + K}. \tag{3.5}$$

But from $C \ge \frac{2K}{1-b}$, we see that $\frac{C}{bC+K} \ge \frac{2}{1+b} > 1$. So that (3.5) is true if R is big enough.

We have now k disjoint domains attracted by 0. Each of them is positively invariant by F since $V_{R,\rho}$ is positively invariant, and, since $x_{n+1} \sim x_n$ when $n \to \infty$, we have always the same branch of $x^{1/k}$. Let D be the attracting domain of 0, we want to prove that if $\zeta \in D$, for n

big enough, $\zeta_n = (x_n, y_n)$ is in one of the $W_{R,\rho,j}$'s, or equivalently that (x_n^k, y_n) is in $U_{R,\rho}$, or that $\left(\frac{1}{x_n^k}, y_n\right)$ is in $V_{R,\rho}$. But $y_n \to 0$ and we have

$$\frac{1}{x_1^k} = \frac{1}{x^k} + 1 + cx^k + O_y(x^{k+1}),$$

so $\operatorname{Re} \frac{1}{x_n^k} \to \infty$ when $x_n \to 0$. So ζ belongs to the union of the increasing sequence of open sets

$$D_j = \bigcup_{n=0}^{\infty} F^{-n}(W_{R,\rho,j}).$$

When F is an isomorphism of $\mathbf{C}^p$, each D_j is connected. In general, we can only say that the D_j's are disjoint and contain the $W_{R,\rho,j}$'s. ■

4. Abel-Fatou's functions

Let F be a semi-attractive germ of $(\mathbf{C}^p, 0)$ such that $F - \mathrm{Id}$ has multiplicity $k+1$ in 0, the coordinates and the notations used here are those introduced in the proof of proposition 3.1. In each attractive petal $W_{R,\rho,j}$, we can define an Abel-Fatou function φ, more precisely, a holomorphic function $\varphi : W_{R,\rho,j} \to \mathbf{C}$ verifying the functional equation

$$\varphi(F(p)) = \varphi(p) + 1, \tag{4.1}$$

in the following way.

To construct φ we first observe that in $V_{R,\rho} = \{(X, y) \in \mathbf{C} \times \mathbf{C}^{p-1};\ \operatorname{Re} X \geq R,\ \|y\| < \rho\}$, F is given (see (3.3)) by

$$\begin{cases} X_1 = X + 1 + c\dfrac{1}{X} + O_y\left(\dfrac{1}{|X|^{1+1/k}}\right) \\ y_1 = g(y) + xh(x,y) = g(y) + O_y\left(\dfrac{1}{|X|^{1/k}}\right) \end{cases}.$$

Following Fatou, define

$$U_n = X_n - n - c \operatorname{Log} X_n.$$

We get

$$\|U_{n+1} - U_n\| = \left\| X_{n+1} - X_n - 1 - c \operatorname{Log} \frac{X_{n+1}}{X_n} \right\| \leq \frac{K}{n^{1+1/k}}.$$

So the series $\sum_{n=0}^{\infty}(U_{n+1}-U_n)$ is uniformly convergent and has a holomorphic bounded sum in $V_{R,\rho}$. Hence

$$U_n(X,y) = U_0 + \sum_{k=0}^{n-1}(U_{k+1} - U_k)$$

has a limit

$$u(X,y) = X - c\,\mathrm{Log}\,X + v(x,y)$$

with $v(,)$ a holomorphic bounded function. The functional equation

$$u(X_1,y_1) = u(X,y) + 1$$

is an immediate consequence of

$$u(X_1,y_1) = \lim_{n\to\infty}(X_{n+1} - n - c\,\mathrm{Log}\,X_{n+1}).$$

In each attractive petal $W_{R,\rho,j}$, the function $\varphi : W_{R,\rho,j} \to \mathbf{C}$ is then defined by

$$\varphi(x,y) = u\left(\frac{1}{x^k}, y\right) \tag{4.2}$$

and verifies $\varphi(f(p)) = \varphi(p) + 1$. Remark that in $W_{R,\rho,j}$, the function φ has the asymptotic expansion

$$\begin{aligned}\varphi(x,y) &= \frac{1}{x^k}(1 - cx^k\,\mathrm{Log}\,x^k + x^k v_1(x^k,y)) \\ &= \frac{1}{x^k}(1 + O(|x^k\,\mathrm{Log}\,x^k|)).\end{aligned} \tag{4.3}$$

5. Global isomorphism of $\mathbf{C}^2$

We now consider the case where F is an isomorphism of $\mathbf{C}^2$, with fixed point 0. We assume that $F'_{(0)}$ is semi-attractive (with eigenvalues 1 and λ s.t. $|\lambda| < 1$) and that $F - \mathrm{Id}$ has multiplicity $k+1$ in 0. In this case the attracting domain has k components given with the notations of proposition (3.1) by

$$D_j = \bigcup_{n=0}^{\infty} F^{-n}(W_{R,\rho,j})$$

for $j = 0, 1, \ldots, k-1$. For a fixed j, the Abel-Fatou's function defined in $W_{R,\rho,j}$ can be extended to D_j in the following way: Let $p \in D_j$, for n big enough, we have $F^n(p) \in W_{R,\rho,j}$. So we can define $\varphi(p)$ by $\varphi(p) = \varphi(F^n(p) - n$ and the definition of $\varphi(p)$ does not depend on the choice of the integer n such that $F^n(p) \in W_{R,\rho,j}$.

Proposition 5.1. *The Abel-Fatou's function $\varphi : D_j \to \mathbf{C}$ is surjective.*

Proof: We have to show (if R has been chosen big enough) that, for all $z \in \mathbf{C}$, there exists $n \in \mathbf{N}$ and $p \in W_{R,\rho,j}$ such that $\varphi(p) = z + n$. Using the definition of φ, and the notations in the proof of proposition (3.1), we have to show that there exists $(X, y) \in V_{R,\rho}$ such that

$$u(X,y) = X - c\operatorname{Log} X + u_1(x,y) = z + n. \tag{5.1}$$

But for every fixed y such that $\|y\| < \rho$, and for n big enough, one can solve (5.1) in z in P_R. This is a consequence of Rouché's theorem, for the equation (5.1) can be written

$$X\left(1 - \frac{1}{X}(c\operatorname{Log} X + u_1(X,y))\right) = z + n,$$

and on the boundary $\operatorname{Re} X = R$, we have $\left|\frac{1}{X}(c\operatorname{Log} X + u_1(x,y))\right| < 1$ (if R is big enough). So the equation (5.1) has the same number of solutions as $X = z + n$. ■

We will now give a proof of the theorem 1.2 stated in the introduction.

Theorem 1.2. *Let F be a global automorphism in $\mathbf{C}^2$ with a fixed point p, such that $F'_{(p)}$ has eigenvalues $\{1, \lambda\}$, with $|\lambda| < 1$, and that $F - \mathrm{Id}$ has a multiplicity $k+1$ in p. The attracting domain of p has then k components and each component is isomorphic to $\mathbf{C}^2$.*

Proof: We first choose coordinates (x, y) such that F takes the form

$$\begin{cases} x_1 = x\left(1 - \frac{1}{k}x^k + a_{2k}x^{2k} + a_{2k+1}(y)x^{2k+1} + \dots\right) \\ y_1 = by + b_1yx + \dots + b_kyx^k + b_{k+1}(y)x^{k+1} + \dots \end{cases}. \tag{5.2}$$

In fact, it is proved in [U] that for all integer m, coordinates (x, y) can be chosen so that y_1 is expressed as

$$y_1 = by + b_1yx + \dots + b_myx^m + b_{m+1}(y)x^{m+1} + \dots,$$

with b_j constant for $0 \le j \le m$.

Let us fix $W = W_{R,\rho,j}$ a component of $D = D_j$ in a neighborhood of 0 chosen as section 3 in these coordinates. We want to prove that the open set

$$D = \bigcup_{n=0}^{\infty} F^{-n}(W) \tag{5.3}$$

is isomorphic to $\mathbf{C}^2$.

We first choose new coordinates in W. Let φ be the Abel-Fatou function. We have seen in (4.3) that

$$\varphi(x,y) = \frac{1}{x^k} - c\,\mathrm{Log}\,x^k + v(x,y),$$

with $v(,)$ a holomorphic bounded function in W. We define new coordinates in $W(s,y)$ where

$$s = \frac{1}{x}(1 - cx^k\,\mathrm{Log}\,x^k + x^k v(x,y))^{1/k} = (\varphi(x,y))^{1/k}. \tag{5.4}$$

So

$$s = \frac{1}{x}(1 + O_y(|x^k\,\mathrm{Log}\,x^k|)$$

and

$$x = \frac{1}{s} + O_y\left(\frac{|\,\mathrm{Log}\,s|}{|s|^{k+1}}\right).$$

In these coordinates F takes the form

$$\begin{cases} s_1 = s\left(1 + \dfrac{1}{s^k}\right)^{1/k} \quad (\text{or } s_1^k = s^k + 1) \\ y_1 = by + b_1 y\dfrac{1}{s} + \cdots + b_k y\dfrac{1}{s^k} + O_y\left(\dfrac{|\,\mathrm{Log}\,s|}{|s|^{k+1}}\right) \end{cases}. \tag{5.5}$$

We now follow the same method as Ueda. We will give a sketch of his proof adding the necessary modifications.

Let $D/\langle F\rangle$ be the quotient manifold of D by the transformation group $\{F^n\}_{n\in Z}$ (this group acts discretely on D, because F^n tends to $0 \notin D$ when n goes to $+\infty$).

Let $\pi : D \to D/\langle F\rangle$ be the projection and $E : \mathbf{C} \to \mathbf{C}^*$, the function $E(z) = \exp(2i\pi z)$. Since the Abel-Fatou function satisfies

$$\varphi(F(p)) = \varphi(p) + 1,$$

one can define $\tilde\varphi : D/\langle F\rangle \to C^*$ such that the following diagram is commutative

$$\begin{array}{ccc} D & \xrightarrow{\ \pi\ } & D/\langle F\rangle \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\tilde\varphi} \\ C & \xrightarrow{\ E\ } & C^* \end{array}. \tag{5.6}$$

Define $B = \varphi(W)$. Let us consider the diagram obtained by restriction of the preceeding one to $\varphi^{-1}(B)$, that is

$$\begin{array}{ccc} \varphi^{-1}(B) & \xrightarrow{\pi_B} & D/\langle F\rangle \\ {\scriptstyle\varphi}\Big\downarrow & & {\scriptstyle\tilde\varphi}\Big\downarrow \\ B & \xrightarrow{E_B} & C^* \end{array} \tag{5.7}$$

As follows from proposition (5.1), we know that $\mathbf{C} = \bigcup_{n=0}^{\infty}(B-n)$, so that the restriction

$$E_B : B \to C^*$$

is surjective. For $s \in B$, Ueda defines a holomorphic family of holomorphic functions $\psi_s : \varphi^{-1}(s) \to \mathbf{C}$ on the fibers of φ which gives to $D/\langle F\rangle \to \mathbf{C}^*$ a fiber bundle structure with fibers isomorphic to $\mathbf{C}$ and with transition group the additive group of holomorphic functions on $\mathbf{C}^*$. This fiber bundle structure is necessarily trivial because $H^1(\mathbf{C}^*, O) = 0$. Lifting this structure to $\varphi : D \to \mathbf{C}$ by E, we get a trivial fiber bundle structure on D and this gives an isomorphism from D to $\mathbf{C}^2$.

The definition of the ψ_s's is obtained by integrating on the fibers $s =$ Constant a holomorphic differential 1-form satisfying a functional equation, that we have now to define. ∎

Definition of a holomorphic family of 1-forms on W.

We will define on W a family of holomorphic differential 1-forms $\{\omega_s\}$ on the fiber of the Abel Fatou function depending holomorphically on s, of the form

$$\omega_s(p) = \eta(s,y)\,dy = \eta(p)\,dy,$$

with η a holomorphic function in $B = \varphi(W)$ satisfying the functional equation

$$\eta(F(p))\frac{\partial y_1}{\partial y}(p) = \eta(p). \tag{5.8}$$

We notice that

$$\frac{\partial y_{n+1}}{\partial y}(p) = \frac{\partial y_n}{\partial y}(F(p))\frac{\partial y_1}{\partial y}(p). \tag{5.9}$$

So that if the sequence $\left\{\frac{\partial y_n}{\partial y}\right\}$ were uniformly converging, its limit would be a good candidate for η. But this is not the case since

$$\begin{aligned} \frac{\partial y_n}{\partial y} &= \frac{\partial y_1}{\partial y}\frac{\partial y_2}{\partial y_1}\cdots\frac{\partial y_n}{\partial y_{n-1}} \\ &= b^n\prod_{h=0}^{n-1}\left(1 + \frac{b_1}{b}\frac{1}{s_h} + \cdots + \frac{b_k}{b}\frac{1}{s_h^k} + O_y\left(\frac{|\operatorname{Log} s_h|}{|s_h|^{k+1}}\right)\right). \end{aligned}$$

So we will replace the sequence $\left\{\frac{\partial y_n}{\partial y}\right\}$ by a sequence $\left\{h_n \frac{\partial y_n}{\partial y}\right\}$ where $\{h_n\}$ is a sequence of holomorphic functions such that

$$h_n(F(p)) = h_{n+1}(p), \tag{5.10}$$

and $\left\{h_n \frac{\partial y_n}{\partial y}\right\}$ is uniformly convergent.

We can take $h_n(p)$ to be of the form

$$h_n(p) = g(s)u(s)u(s_1)\dots u(s_n)$$

with g and u depending only on s, holomorphic in W such that

$$g(s)u(s) = g(s_1) \tag{5.11}$$

and

$$u(s) = b^{-1}\left(1 + \frac{b_1}{b}\frac{1}{s} + \cdots + \frac{b_k}{b}\frac{1}{s^k}\right)^{-1} + O\left(\frac{1}{|s|^{k+1}}\right). \tag{5.12}$$

The condition (5.11) implies indeed (5.10) and the condition (5.12) implies that

$$h_n(p)\frac{\partial y_n}{\partial y} = \prod_{h=0}^{n-1}\left(1 + O_y\left(\frac{|\operatorname{Log} s_h|}{|s_h|^{k+1}}\right)\right) = \prod_{h=0}^{n-1}\left(1 + O\left(\frac{\operatorname{Log} h}{h^{1+1/k}}\right)\right).$$

So that η will be defined by a uniformly convergent infinite product.

We have then to show that there exists a function g holomorphic in B such that $u(s) = g(s_1)/g(s)$ satisfies (5.12). We define g as a product of three functions

$$g = g_1 \cdot g_2 \cdot g_3$$

where $g_1(s) = b^{-s^k}$ (for any choice of $\log b$). In fact, this gives

$$\frac{g_1(s_1)}{g_1(s)} = b^{-1}. \tag{5.13}$$

We then choose a function g_2 satisfying

$$\frac{g_2(s_1)}{g_2(s)} = \left(1 + \frac{b_1}{b}\frac{1}{s} + \cdots + \frac{b_{k-1}}{b}\frac{1}{s^{k-1}}\right)^{-1} + O\left(\frac{1}{s^k}\right). \quad \blacksquare \tag{5.14}$$

The existence of g_2 is proved by the following lemma

Lemma 5.3. *For any* $(c_1, c_2, \dots, c_{k-1}) \in \mathbf{C}^{k-1}$*, there exists a polynomial in* $\frac{1}{s}$

$$P\left(\frac{1}{s}\right) = \frac{a_1}{s} + \frac{a_2}{s^2} + \cdots + \frac{a_{k-1}}{s^{k-1}}$$

such that $h(s) = \exp\left(s^k P\left(\frac{1}{s}\right)\right)$ *satisfies*

$$\frac{h(s_1)}{h(s)} = 1 + \frac{c_1}{s} + \frac{c_2}{s^2} + \cdots + \frac{c_{k-1}}{s^{k-1}} + O\left(\frac{1}{s^k}\right).$$

Proof: From

$$s_1^k = s^k + 1,$$

we get

$$\frac{1}{s_1} = \frac{1}{s} - \frac{1}{k}\frac{1}{s^{k+1}} + O\left(\frac{1}{s^{2k+1}}\right)$$

so for a polynomial P

$$P\left(\frac{1}{s_1}\right) = P\left(\frac{1}{s}\right) - \frac{1}{k}\frac{1}{s^{k+1}}P'\left(\frac{1}{s}\right) + O\left(\frac{1}{s^{2k+1}}\right).$$

From that we deduce

$$\begin{aligned} s_1^k P\left(\frac{1}{s_1}\right) = s^k P\left(\frac{1}{s}\right) &= s^k\left(P\left(\frac{1}{s_1}\right) - P\left(\frac{1}{s}\right)\right) + P\left(\frac{1}{s_1}\right) \\ &= P\left(\frac{1}{s}\right) - \frac{1}{ks}P'\left(\frac{1}{s}\right) + O\left(\frac{1}{s^{k+1}}\right) \\ &= \sum_{j=1}^{k-1}\left(1 - \frac{j}{k}\right)a_j\frac{1}{s^j} + O\left(\frac{1}{s^{k+1}}\right). \end{aligned}$$

We can then compute the a_j's by identifying the expression above with the polynomial part of degree $\le k-1$ in the Taylor expansion of

$$-\operatorname{Log}\left(1 + \frac{c_1}{s} + \frac{c_2}{s^2} + \cdots + \frac{c_{k-1}}{s^{k-1}}\right)$$

at infinity. ■

Choice of g_3**.** It is a consequence of the choice of g_1 and g_2 that it exists c such that

$$\frac{g_1(s_1)}{g_1(s)}\frac{g_2(s_1)}{g_2(s)}b\left(1 + \frac{b_1}{b}\frac{1}{s} + \cdots + \frac{b_k}{b}\frac{1}{s^k}\right) = 1 + \frac{c}{s^k} + O\left(\frac{1}{s^{k+1}}\right).$$

So we need a function g_3 such that

$$\frac{g_3(s_1)}{g_3(s)} = 1 - \frac{c}{s^k} + O\left(\frac{1}{s^{k+1}}\right).$$

We can take g_3 defined by $g_3(s) = s^{-kc} = \exp(-c \operatorname{Log} s^k)$ (with the branch of logarithm which is real on positive numbers). Indeed, we get

$$\frac{g_3(s_1)}{g_3(s)} = \left(1 + \frac{1}{s^k}\right)^{-c} = 1 - \frac{c}{s^k} + O\left(\frac{1}{s^{k+1}}\right).$$

Once ω_s is defined in W, we just follow the construction of Ueda to define the ψ_s. We recall this construction for reader's convenience.

Construction of $\psi_s : \varphi^{-1}(s) \to \mathbf{C}$.

We have seen that if $p \in D$, for n big enough, $s = F^n(p)$ is in B and the fiber $D \cap \varphi^{-1}(s)$ contains a disk $\Delta_\rho = \{y \in \mathbf{C};\ \|y\| < \rho\}$. So we can fist suppose that W is a set of the form $W = B \times \Delta_\rho$ and that we still have $D = \bigcup_{n=0}^{\infty} F^{-n}(W)$. We first define $\psi_s(p)$ when p is in W and $\varphi(p) = s$ by integrating the form $\eta(p)\, dy$ on the fiber $s =$ constant, along a path joining $p_0 = (s, 0)$ to $p = (s, y(p))$ in $\{s\} \times \Delta_\rho$

$$\psi_s(p) = \int_0^{y(p)} \eta(s, y)\, dy.$$

From the functional equation (5.8) verified by η we get

$$\begin{aligned}
\psi_{s_1}(F(p)) &= \int_0^{y(F(p))} \eta(s_1, y_1))\, dy_1 \\
&= \int_{y(F(p_0))}^{y(F(p))} \eta(s_1, y_1)\, dy_1 + \int_0^{y(F(p_0))} \eta(s_1, y)\, dy \\
&= \psi_s(p) + h(s)
\end{aligned}$$

where $h(s) = \int_0^{y(F(p_0))} \eta(s_1, y)\, dy$ is a holomorphic function of s in B.

We get in this way a holomorphic function $\psi : W \to C$ defined by $\psi(p) = \psi_s(p)$ for $s = (\varphi(p))^{1/k}$ such that

$$\psi(p) = \psi_s(p) = \psi_{s_1}(f(p)) - h(s). \tag{5.15}$$

The relation (5.15) allows the extension of the definition of ψ to $\varphi^{-1}(B)$ in the following way: let $p \in \varphi^{-1}(B)$. For n big enough, we have $F^n(p) \in W$ and we define $\psi(p)$ by the formula

$$\psi(p) = \psi(F^n(p)) - (h(s) + h(s_1) + \cdots + h(s_n)) \tag{5.16}$$

where $s_j = (\varphi(f^j(p)))^{1/k}$ for $j = 0, 1, \ldots, n$. It is clear that the definition doesn't depend on n and that the function ψ is holomorphic in $\varphi^{-1}(B)$.

Proposition 5.4 (Ueda). *For all $s \in B$, $\psi_s : \varphi^{-1}(s) \to \mathbf{C}$ is an isomorphism.*

Proof: Let us prove first the injectivity of ψ_s: Let p and $p' \in \varphi^{-1}(s)$ such that $\psi_s(p) = \psi_s(p')$. According to property (5.16), one can assume that p and p' are in W and that we have with s big and $y(p)$ and $y(p')$ small

$$\int_0^{y(p)} \eta(s,y)\,dy = \int_0^{y(p')} \eta(s,y)\,dy.$$

The results is just a consequence of the fact that the function $\int_0^{y(p)} \eta(s,y)\,dy$ is holomorphic in $y(p)$ and has a derivative $\eta(s,0) \neq 0$.

To prove the surjectivity, we remark that $\eta(s,0)_{\sim} b^{-s^k} s^{-kc}$ when $s \to \infty$ and as

$$h(s) = \int_0^{y(F(p_0))} \eta(s_1,y)\,dy$$

with $y(F(p_0)) = y_1(s,0) = \frac{b_{k+1}(0)}{s^{k+1}} + \dots$ (see the form of F in coordinates (s,y)). We have

$$h(s)_{\sim}\eta(s,0)\frac{b_{k+1}(0)}{s^{k+1}}.$$

We deduce from this that the image $\psi_s(W)$ contains a disk centered at 0 with radius $\varepsilon\rho|b^{-s^k}s^{-kc}|$ for some constant $\varepsilon > 0$. The relation (5.16) then implies that the image of $F^{-n}(W) \cap \varphi^{-1}(s)$ contains a disk centered in $\zeta_n = h(s) + h(s_1) + \cdots + h(s_n)$ and with radius $R_n = \varepsilon\rho|b^{-s_n{}^k} s_{n^{-kc}}|$. An elementary calculation proves that R_n et $|\zeta_n|$ tend to $+\infty$ while $\frac{\zeta_n}{R_n} \to 0$, so the union of the disks $D(\zeta_n, R_n)$ contained in the image of $\varphi^{-1}(s)$ is equal to $\mathbf{C}$.

It is then easy using the ψ_s's to define a structure of locally trivial fiber bundle on $D/\langle F\rangle \to \mathbf{C}^*$, with fibers isomorphic to $\mathbf{C}$ and with structure group, the group of translations by holomorphic functions in s. This ends the proof of the existence of an isomorphism from D to $\mathbf{C}^2$. ∎

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