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# MULTIPARAMETER POINTWISE ERGODIC THEOREMS FOR MARKOV OPERATORS ON $L_\infty$

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*Abstract*

Let $P_1, \dots, P_d$ be commuting Markov operators on $L_\infty(X, \mathcal{F}, \mu)$, where $(X, \mathcal{F}, \mu)$ is a probability measure space. Assuming that each $P_i$ is either conservative or invertible, we prove that for every $f$ in $L_p(X, \mathcal{F}, \mu)$ with $1 \leq p < \infty$ the averages

$$A_n f = (n+1)^{-d} \sum_{0 \leq n_i \leq n} P_1^{n_1} P_2^{n_2} \dots P_d^{n_d} f \quad (n \geq 0)$$

converge almost everywhere if and only if there exists an invariant and equivalent finite measure $\lambda$ for which the Radon-Nikodym derivative $v = d\lambda/d\mu$ is in the dual space $L_{p'}(X, \mathcal{F}, \mu)$. Next we study the case in which there exists $p_1$, with $1 \leq p_1 \leq \infty$, such that for every $f$ in $L_p(X, \mathcal{F}, \mu)$ the limit function belongs to $L_{p_1}(X, \mathcal{F}, \mu)$. We give necessary and sufficient conditions for this problem.

## 1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a probability measure space and let $P_i$ $(i = 1, 2, \dots, d)$ be commuting Markov operators defined on $L_\infty(X, \mathcal{F}, \mu)$. In this paper we assume that each $P_i$ is conservative or invertible, and prove that the averages

$$A_n f = (n+1)^{-d} \sum_{0 \leq n_i \leq n} P_1^{n_1} P_2^{n_2} \dots P_d^{n_d} f \quad (n \geq 0)$$

converge almost everywhere for every $f$ in $L_p(X, \mathcal{F}, \mu)$, $1 \leq p < \infty$, if and only if there exists an invariant and equivalent finite measure $\lambda$ for which the Radon-Nikodym derivative $v = d\lambda/d\mu$ satisfies $v \in L_{p'}(\mu)$ with $1/p + 1/p' = 1$. This is a new result, even if $d = 1$. Next we consider the question whether there exists a constant $p_1$, $1 \le p_1 \le \infty$, for which the limit functions belong to $L_{p_1}(X, \mathcal{F}, \mu)$. We give necessary and sufficient conditions for this problem.

Let $(X, \mathcal{F}, \mu)$ be a probability measure space. By a *Markov operator $P$ defined on* $L_\infty(\mu) = L_\infty(X, \mathcal{F}, \mu)$ we mean a linear operator from $L_\infty(\mu)$ into itself such that

(i) $P$ is positive: $Pf \ge 0$ whenever $f \in L_\infty^+(\mu)$,
(ii) $P$ is a contraction: $\|P\|_\infty \le 1$,
(iii) $\lim_n Pf_n = 0$ a.e. on $X$ whenever $f_n \in L_\infty^+(\mu)$ and $f_n \downarrow 0$ a.e. on $X$.

By virtue of (iii) and the Radon-Nikodym theorem we may define an operator on $L_1(\mu) = L_1(X, \mathcal{F}, \mu)$ by the relation

$$\int_B uP \, d\mu = \int_X u(P\mathbf{1}_B) \, d\mu \quad (u \in L_1(\mu) \text{ and } B \in \mathcal{F}),$$

$\mathbf{1}_B$ being the indicator function of $B$. Here we use the same symbol $P$ for the $L_1(\mu)$ operator, but we write it to the right of its variable. $P$ is called *invertible* if $P\mathbf{1} = 1$ and $P$ is a one-to-one onto mapping of $L_\infty(\mu)$, and *conservative* if $f \in L_\infty^+(\mu)$ and $Pf \le f$ imply $Pf = f$. It is known (see e.g. [**4**]) that $P$ is conservative if and only if $\sum_{i-0}^{\infty} uP^i = \infty$ a.e. on $X$ whenever $u \in L_1^+(\mu)$ and $u > 0$ a.e. on $X$. Since $P$ is positive, we may extend the domain of $P$ to the space $M^+(\mu) = M^+(X, \mathcal{F}, \mu)$ of all nonnegative extended real valued measurable functions on $X$, by the relations

$$Pf = \lim_n Pf_n \text{ a.e. where } f_n \in L_\infty^+(\mu) \text{ and } f_n \uparrow f \text{ on } X$$

and

$$uP = \lim_n u_n P \text{ a.e. where } u_n \in L_1^+(\mu) \text{ and } u_n \uparrow u \text{ on } X.$$

It is easily seen that by this process $Pf$ and $uP$ are uniquely determined a.e. on $X$. We then have

$$\langle uP, f \rangle = \int_X (uP) f \, d\mu = \int_X u(Pf) \, d\mu = \langle u, Pf \rangle.$$

In this paper we consider several commuting Markov operators $P_i$ ($i = 1, 2, \ldots, d$) defined on $L_\infty(\mu)$. Throughout the paper we will assume that *each $P_i$ is conservative or invertible.* Associated with these operators we define the averages

$$A_n f = (n+1)^{-d} \sum_{0 \le n_i \le n} P_1^{n_1} P_2^{n_2} \dots P_d^{n_d} f \quad (n \ge 0)$$

and the maximal operator

$$Mf = \sup_{n \ge 0} A_n |f|.$$

Also we define the $\sigma$-field of all invariant subsets of $X$:

$$\mathcal{I} = \{B \in \mathcal{F} : P_i \mathbf{1}_B = \mathbf{1}_B \text{ for all } i = 1, 2, \dots, d\}.$$

We call the system $\{P_1, P_2, \dots, P_d\}$ *ergodic* if $\mathcal{I}$ is the trivial $\sigma$-field.

It is known that if $\mu$ is invariant under the $P_i$, i.e., $\mathbf{1}P_i = 1$ for all $i = 1, 2, \dots, d$ then the sequence $\{A_n f\}$ converges a.e. for every $f$ in $L_p^+(\mu)$, $1 \le p \le \infty$. But, if $\mu$ is not assumed to be invariant, we cannot expect the almost everywhere convergence of $\{A_n f\}$ for every $f$ in $L_p^+(\mu)$, $1 \le p \le \infty$. Therefore the author thinks that it would be of interest to characterize those probability measures $\mu$ for which the sequence $\{A_n f\}$ converges a.e. for every $f$ in $L_p^+(\mu)$, $1 \le p \le \infty$. As is easily seen, such a characterization for $p = \infty$ is the existence of an invariant and equivalent finite measure. Thus we will concern ourselves with the case $1 \le p < \infty$ below. It is interesting to note here that this problem for $d$-parameter *groups* of null preserving point transformations was recently examined by Martín-Reyes [**5**]. Hence our results may be regarded as generalizations (and improvements) of those due to Martín-Reyes. We will give a new characterization which has a connection with the invariant measure problem. That is, we will prove in Section 3 that the sequence $\{A_n f\}$ converges a.e. for every $f$ in $L_p^+(\mu)$ if and only if there exists an invariant and equivalent finite measure $\lambda$ for which the Radon-Nikodym derivative $v = d\lambda/d\mu$ satisfies

$$v \in L_{p'}^+(X, \mathcal{F}, \mu) \text{ with } 1/p + 1/p' = 1.$$

Next, let us suppose that $f^*(x) = \lim_n A_n f(x)$ exists a.e. for all $f$ in $L_p^+(\mu)$, $1 \le p \le \infty$. Do the functions $f^*$ belong to $L_p^+(\mu)$? As is easily seen, this is not true in general, unless $p = \infty$. So the following question arises naturally. Does there exist a constant $p_1$, $1 \le p_1 \le \infty$, such that $f^* \in L_{p_1}^+(\mu)$ for all $f \in L_p^+(\mu)$? We will prove in Section 3 that the limit $f^* = \lim_n A_n f$ exists a.e. and $f^* \in L_{p_1}^+(\mu)$ for all $f$ in $L_p^+(\mu)$ if and only if for any $u \in L_{p_1'}(\mu)$, $1/p_1 + 1/p_1' = 1$, the limit $u_0^* = \lim_n uA_n$ exists a.e and in the norm topology of $L_1(\mu)$ and further we have $u_0^* \in L_{p'}(\mu)$, $1/p + 1/p' = 1$.

In Section 2 we study the special case where $\mu$ is invariant under the $P_i$, and consider the measure $\lambda = V\,d\mu$, where $V$ is a positive measurable function on $X$. Regarding the $P_i$ as commuting Markov operators defined on $L_\infty(X, \mathcal{F}, V\,d\mu)$, we obtain some preliminary results which may be of independent interest by themselves.

In what follows two functions $f$ and $g$ are not distinguished provided that $f = g$ a.e. on $X$, and if $1 \le p \le \infty$ then $p'$ will be its conjugate exponent, i.e. $1/p + 1/p' = 1$.

## 2. The invariant measure case

In this section we assume that the probability measure $\mu$ is invariant under the $P_i$, or equivalently that $\mathbf{1}P_i = 1 = P_i\mathbf{1}$ for all $i = 1, 2, \ldots, d$, and consider the measure $\lambda = V\,d\mu$, where $V$ is a positive measurable function on $X$. It may happen that $\lambda X = \int_X V\,d\mu = \infty$.

**Theorem 2.1.** *Let $P_i$ $(1 \le i \le d)$ be commuting Markov operators on $L_\infty(X, \mathcal{F}, \mu)$, where $(X, \mathcal{F}, \mu)$ is a probability measure space. Assume that $\mu$ is invariant under the $P_i$. Let $V$ be a positive measurable function on $X$, and let $1 \le p < \infty$. Then the following are equivalent.*

(a) *For every $f$ in $L_p^+(V\,d\mu)$, $\lim_n A_n f$ exists and is finite a.e. on $X$.*

(b) *For every $f$ in $L_p^+(V\,d\mu)$, $Mf < \infty$ a.e. on $X$.*

(c) *There exists a positive measurable function $U$ on $X$ such that*

$$\int_{\{Mf>t\}} U\,d\mu \le t^{-p} \int_X f^p V\,d\mu \quad (t > 0,\ f \in L_p^+(V\,d\mu)).$$

(d) *There exists a positive measurable function $U$ on $X$ such that*

$$\liminf_n \int_{\{A_n f>t\}} U\,d\mu \le t^{-p} \int_X f^p V\,d\mu \quad (t > 0,\ f \in L_p^+(V\,d\mu)).$$

(e) $\operatorname{ess\,sup}\{W \in M^+(X, \mathcal{I}, \mu) : W \le V\} > 0$ *a.e. on $X$ (if $p = 1$); $E\{V^{1-p'}|\mathcal{I}\} < \infty$ a.e. on $X$ (if $1 < p < \infty$).*

*Proof:* It is clear that (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d).

(b) $\Rightarrow$ (c). (b) implies that for each $n \ge 0$ and all $f \in L_p^+(V\,d\mu)$, $|A_n f| < \infty$ a.e. on $X$. Thus the operator $A_n$ can be considered to be a continuous mapping from $L_p(V\,d\mu)$ to $L_0(\mu)$, where $L_0(\mu)$ denotes the space of all finite valued measurable functions on $X$, equipped with the topology of the convergence in measure. By this and Banach's principle (cf. e.g. [**3**, p. 2]) we see that the sublinear operator $f \to M|f|$ is continuous from $L_p(V\,d\mu)$ to $L_0(\mu)$. Hence (c) follows from Nikishin's theorem (cf. [**2**, p. 536]).

(d) $\Rightarrow$ (a). Let $f \in L_p^+(V\,d\mu)$, and choose $f_N \in L_1^+(\mu)$, $N = 1, 2, \ldots$, so that $f_N \uparrow f$ on $X$. Since $\mu$ is invariant under the $P_i$, it follows that $\|P_i\|_r = 1$ for all $1 \le r \le \infty$, and hence the classical pointwise ergodic theorem for $d$-parameter semigroups of Dunford-Schwartz operators can be applied to infer that

$$\lim_n A_n f_N = E\{f_N|\mathcal{I}\} \text{ a.e. on } X.$$

Using this we see that the pointwise limit $f^*(x) = \lim_n A_n f(x)$ exists a.e. on $X$ (but may equal to infinity on some subset of $X$) and that

$$f^* = E\{f|\mathcal{I}\} \text{ a.e. on } X.$$

To prove that $f^* < \infty$ a.e. on $X$, let us write $B = \{x : f^*(x) = \infty\}$. Then, since $B \subset \liminf_n \{A_n f > t\}$ for any $t > 0$, we have by Fatou's lemma and (d),

$$\int_B U\,d\mu \le \liminf_n \int_{\{A_n f > t\}} U\,d\mu \le t^{-p} \int_X f^p V\,d\mu.$$

Letting $t \uparrow \infty$, it follows that $\int_B U\,d\mu = 0$, and consequently $\mu B = 0$.

(a) $\Rightarrow$ (e). As in [**8**], we use the ergodic decomposition technique. We first note that for the proof it may be assumed without loss of generality that $(X, \mathcal{F}, \mu)$ is a Lebesgue measure space in the sense of Rokhlin [**6**]. Then using Rokhlin's theory we can find a countable family $\{E_i\}$ of sets in $\mathcal{I}$ such that if $\xi$ denotes the decomposition of $X$ induced by $\{E_i\}$, i.e., $C \in \xi$ has the form

$$C = \bigcap_i E_i(\varepsilon_i)$$

where $\varepsilon_i = \pm 1$, $E_i(1) = E_i$ and $E_i(-1) = X \backslash E_i$, then:

(i) The factor space $(X/\xi,\, \mathcal{F}_\xi, \mu_\xi)$ of $(X, \mathcal{F}, \mu)$ with respect to $\xi$ is a Lebesgue measure space.

(ii) To a.e. $C \in X/\xi$ with respect to $\mu_\xi$ there corresponds a Lebesgue measure $\mu_C$ on $C$ such that if $B \in \mathcal{F}$ then $B \cap C$ is measurable with respect to $\mu_C$ for a.e. $C \in X/\xi$, and the function $h(C) = \mu_C(B \cap C)$ is measurable with respect to $\mu_\xi$ and satisfies, for all $Z \in \mathcal{F}$ of the form $Z = \iota^{-1}(Z/\xi)$, where $\iota : X \to X/\xi$ denotes the canonical mapping,

$$\mu(B \cap Z) = \int_{Z/\xi} h(C) d\mu_\xi(C) = \int_{Z/\xi} \mu_C(B \cap C) d\mu_\xi(C).$$

(iii) To a.e. $C \in X/\xi$ there correspond commuting Markov operators $P(C)_1, P(C)_2, \ldots, P(C)_d$ defined on $L_\infty(C, \mu_C)$ such that the system $\{P(C)_1, P(C)_2, \ldots, P(C)_d\}$ is ergodic, and also such that if $u \in L_1(X, \mathcal{F}, \mu)$ and $f \in L_\infty(X, \mathcal{F}, \mu)$ then, for a.e. $C \in X/\xi$,

$$u_C P(C)_i = (uP_i)_C \text{ and } P(C)_i f_C = (P_i f)_C \text{ for all } i = 1, 2, \ldots, d,$$

where $u_C$, $(uP_i)_C$, $f_C$ and $(P_i f)_C$ denote, respectively, the restriction functions of $u$, $uP_i$, $f$ and $P_i f$ to the set $C$.

To prove the implication (a) $\Rightarrow$ (e) for $p = 1$ we define

$$\tilde{V} = \operatorname{ess\,sup}\{W \in M^+(X, \mathcal{I}, \mu) : W \leq V\},$$

and assuming that $\mu\{\tilde{V} = 0\} > 0$, we derive a contradiction as follows. Since $\{\tilde{V} = 0\} \in \mathcal{I}$, it may be supposed without loss of generality that $\{\tilde{V} = 0\} = X$. Then we use the ergodic decomposition technique. Define the funciton $h_n$ on $X/\xi$ by

$$h_n(C) = \mu_C(\{V < 1/n\} \cap C) \quad (n \geq 1).$$

Since $h_n(C) > 0$ for a.e. $C \in X/\xi$ because $\tilde{V} = 0$ on $X$, if $f_n$ denotes the function on $X$ defined by

$$f_n(x) = n^{-1}(h_n(C))^{-1}\mathbf{1}_{\{V<1/n\}}(x) \quad (x \in C \in X/\xi),$$

then we have

$$\int_C f_n \, d\mu_C = 1/n \quad (C \in X/\xi)$$

and

$$\begin{aligned}\int_X f_n V \, d\mu &= \int_{X/\xi} \left( \int_C f_n V \, d\mu C \right) d\mu_\xi(C) \\ &\leq \int_{X/\xi} (1/n^2) \, d\mu_\xi(C) = 1/n^2.\end{aligned}$$

Therefore the function $f = \sum_{n=1}^{\infty} f_n$ satisfies $f \in L_1^+(V\,d\mu)$. But, since $f_C \notin L_1^+(\mu_C)$ for a.e. $C \in X/\xi$, it follows from (iii) and the classical pointwise ergodic theorem for $d$-parameter semigroups of Dunford-Schwarz operators that for a.e. $C \in X/\xi$,

$$f^*(x) = \lim_n A_n f(x) = \infty \text{ a.e. on } C$$

with respect to the measure $\mu_C$. It follows that $f^*(x) = \infty$ a.e. on $X$ with respect to the measure $\mu$. This contradicts (a).

Next, let us consider the case $1 < p < \infty$, and suppose that the set

$$B = \{x : E\{V^{1-p'}|\mathcal{I}\}(x) = \infty\}$$

is not a null set. Since $B \in \mathcal{I}$, we then suppose without loss of generality that $B = X$, and from this we derive a contradiction as follows. First we note that if $E\{V^{1-p'}|\mathcal{I}\} = \infty$ a.e. on $X$ then

$$\int_C V^{1-p'}\,d\mu_C = \infty \text{ for a.e. } C \in X/\xi.$$

Using this and doing as in the proof of the implication (a) $\Rightarrow$ (b) of Theorem 1 in [**8**] (see especially p. 75 in [**8**]) it is possible to construct a function $f$ in $L_p^+(V\,d\mu)$ so that for a.e. $C \in F/\xi$, where $F$ is a set in $\mathcal{I}$ and satisfies $\mu F > 0$, we have $\int_C f\,d\mu_C = \infty$. Then it follows from (iii) that for a.e. $C \in F/\xi$,

$$f^*(x) = \lim_n A_n f(x) = \infty \text{ a.e. on } C$$

with respect to the measure $\mu_C$. Hence $f^*(x) = \infty$ a.e. on $F$ with respect to the measure $\mu$. This contradicts (a), because $f \in L_p^+(V\,d\mu)$.

(e) $\Rightarrow$ (a). (e) implies the existence of a sequence $\{X_N\}$ of sets in $\mathcal{I}$, with $X_N \uparrow X$, such that if $V_N$ denotes the restriction function of $V$ to $X_N$ then

$$V_N^{-1} \in L_{p'}^+(X_N, V\,d\mu).$$

Here, since $X_N \in \mathcal{I}$, for the proof of (a) it may be supposed without loss of generality that $X_N = X$. Then for any $f \in L_p^+(V\,d\mu)$ the Hölder inequality yields

$$\int_X f\,d\mu = \int_X f\frac{1}{V}V\,d\mu \le \begin{cases} \left(\int_X fV\,d\mu\right)\left\|\frac{1}{V}\right\|_{L_\infty(X,V\,d\mu)} < \infty & (p=1) \\ \left(\int_X f^pV\,d\mu\right)^{1/p}\left(\int_X \left(\frac{1}{V}\right)^{p'}V\,d\mu\right)^{1/p'} < \infty & (1 < p < \infty), \end{cases}$$

so that $f \in L_1^+(\mu)$, and hence (a) follows from the classical pointwise ergodic theorem for $d$-parameter semigroups of Dunford-Schwartz operators. The proof is complete. ■

The following corollary is immediate from the equivalence of (a) and (e) in Theorem 2.1.

**Corollary 2.2.** *Let $P_i$ $(i = 1, 2, \dots, d)$ be as in Theorem 2.1. Suppose in addition that the system $\{P_1, P_2, \dots, P_d\}$ is ergodic. Then $\lim_n A_n f$ exists and is finite a.e. on $X$ for every $f$ in $L_p^+(V\,d\mu)$, $1 \le p < \infty$, if and only if $V^{-1} \in L_{p'}(X, \mathcal{F}, V\,d\mu)$.*

## 3. The general case

**Theorem 3.1.** *Let $(X, \mathcal{F}, \mu)$ be a probability measure space and let $P_i$ $(i = 1, 2, \dots, d)$ be commuting Markov operators on $L_\infty(X, \mathcal{F}, \mu)$. Assume that each $P_i$ is conservative or invertible. If $1 \le p < \infty$, then the following are equivalent.*

(a) *For every $f$ in $L_p^+(\mu)$, $\lim_n A_n f$ exists and is finite a.e. on $X$.*

(b) *For every $f$ in $L_p^+(\mu)$, $Mf < \infty$ a.e. on $X$.*

(c) *There exists a positive measurable function $U$ on $X$ such that*

$$\int_{\{Mf>t\}} U\,d\mu \le t^{-p} \int_X f^p\,d\mu \quad (t > 0, f \in L_p^+(\mu)).$$

(d) *There exist a positive measurable function $U$ on $X$, a positive constant $r$ and a subsequence $\{n(k)\}$ of the sequence $\{n\}$ such that*

$$\int_{\{A_{n(k)}f>t\}} U\,d\mu \le t^{-r} \left( \int_X f^p\,d\mu \right)^{r/p} \quad (t > 0,\ f \in L_p^+(\mu)).$$

(e) *For every $u$ in $L_1(\mu)$, the sequence $\{uA_n\}$ converges in the norm topology of $L_1(\mu)$ and also a.e. on $X$; further to each $v \in L_1^+(\mu)$ with $vP_i = v$ for all $i = 1, 2, \dots, d$ there corresponds a sequence $\{X_N\}$ of sets in $\mathcal{I}$ such that $X_N \uparrow X$ and the restriction function $v_N$ of $v$ to $X_N$ is in $L_{p'}(X_N, \mu)$ for each $N \ge 1$.*

(f) *There exists $v \in L_{p'}^+(\mu)$ with $v > 0$ a.e. on $X$ and $vP_i = v$ for all $i = 1, 2, \dots, d$.*

In order to prove Theorem 3.1 we begin by proving the following

**Lemma 3.2.** *Let $P_i$ $(i = 1, 2, \dots, d)$ be commuting Markov operators on $L_\infty(X, \mathcal{F}, \mu)$, where $(X, \mathcal{F}, \mu)$ is a probability measure space. Assume that each $P_i$ is conservative or invertible. Then the sequence $\{A_n f\}$ converges a.e. on $X$ for every $f$ in $L_\infty(X, \mathcal{F}, \mu)$ if and only if there exists a function $v$ in $L_1^+(X, \mathcal{F}, \mu)$ such that $v > 0$ a.e. on $X$ and $vP_i = v$ for all $i = 1, 2, \dots, d$.*

*Proof:* Suppose the first assertion of the lemma holds. Then for any $u \in L_1(\mu)$ and $f \in L_\infty(\mu)$ the sequence

$$\langle uA_n, f\rangle = \int_X (uA_n) f \, d\mu = \int_X u(A_n f) \, d\mu = \langle u, A_n f\rangle$$

converges to a finite limit as $n \to \infty$. It follows from the Vitali-Hahn-Saks theorem that $\{uA_n\}$ converges weakly in $L_1(\mu)$. Hence by a mean ergodic theorem (c.f. e.g. [**4**, Theorem 2.1.5]), $\{uA_n\}$ converges strongly in $L_1(\mu)$. Let $v \in L_1^+(\mu)$ be the limit function of the sequence $\{\mathbf{1}A_n\}$ in $L_1(\mu)$. Since $vP_i = v$ for each $i$, it follows that the set $B = \{x : v(x) = 0\}$ satisfies $P_i \mathbf{1}_B \le \mathbf{1}_B$ for each $i$. Here if $P_i$ is conservative, then we have $P_i \mathbf{1}_B = \mathbf{1}_B$. On the other hand, if $P_i$ is invertible then, since $vP_i = v = vP_i^{-1}$, we have $P_i \mathbf{1}_B = \mathbf{1}_B$, too. Consequently $B \in \mathcal{I}$, and thus

$$\mu B = \langle 1, A_n \mathbf{1}_B\rangle = \langle \mathbf{1}A_n, \mathbf{1}_B\rangle \to \int_B v \, d\mu = 0.$$

Conversely, if the second assertion holds, then the $P_i$ may be regarded as commuting Markov operators defined on $L_\infty(X, \mathcal{F}, v\,d\mu)$ such that

$$\|P_i\|_{L_1(X, \mathcal{F}, v\,d\mu)} = 1 \quad (1 \le i \le d).$$

Thus we may apply the classical pointwise ergodic theorem for $d$-parameter semigroups of Dunford-Schwartz operators to infer that the first assertion of the lemma holds. ■

*Proof of Theorem 3.1:* (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) are immediate. The proof of (b) $\Rightarrow$ (c) is the same as that of the corresponding part of Theorem 2.1. (f) $\Rightarrow$ (a) follows from the classical $d$-parameter pointwise ergodic theorem, since $L_p(\mu) \subset L_1(v\,d\mu)$.

(d) $\Rightarrow$ (a). We may suppose that $0 < U \le 1$ on $X$. Given an $\varepsilon > 0$, choose $\delta > 0$ so that $\mu\{U < \delta\} < \varepsilon$. Then, since $A_n \mathbf{1}_B \le \mathbf{1}$ on $X$ for any $B \in \mathcal{F}$, (d) implies

$$\begin{aligned}\int_X (A_{n(k)} \mathbf{1}_B) \, d\mu &\le \delta^{-1} \int (A_{n(k)} \mathbf{1}_B) U \, d\mu + \mu\{U < \delta\} \\ &\le \delta^{-1} \left( \int_{\{A_{n(k)} \mathbf{1}_B > t\}} U \, d\mu + t \int_X U \, d\mu \right) + \varepsilon \\ &\le \delta^{-1} t^{-r} (\mu B)^{r/p} + \delta^{-1} t + \varepsilon \quad (t > 0).\end{aligned}$$

Letting $t \downarrow 0$ and then $\mu B \downarrow 0$, we see that

$$\lim_{\mu B \to 0} \sup_{k \geq 1} \int_B \mathbf{1}A_{n(k)}\, d\mu = 0.$$

Since $\mu$ is a probability measure, it follows that the set $\{\mathbf{1}A_{n(k)} : k \geq 1\}$ is weakly sequentially compact in $L_1(\mu)$. Hence, by a mean ergodic theorem, the averages $\mathbf{1}A_n$ converge in the norm topology of $L_1(\mu)$ to some $v \in L_1^+(\mu)$. Then, as in the proof of Lemma 3.2, we see that $v > 0$ a.e. on $X$. Therefore, regarding the $P_i$ as commuting Markov operators defined on $L_\infty(x, \mathcal{F}, v\, d\mu)$ such that $\|P_i\|_{L_1(X,\mathcal{F},v\,d\mu)} = 1$ for all $i = 1, 2 \ldots, d$, we can apply the classical $d$-parameter pointwise ergodic theorem to infer that for every $f$ in $L_p^+(\mu) = L_p^+(v^{-1}v\, d\mu)$ the limit $f^*(x) = \lim_n A_n f(x)$ exists a.e. on $X$ (but may equal to infinity on some subset of $X$) and that

$$f^* = E\{f|X, \mathcal{I}, v\, d\mu)\} \text{ a.e. on } X.$$

To see that $f^* < \infty$ a.e. on $X$, let us write $B = \{x : f^*(x) = \infty\}$. Since $B \subset \liminf_k \{A_{n(k)} f > t\}$ for each $t > 0$, it follows from (d) together with Fatou's lemma that

$$\int_B U\, d\mu \leq \liminf_k \int_{\{A_{n(k)} f > t\}} U\, d\mu \leq t^{-r} \left( \int_X f^p\, d\mu \right)^{r/p}.$$

Letting $t \uparrow \infty$, we have $\int_B U\, d\mu = 0$ and hence $\mu B = 0$.

(a) $\Rightarrow$ (e). Since $L_\infty(\mu) \subset L_p(\mu)$, it follows from Lemma 3.2 that there exists a function $v_0$ in $L_1^+(\mu)$ such that $v_0 > 0$ a.e. on $X$ and $v_0 P_i = v_0$ for each $i$. As before, let us regard the $P_i$ as commuting Markov operators on $L_\infty(X, \mathcal{F}, v_0\, d\mu)$. Then, considering the invertible mapping $u \to u/v_0$ from $L_1(X, \mathcal{F}, \mu)$ onto $L_1(X, \mathcal{F}, v_0\, d\mu)$, we see that for any $u$ in $L_1(\mu)$ the sequence $\{(uA_n)/v_0\}$ converges a.e. on $X$ and also in the norm topology of $L_1(v_0\, d\mu)$. This proves the first assertion of (e). To prove the second assertion, let $v \in L_1^+(\mu)$ be such that $vP_i = v$ for each $i$. Then the set $D = \{x : v(x) > 0\}$ is in $\mathcal{I}$, and as before regarding the $P_i$ as commuting Markov operators on $L_\infty(D, v\, d\mu)$ and noticing that $L_p^+(D, \mu) = L_p^+\left(D, \frac{1}{v} v\, d\mu\right)$, we may apply the equivalence of (a) and (e) in Theorem 2.1 to infer the existence of a sequence $\{D_N\}$ of sets in $\mathcal{I}$, with $D_N \uparrow D$, such that if $v_N$ denotes the restriction function of $v$ to $D_N$ then

$$v_N = \left( \frac{1}{v_N} \right)^{-1} \in L_{p'}\left( D_N, \frac{1}{v} v\, d\mu \right) = L_{p'}(D_N, \mu).$$

(e) $\Rightarrow$ (f) Letting $w = \lim_n \mathbf{1}A_n$ in $L_1(\mu)$, it follows that $wP_i = w$ and $w > 0$ a.e. on $X$. Choose $\{X_N\}$ in $\mathcal{I}$ so that the functions $w_N = w\mathbf{1}_{X_N}$ are in $L_{p'}(\mu)$. It then suffices to define $v = \sum_{N=1}^{\infty} 2^{-N}\|w_N\|_{p'}^{-1}w_N$. The proof is complete. ■

**Theorem 3.3.** *Let $(X, \mathcal{F}, \mu)$ be a probability measure space and let $P_i$ $(i = 1, 2, \ldots, d)$ be commuting Markov operators on $L_\infty(X, \mathcal{F}, \mu)$. Assume that each $P_i$ is conservative or invertible. If $1 \le p,\ p_1 \le \infty$, then the following are equivalent.*

(a) *The limit $f^*(x) = \lim_n A_n f(x)$ exists a.e. on $X$ and $f^* \in L_{p_1}(\mu)$ for all $f$ in $L_p^+(\mu)$.*

(b) *For every $u$ in $L_{p_1'}(\mu)$ the limit $u_0^*(x) = \lim_n uA_n(x)$ exists a.e. on $X$ and in the norm topology of $L_1(\mu)$; further $u_0^*$ is in $L_{p'}(\mu)$.*

(c) *For every $u$ in $L_{p_1'}^+(\mu)$ with $\|u\|_{p_1'} > 0$ we have*

$$0 < \|\liminf_n uA_n\|_{p'} < \infty.$$

(d) *To each $u$ in $L_{p_1'}^+(\mu)$ there corresponds a functional $F_u$ defined on $L_\infty^+(\mu)$ such that*

$$\begin{cases} F_u(f_n) \to 0 & \text{whenever } f_n \downarrow 0 \\ & \text{a.e. on } X (\text{if } p = \infty), \\ F_u(f) \le K_u\|f\|_p, & K_u \text{ being a constant depending} \\ & \text{only on } u (\text{if } 1 \le p < \infty), \end{cases} \tag{1}$$

$$\begin{cases} F_u(tf) = tF_u(f) & \text{for constants } t \ge 0, \text{ and} \\ F_u(f+g) \le F_u(f) + F_u(g), \end{cases} \tag{2}$$

$$0 \le F_u(f) \le F_u(f+g), \tag{3}$$

$$F_u(P_i f) \le F_u(f), \text{ and} \tag{4}$$

$$F_u(f) = \int_X uf\,d\mu \text{ whenever } f = P_i f \text{ for all } i = 1, 2, \ldots, d. \tag{5}$$

*Proof:* (a) $\Rightarrow$ (b). Since $L_\infty(\mu) \subset L_p(\mu)$, it follows from the proof of Lemma 3.2 that the sequence $\{\mathbf{1}A_n\}$ converges in the norm topology of $L_1(\mu)$ to some $v$ in $L_1^+(\mu)$ with $vP_i = v$ for each $P_i$. Since $P_i$ is conservative or invertible by hypothesis, it follows that $v > 0$ a.e. on $X$. Hence, by the classical $d$-parameter pointwise ergodic theorem, we see that if $u \in L_{p_1'}^+(\mu)$ $(\subset L_1^+(\mu))$ then the limit $u_0^*(x) = \lim_n uA_n(x)$ exists a.e. on $X$ and also in the norm topology of $L_1(\mu)$. In order to prove that $u_0^*$ is a function in $L_{p'}^+(\mu)$, let $f$ be any function in $L_p^+(\mu)$ and put $f_N = f \wedge N$. Then we have

$$\begin{aligned}\int_X u_0^* f \, d\mu &= \lim_N \int_X u_0^* f_N \, d\mu = \lim_N \lim_n \langle uA_n, f_N \rangle \\ &= \lim_N \lim_n \langle u, A_n f_N \rangle = \lim_N \langle u, f_N^* \rangle \\ &\le \int_X u f^* \, d\mu \le \|u\|_{p_1'} \|f^*\|_{p_1} < \infty.\end{aligned}$$

It follows that $u_0^* \in L_{p'}^+(\mu)$.

(b) $\Rightarrow$ (c). Obvious.

(c) $\Rightarrow$ (b). There exists $v_0 \in L_1^+(\mu)$ such that $v_0 P_i = v_0$ for all $i = 1, 2, \ldots, d$ and also such that if $v \in L_1^+(\mu)$ satisfies $vP_i = v$ for all $i = 1, 2, \ldots, d$ then $\{v > 0\} \subset \{v_0 > 0\}$. By virtue of (c) it is sufficient to show that $v_0 > 0$ a.e. on $X$. But, since $\{v_0 = 0\}$ is a set in $\mathcal{I}$ and the function $\tilde{u}_0 = \liminf_n uA_n$ $(u \in L_{p_1'}^+(\mu))$ satisfies $\tilde{u}_0 P_i = \tilde{u}_0 \in L_1^+(\mu)$ for all $i = 1, 2, \ldots, d$, this follows immediately from (c).

(b) $\Rightarrow$ (d). For $u \in L_{p_1'}^+(\mu)$, let $F_u$ be the functional on $L_\infty^+(\mu)$ defined by

$$F_u(f) = \int_X u_0^* f \, d\mu \quad (f \in L_\infty^+(\mu)),$$

where $u_0^*(x) = \lim_n uA_n(x)$ a.e. on $X$. Since the sequence $\{uA_n\}$ converges to $u_0^*$ in the norm topology of $L_1(\mu)$, $F_u$ satisfies all the requirements from (1) to (5).

(d) $\Rightarrow$ (a). Let $u \in L_{p_1'}^+(\mu)$ with $u > 0$ a.e. on $X$. We define a functional $q$ on $L_\infty(\mu)$ by putting

$$q(f) = F_u(f^+) \quad (f \in L_\infty(\mu)).$$

Clearly we have $q(f+g) \le q(f) + q(g)$, $q(tf) = tq(f)$ for each constant $t \ge 0$, and $q(P_i f) \le q(f)$ for each $P_i$. On the other hand, since $P_i$ is conservative or invertible by hypothesis, it follows that

$$\{f \in L_\infty(\mu) : P_i f = f \text{ for all } i = 1, 2, \ldots, d\} = L_\infty(X, \mathcal{I}, \mu).$$

Thus, by a variant of the Hahn-Banach theorem (cf. e.g. [**7**, Proposition 10.5], the linear functional $f \to \tilde{F}_u(f) = \int_X uf\,d\mu$ defined on $L_\infty(X, \mathcal{I}, \mu)$ can be extended to a linear functional $F_u^*$ on $L_\infty(\mu) = L_\infty(X, \mathcal{F}, \mu)$ so that

$$F_u^*(f) \leq q(f) \text{ and } F_u^*(P_i f) = P_u^*(f) \tag{6}$$

for all $f \in L_\infty(\mu)$ and $P_i$ $(1 \leq i \leq d)$. $F_u^*$ is positive, because $f \in L_\infty^+(\mu)$ implies $-F_u^*(f) = F_u^*(-f) \leq q(-f) = 0$.

Now, let us set $\lambda(B) = F_u^*(\mathbf{1}_B)$ for $B \in \mathcal{F}$. By (1) and (6), $\lambda$ is a finite measure absolutely continuous with respect to $\mu$ and invariant under the $P_i$. Therefore the function $u_0^* = d\lambda/d\mu \in L_1^+(\mu)$ satisfies $u_0^* P_i = u_0^*$ for all $P_i$, and hence the set $B = \{x : u_o^*(x) = 0\}$ is in $\mathcal{I}$. This and (5) imply $F_u^*(\mathbf{1}_B) = F_u(\mathbf{1}_B) = \int_B u\,d\mu = 0$ and $\mu B = 0$. Thus the $P_i$ can be regarded as commuting Markov operators defined on $L_\infty(X, \mathcal{F}, u_0^*\,d\mu)$ such that

$$\|P_i\|_{L_1(X,\mathcal{F},u_0^*\,d\mu)} = 1 \quad (1 \leq i \leq d).$$

We then apply the classical $d$-parameter pointwise ergodic theorem to infer that for any $f \geq 0$ on $X$ the pointwise limit

$$f^*(x) = \lim_n A_n f(x)$$

exists a.e. on $X$ (but may equal to infinity on some subset of $X$); further we see that if $0 \leq f_n \uparrow f$ a.e. on $X$ then $f_n^* \uparrow f^*$ a.e. on $X$.

Next, let $f \in L_p^+(\mu)$ be fixed arbitrarily, and put $f_N = f \wedge N$. Since $f_N^* \uparrow f^*$ a.e. on $X$ and $f_N^* \in L_\infty^+(X, \mathcal{I}, \mu)$, we see by (5), (6) and an approximation argument that

$$\begin{aligned}
\int_X uf^*\,d\mu &= \lim_N \int_X uf_N^*\,d\mu = \lim_N \int_X u_0^* f_N^*\,d\mu \\
&= \lim_N \left( \lim_n \int_X u_0^*(A_n f_N)\,d\mu \right) = \lim_N \left( \lim_n \int_X (u_0^* A_n) f_N\,d\mu \right) \\
&= \lim_N \int_X u_0^* f_N\,d\mu \leq \lim_N F_u(f_N) \leq \lim_N K_u \|f_N\|_p \\
&\leq K_u \|f\|_p < \infty \quad (u \in L_{p_1'}^+(\mu) \text{ with } u > 0 \text{ a.e. on } X).
\end{aligned}$$

This proves that $f^* \in L_{p_1}^+(\mu)$, completing the proof. ■

**Remark.** Using the duality relation between Lorenz spaces $L(p,q)$ and $L(p',q')$, where $1 < p,\, q < \infty$, it is possible to generalize Theorem 3.3 to $L(p,q)$ spaces. For this see [**9**], in which one-parameter semigroups of null preserving point transformations are studied.

As a direct corollary to Theorems 3.1 and 3.3 we have

**Theorem 3.4.** *Let $(X, \mathcal{F}, \mu)$ be a probability measure space and let $P_i$ $(i = 1, 2, \ldots, d)$ be commuting Markov operators on $L_\infty(X, \mathcal{F}, \mu)$. Assume that each $P_i$ is conservative or invertible, and that the system $\{P_1, P_2, \ldots, P_d\}$ is ergodic. If $1 \le p < \infty$, then the following are equivalent.*

(a) *For every $f$ in $L_p^+(\mu)$, $\lim_n A_n f$ exists and is finite a.e. on $X$.*

(b) *There exists a function $v$ in $L_{p'}^+(\mu)$ such that $v > 0$ a.e. on $X$ and $vP_i = v$ for all $i = 1, 2, \ldots, d$.*

(c) *For every $u \in L_1^+(\mu)$ with $\|u\|_1 > 0$, the limit $u_0^*(x) = \lim_n uA_n(x)$ exists a.e. on $X$, $u_0^* \in L_{p'}^+(\mu)$ and $\|u_0^*\|_{p'} > 0$.*

(d) *For every $u \in L_\infty^+(\mu)$ with $\|u\|_\infty > 0$, the limit $u_0^*(x) = \lim_n uA_n(x)$ exists a.e. on $X$, $u_0^* \in L_{p'}^+(\mu)$ and $\|u_0^*\|_{p'} > 0$.*

(e) *There exists a functional $F$ defined on $L_\infty^+(\mu)$ such that*

$$F(1) = 1 \text{ and } F(f) \le K\|f\|_p,\ K \text{ being a constant,} \tag{7}$$

$$F(tf) = tF(f) \text{ for constants } t \ge 0, \text{ and } F(f+g) \le F(f) + F(g), \tag{8}$$

$$0 \le F(f) \le F(f+g), \tag{9}$$

$$F(P_i f) \le F(f). \tag{10}$$

**Final remark.** If we don't assume that the $P_i$ commute, then the equivalence of (a) and (b) in Theorem 3.4 does not hold at least for $p = 1$, under the notation

$$A_n f = (n+1)^{-d} \sum_{0 \le n_i \le n} P_1^{n_1} P_2^{n_2} \ldots P_d^{n_d} f$$

We prove this by constructing a counterexample with $d = 2$. Let $(X, \mathcal{F}, \mu)$ be a nonatomic probability measure space and let $\tau$ be an invertible, ergodic and measure preserving transformation on $(X, \mathcal{F}, \mu)$. Let $h$ be a function on $X$, with $h \ge 1$ on $X$, such that $h \in L_1(\mu)$ and $h(\log h) \notin L_1(\mu)$. As is well-known (see e.g. [**4**, p. 54], we then have

$$\sup_{n \ge 0} (n+1)^{-1} \sum_{i=0}^{n} h \circ \tau^i \notin L_1(\mu).$$

On the other hand, since $\lim_n (n+1)^{-1} \sum_{i=0}^{n} h \circ \tau^i = \int_X h\, d\mu$ a.e. on $X$ and in the norm topology of $L_1(\mu)$, there exists a sub-$\sigma$-field $\mathcal{B}$ such that the sequence

$$E\left\{(n+1)^{-1} \sum_{i=0}^{n} h \circ \tau^i | \mathcal{B}\right\} \quad (n \geq 0)$$

is a.e. nonconvergent (cf. Theorem 4.3 in [**1**]). Here if we define

$$P_1 f = E\{f|B\} \text{ and } P_2 f = f \circ \tau \quad (f \in L_\infty(\mu))$$

then $P_1$ and $P_2$ are conservative Markov operators on $L_\infty(\mu)$ such that $\mathbf{1}P_1 = \mathbf{1}P_2 = 1$. Thus (b) in Theorem 3.4 holds with $v = 1$ on $X$. But (a) in Theorem 3.4 does not hold for $p = 1$, because the averages

$$A_n h = \frac{n}{n+1} E\left\{(n+1)^{-1} \sum_{i=0}^{n} h \circ \tau^i | \mathcal{B}\right\} + (n+1)^{-2} \sum_{i=0}^{n} h \circ \tau^i$$

are a.e. nonconvergent.

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