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# NORMAL BASES FOR THE SPACE OF CONTINUOUS FUNCTIONS DEFINED ON A SUBSET OF $\mathbb{Z}_p$

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*Abstract*

Let $K$ be a non-archimedean valued field which contains $\mathbb{Q}_p$ and suppose that $K$ is complete for the valuation $|\cdot|$, which extends the $p$-adic valuation. $V_q$ is the closure of the set $\{aq^n | n = 0, 1, 2, \dots\}$ where $a$ and $q$ are two units of $\mathbb{Z}_p$, $q$ not a root of unity. $C(V_q \to K)$ is the Banach space of continuous functions from $V_q$ to $K$, equipped with the supremum norm. Our aim is to find normal bases $(r_n(x))$ for $C(V_q \to K)$, where $r_n(x)$ does not have to be a polynomial.

## 1. Introduction

The main aim of this paper is to find normal bases $(r_n(x))$ for the space of continuous functions on $V_q$, where $r_n(x)$ does not have to be a polynomial.

Therefore we start by recalling some definitions and some previous results.

Let $E$ be a non-archimedean Banach space over a non-archimedean valued field $L$.

Let $f_1, f_2, \dots$ be a finite or infinite sequence of elements of $E$. We say that this sequence is orthogonal if $\|\alpha_1 f_1 + \cdots + \alpha_k f_k\| = \max\{\|\alpha_i f_i\| : i = 1, \dots, k\}$ for all $k$ in $\mathbb{N}$ (or for all $k$ that do not exceed the length of the sequence) and for all $\alpha_1, \dots, \alpha_k$ in $L$. If the sequence is infinite, it follows that $\left\|\sum_{i=1}^{\infty} \alpha_i f_i\right\| = \max\{\|\alpha_i f_i\| : i = 1, 2, \dots\}$ for all $\alpha_1, \alpha_2, \dots$ in $L$ for which $\lim_{i\to\infty} \alpha_i f_i = 0$. An orthogonal sequence $f_1, f_2, \dots$ is called orthonormal if $\|f_i\| = 1$ for all $i$.

This leads us to the following definition:

If $E$ is a non-archimedean Banach space over a non-archimedean valued field $L$, then a family $(f_i)$ of elements of $E$ is a (ortho)normal basis of $E$ if the family $(f_i)$ is orthonormal and also a basis.

An equivalent formulation is (see [**1**, Propositions 50.4 and 50.6])

If $E$ is a non-archimedean Banach space over a non-archimedean valued field $L$, then a family $(f_i)$ of elements of $E$ is a (ortho)normal basis of $E$ if each element $x$ of $E$ has a unique representation $x = \sum_i x_i f_i$ where $x_i \in L$ and $x_i \to 0$ if $i \to \infty$, and if the norm of $x$ is the supremum of the norms of $x_i$.

Let $\mathbb{Z}_p$ be the ring of $p$-adic integers, $\mathbb{Q}_p$ the field of $p$-adic numbers, and $K$ is a non-archimedean valued field, $K$ containing $\mathbb{Q}_p$, and we suppose that $K$ is complete for the valuation $|\cdot|$, which extends the $p$-adic valuation. Let $a$ and $q$ be two units of $\mathbb{Z}_p$, $q$ not a root of unity. We define $V_q$ to be the closure of the set $\{aq^n | n = 0, 1, 2, \dots\}$. The set $V_q$ has been described in [**3**]. Let $C(V_q \to K)$ (resp. $C(\mathbb{Z}_p \to K)$) be the Banach space of continous functions from $V_q$ to $K$ (resp. $\mathbb{Z}_p$ to $K$) equipped with the supremum norm. $\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{N}_0$ is the set of natural numbers without zero.

We introduce the following:

If $x$ is an element of $\mathbb{Q}_p$, $x$ can be written in the following way: $x = \sum_{j=-\infty}^{+\infty} a_j p^j$ where $a_{-i}$ is zero for $i$ sufficiently large ($i \in \mathbb{N}$) (see [**1**, section 3 and section 4]). This is called the Henseldevelopment of the $p$-adic integer $x$. We then define the $p$-adic entire part $[x]_p$ of $x$ by $[x]_p = \sum_{j=-\infty}^{-1} a_j p^j$ and we put $x_n = p^n [p^{-n} x]_p = \sum_{j=-\infty}^{n-1} a_j p^j$ ($n \in \mathbb{N}$).

We write $m \lhd x$, if $m$ is one of the numbers $x_0, x_1, \dots$ . We then say that "$m$ is an initial part of $x$" or "$x$ starts with $m$" (see [**1**, section 62]).

If $n$ belongs to $\mathbb{N}_0$, $n = \sum_{j=0}^{s} a_j p^j$ where $a_s \neq 0$, then we put $n_- = \sum_{j=0}^{s-1} a_j p^j$. We remark that $n_- \lhd n$.

In [**1**, Theorem 62.2], we find the following result which is due to van der Put:

**Theorem.**

*The functions $g_0, g_1, \dots$ defined by*

$$\begin{aligned} g_n(x) &= 1 \quad \textit{if } n \lhd x, \\ &= 0 \quad \textit{otherwise}, \end{aligned}$$

*form a normal basis for* $C(\mathbb{Z}_p \to K)$. *If* $f$ *is an element of* $C(\mathbb{Z}_p \to K)$, *then* $f$ *can be written as a uniformly convergent series* $f(x) = \sum_{k=0}^{\infty} \gamma_k g_k(x)$ *where* $\gamma_0 = f(0)$ *and* $\gamma_n = f(n) - f(n_-)$ *if* $n \in \mathbb{N}_0$.

We now survey the content of this paper:

In Theorem 1 of section 2, our aim is to find a basis $(e_n(x))$ analogous to van der Put's basis, but with the space $C(\mathbb{Z}_p \to K)$ replaced by $C(V_q \to K)$. If $f$ is an element of $C(V_q \to K)$, then there exist elements $a_k$ of $K$ such that $f(x) = \sum_{k=0}^{\infty} a_k e_k(x)$ where the series on the right-hand-side is uniformly convergent. We are able to give an expression for the coefficients $a_k$.

In Theorem 2 of section 3, we prove the following result:

Define $r_n(x) = \sum_{j=0}^{n} c_{n;j} e_j(x)$, $c_{n;j} \in K$, $c_{n;n} \neq 0$ ($(e_n(x))$ as in Theorem 1 below).

Then $(r_n(x))$ forms a normal basis for $C(V_q \to K)$ if and only if for all $n$ $\|r_n\| = 1$ and $|c_{n;n}| = 1$.

In Theorem 3 of section 3, we give an extension of Theorem 2:

Let $(r_n(x))$ be such a sequence which forms a normal basis for $C(V_q \to K)$, and let $(s_n(x))$ be a sequence such that $s_n(x) = \sum_{j=0}^{n} d_{n;j} r_j(x)$, $d_{n;j} \in K$, $d_{n;n} \neq 0$. Then $(s_n(x))$ forms a normal basis for $C(V_q \to K) \Leftrightarrow$ $\|s_n\| = 1$, $|d_{n;n}| = 1 \Leftrightarrow |d_{n;j}| \leq 1$, $|d_{n;n}| = 1$.

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## 2. Proof of the first theorem

We start with some lemmas and some definitions.

**Definition.**
If $b$ and $x$ are elements of $\mathbb{Z}_p$, $b \equiv 1 \pmod{p}$, then we put $b^x = \lim_{n \to x} b^n$.
The mapping: $\mathbb{Z}_p \to \mathbb{Z}_p : x \to b^x$ is continuous.
For more details, we refer the reader to [**1**, section 32].

**Notation.**
Take $m \geq 1$, $m$ the smallest integer such that $q^m \equiv 1 \pmod{p}$.
We have $1 \leq m \leq p - 1$ and $(q^m)^x$ is defined for all $x$ in $\mathbb{Z}_p$.

**Definition.**

Let $k$ be a natural number prime to $p$. We denote by $\mathbb{Z}_p(k)$ the projective limit $\mathbb{Z}_p(k) = \varprojlim_j (\mathbb{Z}/kp^j\mathbb{Z}) \cong (\mathbb{Z}/k\mathbb{Z}) \times \mathbb{Z}_p$.

In the following lemma we use the fact that $\mathbb{Z}_p(m) = (\mathbb{Z}/m\mathbb{Z}) \times \mathbb{Z}_p$ to denote an element of $\mathbb{Z}_p(m)$ as $x = (r, y)$. Also, if $n \in \mathbb{N}$, $n = r + mk$ $(0 \leq r < m)$ then the map $n \to (r, k)$ imbeds $\mathbb{N}$ in $\mathbb{Z}_p(m)$.

**Lemma 1.**

*The mapping $\varphi : \mathbb{Z}_p(m) \to V_q : (r, y) \to aq^r(q^m)^y$ is a homeomorphism.*

The proof of this lemma can be found in [**2**, p. 377].

**Corollary.**

*If $q \equiv 1 \pmod{p}$, i.e. $m = 1$, then the mapping: $\mathbb{Z}_p \to V_q : x \to aq^x$ is a homeomorphism.*

Let $\beta$ be an element of $\mathbb{Z}_p \backslash \{0\}$. We want to know the valuation of the $p$-adic integer $(q^m)^\beta - 1$. Therefore we need two lemmas:

The following lemmas (2 and 3) are proved in [**3**]:

**Lemma 2.**

*Let $\alpha$ be an element of $\mathbb{Z}_p$, $\alpha \equiv 1 \pmod{p^r}$, $\alpha \not\equiv 1 \pmod{p^{r+1}}$ $r \geq 1$.*

*If $(p, r) \neq (2, 1)$, $\beta \in \mathbb{Z}_p \backslash \{0\}$ then $\alpha^\beta \equiv 1 \pmod{p^{r+\mathrm{ord}_p \beta}}$, $\alpha^\beta \not\equiv 1 \pmod{p^{r+1+\mathrm{ord}_p \beta}}$.*

**Corollary.**

*Let $q^m \equiv 1 \pmod{p^{k_0}}$, $q^m \not\equiv 1 \pmod{p^{k_0+1}}$. If $(p, k_0) \neq (2, 1)$, $\beta \in \mathbb{Z}_p \backslash \{0\}$ then $(q^m)^\beta \equiv 1 \pmod{p^{k_0+\mathrm{ord}_p \beta}}$, $(q^m)^\beta \not\equiv 1 \pmod{p^{k_0+1+\mathrm{ord}_p \beta}}$.*

In Lemma 2 we excluded the case where $(p, r) = (2, 1)$. This case will be handled in the following lemma:

**Lemma 3.**

*Let $\alpha$ be an element of $\mathbb{Z}_2$, $\alpha \equiv 3 \pmod{4}$. Define a natural number $n$ by $\alpha = 1 + 2 + 2^2\varepsilon$, $\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \dots$, $\varepsilon_0 = \varepsilon_1 = \dots = \varepsilon_{n-1} = 1$, $\varepsilon_n = 0$.*

*If $\beta \in \mathbb{Z}_2 \backslash \{0\}$, $\mathrm{ord}_2 \beta = 0$ then $\alpha^\beta \equiv 1 \pmod{2}$, $\alpha^\beta \not\equiv 1 \pmod{4}$.*

*If* $\beta \in \mathbb{Z}_2\backslash\{0\}$, $\operatorname{ord}_2 \beta = k \geq 1$ *then* $\alpha^\beta \equiv 1 \pmod{2^{n+2+\operatorname{ord}_2 \beta}}$, $\alpha^\beta \not\equiv 1 \pmod{2^{n+3+\operatorname{ord}_2 \beta}}$.

**Corollary.**

*If* $q \equiv 3 \pmod 4$, *we define a natural number* $N$ *by* $q = 1 + 2 + 2^2\varepsilon$, $\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots$, $\varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_{N-1} = 1$, $\varepsilon_N = 0$.

*If* $\beta \in \mathbb{Z}_2\backslash\{0\}$, $\operatorname{ord}_2 \beta = 0$ *then* $q^\beta \equiv 1 \pmod 2$, $q^\beta \not\equiv 1 \pmod 4$.

*If* $\beta \in \mathbb{Z}_2\backslash\{0\}$, $\operatorname{ord}_2 \beta = k \geq 1$ *then* $q^\beta \equiv 1 \pmod{2^{N+2+\operatorname{ord}_2 \beta}}$, $q^\beta \not\equiv 1 \pmod{2^{N+3+\operatorname{ord}_2 \beta}}$.

We remark that is possible to write each $x$ and element of $V_q$ in the following way: $x = aq^{i_x}(q^m)^{\alpha_x}$ where $i_x$ is a natural number, $0 \leq i_x < m$, and where $\alpha_x$ is an element of $\mathbb{Z}_p$. This immediately follows from Lemma 1. This leads us to the following definition:

**Definition.**

We now define a sequence of functions $e_k$ in the following way. Write $k(\in \mathbb{N})$ in the form $k = i + mj$, $0 \leq i < m$ $(i, j \in \mathbb{N})$. The functions $e_k$ are defined by

$$\begin{aligned} e_k(x) = e_{i+mj}(x) &= 1 \quad \text{if } x = aq^{i_x}(q^m)^{\alpha_x} \text{ where } i_x = i,\ j \lhd \alpha_x. \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Let us use the notation $B(b, r^-)$ for the 'open' disc with radius $r$ and with center $b$, i.e. $B(b, r^-) = \{x \in V_q \mid |x - b| < r\}$, and $B(b, r)$ for the 'closed' disc with radius $r$ and with center $b$, i.e. $B(b, r) = \{x \in V_q \mid |x - b| \leq r\}$.

In the following lemmas we will show that the functions $e_k(x)$ are characteristic functions of discs. There exists a $k_0$ such that $q^m \equiv 1 \pmod{p^{k_0}}$, $q^m \not\equiv 1 \pmod{p^{k_0+1}}$. We distinguish two cases: $(p, k_0) \neq (2, 1)$ (Lemma 4), and $(p, k_0) = (2, 1)$ i.e. $q \equiv 3 \pmod 4$ (Lemma 5). If we use the same notation in Lemmas 4 and 5 as in the definition, we have

**Lemma 4.**

*Let* $q^m \equiv 1 \pmod{p^{k_0}}$, $q^m \not\equiv 1 \pmod{p^{k_0+1}}$ *and suppose* $(p, k_0) \neq (2, 1)$.

*If* $0 \leq i < m$ *then* $e_i(x)$ *is the characteristic function of the closed disc* $B(aq^i, p^{-k_0})$, *and if* $0 \leq i < m$, $j \geq 1$ *then* $e_k(x) = e_{i+jm}(x)$ *is the characteristic function of the open disc* $B\left(aq^i(q^m)^j, \left(\frac{p^{-k_0}}{j}\right)^-\right)$.

*Proof:*

Let $j = \sum_{i=0}^{s} a_i p^i$ be the Henseldevelopment of $j \in \mathbb{N}_0$, with $a_s$ different from zero.

If we use the notation $x = aq^{i_x}(q^m)^{\alpha_x}$ $(0 \leq i_x < m)$ for an element $x$ of $V_q$, we will show the following:

a) if $0 \leq i < m : |x - aq^i| \leq p^{-k_0}$ if and only if $i_x = i$.

b) if $0 \leq i < m,\ j \geq 1 : |x - aq^i(q^m)^j| < \frac{p^{-k_0}}{j}$ if and only if $i_x = i$, $j \lhd \alpha_x$.

We first prove a). If $i_x = i$, then $|x - aq^i| = |aq^{i_x}(q^m)^{\alpha_x} - aq^i| = |(q^m)^{\alpha_x} - 1| \leq p^{-k_0}$ by the corollary to Lemma 2.

If $i_x \neq i$, then

$$\begin{aligned}|x - aq^i| &= |aq^{i_x}(q^m)^{\alpha_x} - aq^i| \\ &= \max\{|aq^{i_x}(q^m)^{\alpha_x} - aq^{i_x}|, |aq^{i_x} - aq^i|\} = 1,\end{aligned}$$

since $|aq^{i_x}(q^m)^{\alpha_x} - aq^{i_x}| \leq p^{-k_0}$, $|aq^{i_x} - aq^i| = 1$. This proves a).

Now we prove b).

Suppose $i_x = i$, $j \lhd \alpha_x$. Then $|x - aq^i(q^m)^j| = |(q^m)^{\alpha_x - j} - 1| \leq p^{-k_0-(s+1)}$ by the corollary following Lemma 2, since $j$ is an initial part of $\alpha_x$. Since $j$ is strictly smaller than $p^{(s+1)}$, we conclude that $|x - aq^i(q^m)^j| < \frac{p^{-k_0}}{j}$.

For the converse, suppose $|x - aq^i(q^m)^j| < \frac{p^{-k_0}}{j}$. Then we must have that $i_x$ equals $i$, since otherwise $|x - aq^i(q^m)^j| = 1$:

$$\begin{aligned}|x - aq^i(q^m)^j| &= |aq^{i_x}(q^m)^{\alpha_x} - aq^i(q^m)^j)| \\ &= \max\{|aq^{i_x}(q^m)^{\alpha_x} - aq^{i_x}|, |aq^{i_x} - aq^i|, |aq^i - aq^i(q^m)^j|\} \\ &= 1\end{aligned}$$

since $|aq^{i_x}(q^m)^{\alpha_x} - aq^{i_x}| \leq p^{-k_0}$, $|aq^i - aq^i(q^m)^j| \leq p^{-k_0}$ (corollary to Lemma 2) and $|aq^{i_x} - aq^i| = 1$ if $i_x$ is different from $i$.

So we have $|(q^m)^{\alpha_x - j} - 1| < \frac{p^{-k_0}}{j}$ and from this it follows that $|(q^m)^{\alpha_x - j} - 1| \leq p^{-k_0-(s+1)}$ since $j$ is at least $p^s$. This means that $\mathrm{ord}_p(\alpha_x - j)$ is at least $s+1$ (again by the corollary to Lemma 2) and so we conclude that $j$ is an initial part of $\alpha_x$. ∎

**Lemma 5.**

*If $q \equiv 3 \pmod 4$, with $q = 1 + 2 + 2^2\varepsilon$, where $\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \dots$, $\varepsilon_0 = \varepsilon_1 = \dots = \varepsilon_{N-1} = 1$, $\varepsilon_N = 0$, then $e_0(x)$ is the characteristic* *function of $V_q$, and $e_j(x)$ is the characteristic function of the open disc $B\left(aq^j, \left(\frac{2^{-(N+2)}}{j}\right)^-\right)$ if $j \geq 1$.*

*Proof:*

In this case $m$ equals one and we use the notation $x = aq^{\alpha_x}$ for an element $x$ of $V_q$.

It is clear that $e_0(x)$ is the characteristic function of $V_q$.

If $j$ is at least one, we prove: $|x - aq^j| < \frac{2^{-(N+2)}}{j}$ if and only if $j \lhd \alpha_x$.

Suppose $j \lhd \alpha_x$. Then $|x - aq^j| = |q^{\alpha_x - j} - 1| \leq 2^{-(N+2)-(s+1)}$ (corollary following Lemma 3), and since $j$ is strictly smaller than $2^{s+1}$, we conclude $|x - aq^j| < \frac{2^{-(N+2)}}{j}$.

For the converse, suppose $|x - aq^j| < \frac{2^{-(N+2)}}{j}$. Then $|q^{\alpha_x - j} - 1| < \frac{2^{-(N+2)}}{j}$ and so $|q^{\alpha_x - j} - 1| \leq 2^{-(N+2)-(s+1)}$ since $j$ is at least $2^s$. By the corollary to Lemma 3, we have that $\text{ord}_2(\alpha_x - j)$ is at least $s+1$ and so $j$ is an initial part of $\alpha_x$. ■

**Corollary.**

*The functions $(e_k(x))$ are continuous functions on $V_q$.*

In the following theorem we prove that the sequence $(e_k(x))$ forms a normal basis for $C(V_q \to K)$. This implies that if $f$ is an element of $C(V_q \to K)$, there exists elements $a_k$ of $K$ such that $f(x) = \sum_{k=0}^{\infty} a_k e_k(x)$ where the right-hand-side is uniformly convergent. We are able to give an expression for the coefficients $a_k$. The proof of this theorem is analogous to the proof of Theorem 62.2 in [**1**].

**Theorem 1.**

*The functions $(e_k(x))$ form a normal basis for $C(V_q \to K)$. If $f$ is an element of $C(V_q \to K)$ then $f$ can be written as a uniformly convergent series $f(x) = \sum_{k=0}^{\infty} a_k e_k(x)$ where*

$$(*) \quad \begin{array}{ll} a_k = f(aq^k) & \text{if } 0 \leq k < m \\ a_k = a_{i+jm} = f(aq^i(q^m)^j) - f(aq^i(q^m)^{j-}) & \text{if } 0 \leq i < m,\ j > 0. \end{array}$$

*Proof:*

Let $f$ be an element of $C(V_q \to K)$, and let $a_k$ be defined as $a_k = f(aq^k)$ if $0 \leq k < m$, $a_k = a_{i+jm} = f(aq^i(q^m)^j) - f(aq^i(q^m)^{j-})$ if $0 \leq i < m$, $j > 0$.

We first prove that $a_k$ tends to zero if $k$ tends to infinity: for all $\varepsilon > 0$, there exists a $J$ such that $k > J$ implies $|a_k| \leq \varepsilon$. To prove this, we distinguish two cases:

i) Let $q^m \equiv 1 \pmod{p^{k_0}}$, $q^m \not\equiv 1 \pmod{p^{k_0+1}}$, with $(p, k_0) \neq (2, 1)$.

Since the function $f$ is continuous on $V_q$, it is uniformly continuous on $V_q$, and so there exist an $S$, such that $|x - y| \leq p^{-(k_0+S)}$ implies $|f(x) - f(y)| < \varepsilon$. We then put $J = p^S m$.

If $k > J$, and $k$ equals $i + jm$ with $0 \leq i < m$, then we have that $j \geq p^S$ and so (corollary to Lemma 2) $|aq^i(q^m)^j - aq^i(q^m)^{j_-}| = |(q^m)^{j-j_-} - 1| \leq p^{-(k_0+S)}$ and this implies that $|a_k| = |f(aq^i(q^m)^j) - f(aq^i(q^m)^{j_-})| < \varepsilon$.

ii) Let $q \equiv 3 \pmod 4$, $q = 1 + 2 + 2^2\varepsilon$, $\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \dots$, $\varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_{N-1} = 1$, $\varepsilon_N = 0$. We remark that $m$ equals one in this case.

Since the function $f$ is continous on $V_q$, it is uniformly continuous on $V_q$, and so there exist an $S$, such that $|x - y| \leq 2^{-(N+2+S)}$ implies $|f(x) - f(y)| < \varepsilon$. We then put $J = 2^S$.

If $k > J$, then (corollary to Lemma 3) $|q^k - q^{k_-}| = |q^{k-k_-} - 1| \leq 2^{-(N+2+S)}$ and this implies that $|a_k| = |f(q^k) - f(q^{k_-})| < \varepsilon$.

We conclude that $a_k$ tends to zero if $k$ tends to infinity.

If $f$ is an element of $C(V_q \to K)$, we introduce a function $g(x)$ defined by $g(x) = \sum_{k=0}^{\infty} a_k e_k(x)$ with $a_k$ as in $(*)$. Since $\|a_k e_k\| \leq |a_k| \to 0$, the series on the right-hand-side converges uniformly, so the function $g$ is continous as a uniformly limit of continuous functions. We can prove that $g(aq^k) = f(aq^k)$ if $0 \leq k < m$ and that $g(aq^i(q^m)^j) - g(aq^i(q^m)^{j_-}) = f(aq^i(q^m)^j) - f(aq^i(q^m)^{j_-})$ for $0 \leq i < m$, $j > 0$. Then we have $g(aq^k) = f(aq^k)$ for all natural numbers $k$ and by continuity, we conclude that $f(x) = g(x)$.

So we have $f(x) = \sum_{k=0}^{\infty} a_k e_k(x)$, with $a_k$ as in $(*)$.

It is clear that $\|f\| \leq \max_{0 \leq k}\{|a_k|\}$, but we also have $|f(aq^k)| \leq \|f\|$ and $|(aq^i(q^m)^j) - f(aq^i(q^m)^{j_-})| \leq \|f\|$, so we conclude $\|f\| = \max_{0 \leq k}\{|a_k|\}$.

Finally we prove the uniqueness of the coefficients.

If $f(x) = \sum_{k=0}^{\infty} a_k e_k(x) = \sum_{k=0}^{\infty} b_k e_k(x)$, then $\sum_{k=0}^{\infty} (a_k - b_k) e_k(x) = 0$. So $\max_{0 \leq k}\{|a_k - b_k|\} = 0$, from which it follows that $a_k = b_k$ for all $k$. This proves the theorem. ■

## 3. More bases for $C(V_q \to K)$

We can make more normal bases, using the basis $(e_k(x))$ of Theorem 1:

**Theorem 2.**

*Let $(e_n(x))$ be as above, and define $r_n(x) = \sum_{j=0}^{n} c_{n;j} e_j(x)$, $c_{n;j} \in K$, $c_{n;n} \neq 0$. Then $(r_n(x))$ forms a normal basis for $C(V_q \to K)$ if and only if $\|r_n\| = 1$ and $|c_{n;n}| = 1$ for all $n$.*

The proof of this theorem will not be given here, since it is analogous to the proof of Theorem 2 in [**3**].

**Remark.**

An analogous result can be found on the space $C(\mathbb{Z}_p \to K)$, if we replace the sequence $(e_n(x))$ by the van der Put basis $(g_n(x))$ from the introduction.

We can extend Theorem 2 to the following:

**Theorem 3.**

*Let $(r_n(x))$ be a sequence as found in Theorem 2, which forms a normal basis for $C(V_q \to K)$, and let $(s_n(x))$ be a sequence such that $s_n(x) = \sum_{j=0}^{n} d_{n;j} r_j(x)$, $d_{n;j} \in K$, $d_{n;n} \neq 0$.*

*Then the following are equivalent:*

i) $(s_n(x))$ *forms a normal basis for* $C(V_q \to K)$.
ii) $\|s_n\| = 1$, $|d_{n;n}| = 1$.
iii) $|d_{n;j}| \leq 1$, $|d_{n;n}| = 1$.

*Proof:*

i) $\Leftrightarrow$ ii) follows from Theorem 2, using the expression $r_n(x) = \sum_{j=0}^{n} c_{n;j} e_j(x)$, and ii) $\Leftrightarrow$ iii) follows from the fact that $(r_n(x))$ forms a normal basis. ■

**Examples.**

1) If a sequence $(r_n(x))$, as found in Theorem 2, forms a normal basis of $C(V_q \to K)$, then so does $(s_n(x))$, where $s_n(x) = r_0(x) + r_1(x) + \cdots + r_n(x)$: apply iii).
2) If we put for $0 \leq i < m$,

$$\begin{aligned} r_i(x) &= 1 \quad \text{if } x = aq^{i_x}(q^m)^{\alpha_x} \text{ where } i_x = i \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

anf for $k \geq m$ we put

$$\begin{aligned} r_k(x) = r_{i+mj}(x)\,(0 \leq i < m) &= 1 \quad \text{if } x = aq^{i_x}(q^m)^{\alpha_x} \\ &\qquad \text{where } i_x = i,\ j \not\lhd \alpha_x. \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

then $(r_n(x))$ forms a normal basis for $C(V_q \to K)$. We can apply iii) since $r_i(x) = e_i(x)$ for $0 \leq i < m$, $r_k(x) = e_i(x) - e_k(x)$ for $k = i+mj$, $0 \leq i < m$, $j > 0$. If $f \in C(V_q \to K)$, then there exists a uniformly convergent expansion of the form $f(x) = \sum_{k=0}^{\infty} c_k r_k(x)$, where

$$\begin{aligned} c_k &= c_{i+jm} \\ &= f(aq^i(q^m)^{j-}) - f(aq^i(q^m)^j) \quad \text{if } 0 \leq i < m,\ j > 0, \text{ and} \\ c_i &= f(aq^i) - \sum_{j=1}^{\infty} c_{i+jm} \qquad\qquad \text{if } 0 \leq i < m. \end{aligned}$$

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