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ON EXTENSIONS OF PSEUDO-INTEGERS

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Abstract

An abelian group A is *pseudofree of rank* k if the p-localization of A is isomorphic to the p-localization of $\mathbb{Z}^k$ for all primes p, i.e. $A_p \cong \mathbb{Z}_p^k$ for all primes p. If $k = 1$, we call A a group of pseudo-integers. We may assume, in this case, that $\mathbb{Z} \subseteq A \subset \mathbb{Q}$ and that for any prime p_i there is a maximal exponent r_i so that $\frac{1}{p_i^{r_i}} \in A$. The group A is then generated by $\{\frac{1}{p_i^{r_i}} \mid p_i \text{ is a prime}\}$ and we write $A = \langle \frac{1}{p_i^{r_i}} \rangle$. We say a pseudofree group of rank k is *completely decomposable* if it can be written as the direct sum of k groups of pseudo-integers.

If A is pseudofree of rank k, B is pseudofree of rank l, and $B \rightarrowtail E \twoheadrightarrow A$ is an extension in $\operatorname{Ext}(A, B)$ then E must be pseudofree of rank $k+l$. In this paper, we consider $\operatorname{Ext}(\bar{P}, P)$ for $P = \langle \frac{1}{p_i^{r_i}} \rangle$ and $\bar{P} = \langle \frac{1}{p_i^{\bar{r}_i}} \rangle$ groups of pseudo-integers. We determine when it is possible, in terms of the defining exponents of P and $\bar{P}$, for $\operatorname{Ext}(\bar{P}, P)$ to contain certain extensions (which we'll call *nontrivial*) where E is completely decomposable as a pseudofree group of rank 2. We find that $\operatorname{Ext}(\bar{P}, P)$ contains such extensions if and only if $r_i \leq \bar{r}_i$ almost everywhere and $r_i < \bar{r}_i$ for an infinite number of primes.

0. Introduction.

In [**1**] Casacuberta and Hilton introduce the concept of the *extended genus* of a nilpotent group N, denoted $\operatorname{EG}(N)$. It is defined to be the set of isomorphism classes of nilpotent groups M so that the p-localizations of M and N are isomorphic for all primes p, i.e. $M_p \cong N_p$ for all primes p. If A is a finitely-generated abelian group, they show that to study $\operatorname{EG}(A)$ it is only necessary to examine $\operatorname{EG}(\mathbb{Z}^k)$ where k is the torsionfree rank of A. The extended genus of $\mathbb{Z}$ is completely described in [**2**]. There it is indicated that if $A \in \operatorname{EG}(\mathbb{Z})$ then A is similar in many ways to $\mathbb{Z}$ and hence A is called a group of *pseudo-integers*. We will adopt this terminology and, accordingly, if $A \in \operatorname{EG}(\mathbb{Z}^k)$ we will say that A is *pseudofree of rank* k. As in [**1**], we define a pseudofree group

A of rank k to be *completely decomposable* if it is the direct sum of k groups of pseudo-integers, i.e. $A = \bigoplus_{i=1}^{k} A_i$ where each A_i is a group of pseudo-integers.

In this paper, we investigate $\mathrm{Ext}(\bar{P}, P)$ for P and $\bar{P}$ groups of pseudo-integers. In [**5**], the structure of $\mathrm{Hom}(\bar{P}, P)$ and $\mathrm{Ext}(\bar{P}, P)$ as abelian groups is identified, while in [4] it is shown that, for certain $\bar{P}$ and P, $\mathrm{Ext}(\bar{P}, P)$ contains extensions $P \rightarrowtail E \twoheadrightarrow \bar{P}$ where E is not completely decomposable as a pseudofree group of rank 2. Here we should note that for any extension $P \rightarrowtail E \twoheadrightarrow \bar{P}$ of $\bar{P}$ by P, E must be pseudofree of rank 2 since p-localization is exact and $\mathbb{Z}_p$ is a p.i.d. In this paper, we determine when it is possible for $\mathrm{Ext}(\bar{P}, P)$ to contain extensions $P \rightarrowtail E \twoheadrightarrow \bar{P}$ where E is a certain type of completely decomposable pseudofree group of rank 2, i.e. $E \cong B_1 \oplus B_2$ for particular groups of pseudo-integers B_1 and B_2.

In Section 1, we define the type of completely decomposable extension (which we will call *nontrivial*) that we hope to find by making clear our restrictions on B_1 and B_2. We describe, in Section 2, when P embeds into $B_1 \oplus B_2$ for arbitrary groups of pseudo-integers P, B_1, and B_2. Since we want the quotient resulting from such an embedding to be isomorphic to $\bar{P}$ (and hence torsionfree), in Section 3 we characterize the torsion subgroup of the quotient resulting from a given embedding of P into $B_1 \oplus B_2$. Finally, in Section 4, we have a theorem which allows us to determine when such a quotient is in fact isomorphic to $\bar{P}$. We then find exactly when $\mathrm{Ext}(\bar{P}, P)$ contains nontrivial completely decomposable extensions in terms of defining characteristics of $\bar{P}$ and P.

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1. Pseudo-Integers and Extensions.

For A, B abelian groups, we state an interpretation of $\mathrm{Ext}(A, B)$ which may be found in [**3**]. The extensions $B \rightarrowtail E_1 \twoheadrightarrow A$ and $B \rightarrowtail E_2 \twoheadrightarrow A$ are said to be equivalent if there exists $\psi : E_1 \to E_2$ so that the following diagram commutes:

$$\begin{array}{ccccc} B & \rightarrowtail & E_1 & \twoheadrightarrow & A \\ \| & & \downarrow \psi & & \| \\ B & \rightarrowtail & E_2 & \twoheadrightarrow & A \end{array}$$

Since ψ is necessarily an isomorphism, the above relation is an equivalence relation and $\mathrm{Ext}(A, B)$ is regarded as the set of equivalence classes of extensions. It may be shown to have an abelian group structure with zero element the equivalence class of the extension $B \rightarrowtail B \oplus A \twoheadrightarrow A$ where B embeds naturally into $B \oplus A$ and $B \oplus A$ projects naturally onto A.

In this paper we shall be concerned with certain extensions of the form $P \rightarrowtail E \twoheadrightarrow \bar{P}$ where P and $\bar{P}$ are groups of pseudo-integers and E is necessarily pseudofree of rank 2. However, before we consider particular extensions in $\mathrm{Ext}(\bar{P}, P)$ we will note some results previously obtained on the algebraic structure of $\mathrm{Hom}(\bar{P}, P)$ and $\mathrm{Ext}(\bar{P}, P)$ as abelian groups.

We first introduce notation to be used throughout the paper. In [**2**] it is demonstrated that if P is a group of pseudo-integers then we may assume $\mathbb{Z} \subseteq P \subseteq \mathbb{Q}$ and for any prime p_i there is a maximal exponent $r_i \geq 0$ so that $\frac{1}{p_i^{r_i}} \in P$. Moreover, as may be easily shown, P is then generated by the set $\{\frac{1}{p_i^{r_i}} \mid p_i \text{ is a prime}\}$ and the elements of P are represented by reduced fractions $\frac{a}{b}$ where b has prime power factors $p_i^{l_i}$, with $l_i \leq r_i$. We will henceforth denote P by $\langle \frac{1}{p_i^{r_i}} \rangle$ and assume all fractions mentioned to be reduced.

We will often employ the following result (also from [**2**]) concerning the isomorphism problem for groups of pseudo-integers.

Theorem. *Assume $P = \langle \frac{1}{p_i^{r_i}} \rangle$ and $H = \langle \frac{1}{p_i^{k_i}} \rangle$ are groups of pseudo-integers. Then $P \cong H$ if and only if $r_i = k_i$ almost everywhere.*

Now suppose $P = \langle \frac{1}{p_i^{r_i}} \rangle$ and $\bar{P} = \langle \frac{1}{p_i^{\bar{r}_i}} \rangle$ are groups of pseudo-integers. The above theorem implies that to determine $\mathrm{Hom}(\bar{P}, P)$ one need only consider the case where $\bar{r}_i \geq r_i$ for infinitely many i and the case where $\bar{r}_i \leq r_i$ everywhere. The following results are then obtained in [**5**] which show that $\mathrm{Hom}(\bar{P}, P)$ is either trivial or another group of pseudo-integers.

Theorem. *If $\bar{r}_i > r_i$ for infinitely many i then $Hom(P, P) = 0$.*

Theorem. *If $\bar{r}_i \leq r_i$ for all i then $Hom(\bar{P}, P) \cong \langle \frac{1}{p_i^{l_i}} \rangle$ where $l_i = r_i - \bar{r}_i$.*

A special case of a theorem proved for pseudofree groups of arbitrary rank then describes $\mathrm{Ext}(\bar{P}, P)$.

Theorem. *Suppose $\bar{P}$ and P are groups of pseudo-integers. Then $Ext(\bar{P}, P) \cong V \oplus (\mathbb{Q}/\mathbb{Z})^{1-l}$ where V is a $\mathbb{Q}$-vector space of rank c and l is the rank of $Hom(\bar{P}, P)$.*

In this paper, we shall consider extensions $P \rightarrowtail E \twoheadrightarrow \bar{P}$ in $\mathrm{Ext}(\bar{P}, P)$ where E is completely decomposable as a pseudofree group of rank 2, i.e. $E \cong B_1 \oplus B_2$ for B_1 and B_2 groups of pseudo-integers. We will sometimes refer to E itself as the extension. Now it is clear that one such extension is always present, i.e. the zero extension $P \rightarrowtail P \oplus \bar{P} \twoheadrightarrow \bar{P}$ in $\mathrm{Ext}(\bar{P}, P)$. So we wish to determine when $\mathrm{Ext}(\bar{P}, P)$ contains other, less obvious completely decomposable extensions. We therefore define a completely decomposable extension $P \rightarrowtail B_1 \oplus B_2 \twoheadrightarrow \bar{P}$ to be *trivial* if $P \cong B_1$, $\bar{P} \cong B_2$ or $P \cong B_2$, $\bar{P} \cong B_1$ and *nontrivial* otherwise. As we see from the following theorem, our definition guarantees that a nontrivial extension will not belong to the zero element of $\mathrm{Ext}(\bar{P}, P)$. Hence our goal will be to describe when $\mathrm{Ext}(\bar{P}, P)$ contains nontrivial completely decomposable extensions.

Theorem 1.1. *If $P \rightarrowtail B_1 \oplus B_2 \twoheadrightarrow \bar{P}$ is nontrivial in $Ext(\bar{P}, P)$ then the extension is also nonzero in $Ext(\bar{P}, P)$.*

Proof: Suppose $P \rightarrowtail B_1 \oplus B_2 \twoheadrightarrow \bar{P}$ does belong to the zero element in $\mathrm{Ext}(\bar{P}, P)$. This implies that $P \rightarrowtail B_1 \oplus B_2 \twoheadrightarrow \bar{P}$ is equivalent to the natural extension $P \rightarrowtail P \oplus \bar{P} \twoheadrightarrow \bar{P}$. Hence $B_1 \oplus B_2$ is necessarily isomorphic to $P \oplus \bar{P}$. Now in [4] it is shown that the decomposition of a pseudofree group of rank 2 must be unique, i.e. if $P \oplus \bar{P} \cong B_1 \oplus B_2$ then $P \cong B_1$, $\bar{P} \cong B_2$ or $P \cong B_2$, $\bar{P} \cong B_1$. Thus we have a contradiction since we assumed our extension to be nontrivial. ∎

2. Embeddings.

Since we wish to find extensions of the form $P \rightarrowtail B_1 \oplus B_2 \twoheadrightarrow \bar{P}$, we must first determine when it is possible to embed P into $B_1 \oplus B_2$. We note that since P, B_1, B_2 are subgroups of $\mathbb{Q}$, to describe an embedding of P into $B_1 \oplus B_2$ we need only indicate the ordered pair to which $1 \in P$ is sent. For, if $1 \mapsto (\frac{u_1}{v_1}, \frac{u_2}{v_2})$ in $B_1 \oplus B_2$ then $\frac{a}{b} \mapsto (\frac{u_1 a}{v_1 b}, \frac{u_2 a}{v_2 b})$ for any $\frac{a}{b} \in P$.

We shall consider only embeddings of the form $1 \mapsto (\frac{u_1}{v_1}, \frac{u_2}{v_2})$ where $u_1 \neq 0$ and $u_2 \neq 0$ as otherwise, either the quotient has torsion (and hence could not be isomorphic to $\bar{P}$) or we are led to a trivial extension. For suppose that $P \rightarrowtail B_1 \oplus B_2 \twoheadrightarrow \bar{P}$ is an extension in $\mathrm{Ext}(\bar{P}, P)$ so that P embeds into $B_1 \oplus B_2$ with the embedding $1 \mapsto (\frac{u_1}{v_1}, 0)$ for some $u_1 \neq 0$. Now $\frac{B_1 \oplus B_2}{P} \cong B_1/P \oplus B_2$ and $\frac{B_1 \oplus B_2}{P} \cong \bar{P}$. But B_1/P is a torsion group and hence must be trivial which indicates that $B_1 \cong P$. Now B_2 must then be isomorphic to $\bar{P}$ and we see that the extension is trivial. Of course an embedding of the form $1 \mapsto (0, \frac{u_2}{v_2})$ for some $u_2 \neq 0$ yields a similar result.

The next two theorems address the problem of embedding P into $B_1 \oplus B_2$. The first describes when a specific homomorphism from P to $\mathbb{Q}\oplus\mathbb{Q}$ is an embedding and the second shows when it is possible, in terms of the exponents of P, B_1, and B_2, to embed P into $B_1 \oplus B_2$. For $n \in \mathbb{Z}$ and p_i a prime, let $v_{p_i}(n)$ be the usual p_i-valuation of n, i.e. the highest power of p_i that divides n. Then for $\frac{u}{v}$ in $\mathbb{Q}$, we define $v_{p_i}(\frac{u}{v}) = v_{p_i}(u) - v_{p_i}(v)$. Finally, let $P = \langle \frac{1}{p_i^{r_i}} \rangle$, $B_1 = \langle \frac{1}{p_i^{m_i}} \rangle$, and $B_2 = \langle \frac{1}{p_i^{n_i}} \rangle$ represent the groups of pseudo-integers.

Theorem 2.1. *The embedding of P into $\mathbb{Q} \oplus \mathbb{Q}$ defined by $1 \mapsto (\frac{u_1}{v_1}, \frac{u_2}{v_2})$ is an embedding of P into $B_1 \oplus B_2$ if and only if $r_i - v_{p_i}(\frac{u_1}{v_1}) \leq m_i$ and $r_i - v_{p_i}(\frac{u_2}{v_2}) \leq n_i$ for all i.*

Proof: Now $1 \mapsto (\frac{u_1}{v_1}, \frac{u_2}{v_2})$ restricts to an embedding of P into $B_1 \oplus B_2$ if and only if $\frac{1}{p_i^{r_i}} \mapsto (\frac{u_1}{v_1 p_i^{r_i}}, \frac{u_2}{v_2 p_i^{r_i}}) \in B_1 \oplus B_2$ for all i. But $\frac{u_1}{v_1 p_i^{r_i}} \in B_1 \iff v_{p_i}(\frac{u_1}{v_1 p_i^{r_i}}) \geq -m_i \iff v_{p_i}(\frac{u_1}{v_1}) - r_i \geq -m_i \iff r_i - v_{p_i}(\frac{u_1}{v_1}) \leq m_i$. Similarly, $\frac{u_2}{v_2 p_i^{r_i}} \in B_2 \iff r_i - v_{p_i}(\frac{u_2}{v_2}) \leq n_i$. ■

Note that, in particular, $1 \mapsto (1,1)$ defines an embedding of P into $B_1 \oplus B_2$ if and only if $r_i \leq m_i$ and $r_i \leq n_i$ for all i.

Theorem 2.2. *The group of pseudo-integers P embeds into $B_1 \oplus B_2$ if and only if $r_i \leq m_i$ and $r_i \leq n_i$ almost everywhere.*

Proof: Assume P embeds into $B_1 \oplus B_2$ and let $1 \mapsto (\frac{u_1}{v_1}, \frac{u_2}{v_2})$ be an embedding. Since $v_{p_i}(\frac{u_1}{v_1}) = 0$ and $v_{p_i}(\frac{u_2}{v_2}) = 0$ for all but a finite number of i, it is clear from Theorem 2.1 that $r_i \leq m_i$, $r_i \leq n_i$ almost everywhere.

If we suppose that $r_i \leq m_i$ and $r_i \leq n_i$ almost everywhere, we can easily construct an embedding. Let $S_1 = \{j \mid r_j > m_j\}$ and $S_2 = \{k \mid r_k > n_k\}$ and consider the embedding of B into $\mathbb{Q} \oplus \mathbb{Q}$ defined by $1 \mapsto (\prod_{j \in S_1} p_j^{r_j - m_j}, \prod_{k \in S_2} p_k^{r_k - n_k})$. It can be readily verified that $r_i - v_{p_i}(\prod_{j \in S_1} p_j^{r_j - m_j}) \leq m_i$ and $r_i - v_{p_i}(\prod_{k \in S_2} p_k^{r_k - m_k}) \leq n_i$ for all i and hence, by Theorem 2.1, we have an embedding of P into $B_1 \oplus B_2$. ■

3. Torsion Subgroups of Quotients.

Recall that our goal is to determine when it is possible to find non-trivial completely decomposable extensions in $\operatorname{Ext}(\bar{P}, P)$ i.e. extensions of the form $P \rightarrowtail B_1 \oplus B_2 \twoheadrightarrow \bar{P}$ where certain conditions are placed on B_1, B_2. Since, for such an extension to exist, the quotient $\frac{B_1 \oplus B_2}{P}$ must be isomorphic to $\bar{P}$, we must determine when it is possible to embed P into $B_1 \oplus B_2$ so that the resulting quotient is torsionfree. Hence in this section we characterize the torsion subgroup of $\frac{B_1 \oplus B_2}{P}$ with respect to a given embedding. This characterization, given in the next two theorems, is essential to our determination of the existence of completely decomposable extensions in $\mathrm{Ext}(\bar{P}, P)$. Given an embedding of P into $B_1 \oplus B_2$, we will use Q to represent the resulting quotient of $B_1 \oplus B_2$ by P, i.e. $Q = \frac{B_1 \oplus B_2}{P}$. We will represent the torsion subgroup of this quotient by TQ. Also, we will assume $P = \langle \frac{1}{p_i^{r_i}} \rangle$, $B_1 = \langle \frac{1}{p_i^{m_i}} \rangle$, and $B_2 = \langle \frac{1}{p_i^{n_i}} \rangle$.

Theorem 3.1. *Suppose P embeds into $B_1 \oplus B_2$ with the embedding defined by $1 \mapsto (1,1)$. Then $TQ \cong \bigoplus_{p_i} \mathbb{Z}/p_i^{l_i - r_i}\mathbb{Z}$ where $l_i = \min(m_i, n_i)$.*

Proof: We first show that $TQ = \frac{\{(\frac{a}{b}, \frac{a}{b}) \mid \frac{a}{b} \in B_1 \cap B_2\}}{P}$. Suppose $[(\frac{a_1}{b_1}, \frac{a_2}{b_2})]$ is a coset in TQ. There must exist an integer k so that $k(\frac{a_1}{b_1}, \frac{a_2}{b_2}) \in \operatorname{im} P = \{(x,x) \mid x \in P\}$ (since $1 \mapsto (1,1)$ gives the embedding). Hence $(\frac{a_1}{b_1}, \frac{a_2}{b_2}) = (\frac{x}{k}, \frac{x}{k})$ for some $x \in P$ and $[(\frac{a_1}{b_1}, \frac{a_2}{b_2})] \in \frac{\{(\frac{a}{b}, \frac{a}{b}) \mid \frac{a}{b} \in B_1 \cap B_2\}}{P}$. Now assume $[(\frac{a}{b}, \frac{a}{b})]$ is a coset in $\frac{\{(\frac{a}{b}, \frac{a}{b}) \mid \frac{a}{b} \in B_1 \cap B_2\}}{P}$. Since $b(\frac{a}{b}, \frac{a}{b}) = (a, a) \in \operatorname{im} P$ in $B_1 \oplus B_2$, $[(\frac{a}{b}, \frac{a}{b})] \in TQ$.

So $TQ = \frac{\{(\frac{a}{b}, \frac{a}{b}) \mid \frac{a}{b} \in B_1 \cap B_2\}}{P} \cong \frac{B_1 \cap B_2}{P}$. Now $B_1 \cap B_2 = \langle \frac{1}{p_i^{l_i}} \rangle$ where $l_i = \min(m_i, n_i)$. Hence $\frac{B_1 \cap B_2}{P} = \langle \frac{1}{p_i^{l_i}} \rangle / P \cong \bigoplus_{p_i} \mathbb{Z}/p_i^{l_i - r_i}\mathbb{Z}$ since a torsion abelian group is the direct sum of its p_i-torsion subgroups. Note that $l_i - r_i = \min(m_i, n_i) - r_i \geq 0$ for all i, since $r_i \leq m_i$ and $r_i \leq n_i$ for all i by Theorem 2.1. ■

Now we use Theorem 3.1 to determine the torsion subgroup resulting from an arbitrary embedding.

Theorem 3.2. *Suppose P embeds into $B_1 \oplus B_2$ with the embedding defined by $1 \mapsto (\frac{u_1}{v_1}, \frac{u_2}{v_2})$. Then $TQ \cong \bigoplus_{p_i} \mathbb{Z}/p_i^{l_i' - r_i}\mathbb{Z}$ where $l_i' = \min(m_i + v_{p_i}(\frac{u_1}{v_1}), n_i + v_{p_i}(\frac{u_2}{v_2}))$.*

Proof: Consider the following commutative diagram

$$\begin{array}{ccc} 1 & \mapsto & (\frac{u_1}{v_1}, \frac{u_2}{v_2}) \\ P & \rightarrowtail & B_1 \oplus B_2 \\ \| & & \downarrow \frac{v_1}{u_1} \oplus \frac{v_2}{u_2} \\ P & \rightarrowtail & B_1' \oplus B_2' \\ 1 & \mapsto & (1,1) \end{array}$$

where $B_1' = \langle \frac{1}{p_i^{m_i'}} \rangle$ with $m_i' = m_i + v_{p_i}(\frac{u_1}{v_1})$, $B_2' = \langle \frac{1}{p_i^{n_i'}} \rangle$ with $n_i' = n_i + v_{p_i}(\frac{u_2}{v_2})$, and $\frac{v_1}{u_1} \oplus \frac{v_2}{u_2} : B_1 \oplus B_2 \cong B_1' \oplus B_2'$ is the isomorphism defined as multiplication by $\frac{v_1}{u_1}$ in the first coordinate and by $\frac{v_2}{u_2}$ in the second. Note that $1 \mapsto (1,1)$ does define an embedding of P into $B_1' \oplus B_2'$. For, since $1 \mapsto (\frac{u_1}{v_1}, \frac{u_2}{v_2})$ is an embedding of P into $B_1 \oplus B_2$, $r_i \leq m_i + v_{p_i}(\frac{u_1}{v_1}) = m_i'$ and $r_i \leq n_i + v_{p_i}(\frac{u_2}{v_2}) = n_i'$ for all i by Theorem 2.1. Thus $1 \mapsto (1,1)$ is an embedding of P into $B_1' \oplus B_2'$ also by Theorem 2.1. Let $Q' = \frac{B_1' \oplus B_2'}{P}$ be the quotient resulting from this embedding. Now $Q \cong Q'$ since $\frac{v_1}{u_1} \oplus \frac{v_2}{u_2}$ induces an isomorphism on the quotients. Hence $TQ \cong TQ' \cong \bigoplus_{p_i} \mathbb{Z}/p_i^{l_i'-r_i}\mathbb{Z}$ where $l_i' = \min(m_i', n_i')$ by Theorem 3.1, and the theorem is proved. ■

The theorems in the remainder of this section are not necessary to prove our eventual result on the existence of completely decomposable extensions in $\mathrm{Ext}(\bar{P}, P)$. However, given that P, B_1, and B_2 are groups of pseudo-integers so that it is possible to embed P into $B_1 \oplus B_2$, the theorems provide insight into the nature of the torsion subgroups of $Q = \frac{B_1 \oplus B_2}{P}$ which occur with respect to different embeddings. For, given one such torsion subgroup, we are able to determine all other torsion subgroups which it is possible to obtain. We also find, in terms of the exponents of P, B_1, and B_2, when it is possible to embed P with a torsionfree quotient. Moreover, we describe under what conditions a given embedding will have a quotient that is torsionfree. Again, we let $P = \langle \frac{1}{p_i^{r_i}} \rangle$, $B_1 = \langle \frac{1}{p_i^{m_i}} \rangle$, and $B_2 = \langle \frac{1}{p_i^{n_i}} \rangle$ represent the groups of pseudo-integers.

Theorem 3.3. *Suppose P embeds into $B_1 \oplus B_2$ with $TQ \cong \bigoplus_{p_i} \mathbb{Z}/p_i^{k_i}$, $k_i \geq 0$. Let $\bigoplus_{p_i} \mathbb{Z}/p_i^{k_i'}$ with $k_i' \geq 0$ be a torsion group. There exists an embedding of P into $B_1 \oplus B_2$ so that $TQ \cong \bigoplus_{p_i} \mathbb{Z}/p_i^{k_i'}$ if and only if $k_i = k_i'$ almost everywhere.*

Proof: Let $1 \mapsto (\frac{u_1}{v_1}, \frac{u_2}{v_2})$ be an embedding of P into $B_1 \oplus B_2$ so that $TQ \cong \bigoplus_{p_i} \mathbb{Z}/p_i^{k_i}\mathbb{Z}$. We know that $r_i - v_{p_i}(\frac{u_1}{v_1}) \leq m_i$ and $r_i - v_{p_i}(\frac{u_2}{v_2}) \leq n_i$ for all i (Theorem 2.1). Also, by Theorem 3.2, $k_i = l_i - r_i$ where $l_i = \min(m_i + v_{p_i}(\frac{u_1}{v_1}), n_i + v_{p_i}(\frac{u_2}{v_2}))$.

Now assume $1 \mapsto (\frac{a_1}{b_1}, \frac{a_2}{b_2})$ is an embedding of P into $B_1 \oplus B_2$ so that $TQ \cong \bigoplus_{p_i} \mathbb{Z}/p_i^{k_i'}\mathbb{Z}$. As above, $k_i' = l_i' - r_i$ where $l_i' = \min(m_i + v_{p_i}(\frac{a_1}{b_1}), n_i +$

$v_{p_i}(\frac{a_2}{b_2}))$. Since $v_{p_i}(\frac{a_1}{b_1}) = v_{p_i}(\frac{a_2}{b_2}) = v_{p_i}(\frac{u_1}{v_1}) = v_{p_i}(\frac{u_2}{v_2}) = 0$ for almost all i, $l_i = l_i' = \min(m_i, n_i)$ almost everywhere. Hence $k_i = k_i'$ almost everywhere.

Now suppose $k_i = k_i'$ almost everywhere. Let $S_1 = \{i \mid k_i' > k_i\}$, $S_2 = \{i \mid k_i > k_i'\}$, $u = \prod_{i \in S_1} p_i^{k_i' - k_i}$, and $v = \prod_{i \in S_2} p_i^{k_i - k_i'}$. Note that if $i \in S_1 \cup S_2$ then $v_{p_i}(\frac{u}{v}) = k_i' - k_i$ and otherwise $v_{p_i}(\frac{u}{v}) = 0$. We can show that $1 \mapsto (\frac{uu_1}{vv_1}, \frac{uu_2}{vv_2})$ is an embedding of P into $B_1 \oplus B_2$. For if $i \notin S_1 \cup S_2$, $r_i - v_{p_i}(\frac{uu_1}{vv_1}) = r_i - v_{p_i}(\frac{u_1}{v_1}) \leq m_i$ and $r_i - v_{p_i}(\frac{uu_2}{vv_2}) = r_i - v_{p_i}(\frac{u_2}{v_2}) \leq n_i$. If $i \in S_1 \cup S_2$, $r_i - v_{p_i}(\frac{uu_1}{vv_1}) = r_i - v_{p_i}(\frac{u_1}{v_1}) - (k_i' - k_i) = r_i - v_{p_i}(\frac{u_1}{v_1}) - k_i' + \min(m_i + v_{p_i}(\frac{u_1}{v_1}), n_i + v_{p_i}(\frac{u_2}{v_2})) - r_i \leq -v_{p_i}(\frac{u_1}{v_1}) - k_i' + m_i + v_{p_i}(\frac{u_1}{v_1}) = m_i - k_i' \leq m_i$ as $k_i' \geq 0$. An analogous argument shows that $r_i - v_{p_i}(\frac{uu_2}{vv_2}) \leq n_i$. Hence, we have an embedding by Theorem 2.1.

We see that the torsion subgroup of Q with this embedding is $\bigoplus_{p_i} \mathbb{Z}/p_i^{k_i'}\mathbb{Z}$. From Theorem 3.2, $TQ \cong \bigoplus_{p_i} \mathbb{Z}/p_i^{l_i' - r_i}$ where $l_i' = \min(m_i + v_{p_i}(\frac{uu_1}{vv_1}), n_i + v_{p_i}(\frac{uu_2}{vv_2})) = l_i + v_{p_i}(\frac{u}{v})$. If $i \notin S_1 \cup S_2$, then $l_i' = l_i$ and $l_i' - r_i = l_i - r_i = k_i = k_i'$. If $i \in S_1 \cup S_2$, $l_i' = l_i + v_{p_i}(\frac{u}{v}) = l_i + (k_i' - k_i) = k_i' + r_i$ and hence $l_i' - r_i = k_i'$. ■

Corollary 3.4. *Let P, B_1, and B_2 be groups of pseudo- integers so that P embeds into $B_1 \oplus B_2$. Then TQ is finite for all embeddings of P into $B_1 \oplus B_2$ or TQ is infinite for all embeddings of P into $B_1 \oplus B_2$. Moreover, if TQ is always finite then P embeds into $B_1 \oplus B_2$ with torsionfree quotient.*

For the next theorems we establish the following notation. Let $\sigma^+(P) = \{i \mid r_i > 0\}$ and $B_1 \backslash P = \langle \frac{1}{p_i^{s_i}} \rangle$ where $s_i = \max(m_i - r_i, 0)$.

Theorem 3.5. *Suppose P embeds into $B_1 \oplus B_2$. There exists an embedding of P into $B_1 \oplus B_2$ so that Q is torsionfree if and only if $\sigma^+(B_1 \backslash P) \cap \sigma^+(B_2 \backslash P)$ is finite.*

Proof: Suppose $\sigma^+(B_1 \backslash P) \cap \sigma^+(B_2 \backslash P)$ is infinite. Let $1 \mapsto (\frac{u_1}{v_1}, \frac{u_2}{v_2})$ be an arbitrary embedding of P into $B_1 \oplus B_2$. Select p_i so that $i \in \sigma^+(B_1 \backslash P) \cap \sigma^+(B_2 \backslash P)$ and $v_{p_i}(v_1) = v_{p_i}(v_2) = 0$. Then note that $x = (\frac{u_1}{v_1 p_i^{r_i+1}}, \frac{u_2}{v_2 p_i^{r_i+1}}) \in B_1 \oplus B_2$. Certainly $x \notin \operatorname{im} P \subseteq B_1 \oplus B_2$ but $p_i x$ is the image of $\frac{1}{p_i^{r_i}} \in P$ in $B_1 \oplus B_2$. Hence Q has torsion.

Now assume that $\sigma^+(B_1 \backslash P) \cap \sigma^+(B_2 \backslash P)$ is finite. Let $S_1 = \{i \mid r_i > m_i \text{ or } r_i > n_i\}$ and note that, by Theorem 2.2, S_1 is finite. Let $S_2 = \{i \mid r_i < m_i \text{ and } r_i < n_i\}$ and also note that S_2 is finite since

$\sigma^+(B_1\backslash P)\cap\sigma^+(B_2\backslash P)$ is finite. Now if $i\notin S_1\cup S_2$ then $r_i\le m_i$, n_i and $r_i=m_i$ or $r_i=n_i$. Let $1\mapsto(\frac{u_1}{v_1},\frac{u_2}{v_2})$ be an embedding of P into $B_1\oplus B_2$. By Theorem 3.2, $TQ\cong\bigoplus_{p_i}\mathbb{Z}p_i^{l_i-r_i}$ where $l_i=\min(m_i+v_{p_i}(\frac{u_1}{v_1}),n_i+v_{p_i}(\frac{u_2}{v_2}))$. Let $S_3=\{i\mid v_{p_i}(\frac{u_1}{v_1})\neq 0$ or $v_{p_i}(\frac{u_2}{v_2})\neq 0\}$ so that if $i\notin S_3$ then $l_i=\min(m_i,n_i)$. Now if $i\notin S_1\cup S_2\cup S_3$ then $l_i=\min(m_i,n_i)=r_i$ and hence TQ has only a finite number of nontrivial p-components. Thus there exists an embedding of P into $B_1\oplus B_2$ so that the torsion subgroup of the quotient is trivial (Theorem 3.3). ■

So given that P embeds into $B_1\oplus B_2$ with torsionfree quotient, we know that $r_i\le m_i$, n_i for almost all i and $\sigma^+(B_1\backslash P)\cap\sigma^+(B_2\backslash P)$ is finite. The following theorem enables us to decide when a given embedding will have a torsionfree quotient in the special case where $r_i\le m_i$, n_i for all i and $\sigma^+(B_1\backslash P)\cap\sigma^+(B_2\backslash P)$ is empty. Although we will not do so explicitly here, we could use a commutative diagram as in the proof of Theorem 3.2 to extend the result to the more general case.

Theorem 3.6. *Suppose P embeds into $B_1\oplus B_2$ with torsionfree quotient. Assume also that $r_i\le m_i$, n_i for all i and $\sigma^+(B_1\backslash P)\cap\sigma^+(B_2\backslash P)$ is empty. Let $1\mapsto(\frac{u_1}{v_1},\frac{u_2}{v_2})$ determine an embedding of P into $B_1\oplus B_2$. Then Q has torsion if and only if there exists p_i so that $p_i\mid u_1$ or γ_1 holds and $p_i\mid u_2$ or γ_2 holds where γ_1 is the condition that $i\in\sigma^+(B_1\backslash P)$ and $v_{p_i}(v_1)<m_i-r_i$, γ_2 is the condition that $i\in\sigma^+(B_2\backslash P)$ and $v_{p_i}(v_2)<n_i-r_i$.*

Now suppose Q has torsion and let $[(\frac{a_1}{b_1},\frac{a_2}{b_2})]$ be a nontrivial coset of finite order. Hence there exists $k\in\mathbb{Z}$ ($k>1$), $\frac{c}{d}\in P$, so that $k(\frac{a_1}{b_1},\frac{a_2}{b_2})=(\frac{cu_1}{dv_1},\frac{cu_2}{dv_2})\in B_1\oplus B_2$. So $\frac{a_1}{b_1}=\frac{cu_1}{kdv_1}$ and $\frac{a_2}{b_2}=\frac{cu_2}{kdv_2}$, where we may assume $\frac{c}{kd}$ to be reduced (if it were not, we would replace $\frac{c}{kd}$ with its reduced form). Since our chosen coset was nontrivial, $\frac{c}{kd}\notin P$ and hence there exists p_i so that $v_{p_i}(kd)>r_i$. Now $(\frac{a_1}{b_1},\frac{a_2}{b_2})=(\frac{cu_1}{kdv_1},\frac{cu_2}{kdv_2})\in B_1\oplus B_2$ and $\sigma^+(B_1\backslash P)\cap\sigma^+(B_2\backslash P)$ is empty so $p_i\mid u_1$ or $p_i\mid u_2$. We assume $p_i\mid u_1$ and note that if $p_i\mid u_2$ also, there is nothing more to prove. Otherwise, since $v_{p_i}(kd)>r_i$ and $\frac{cu_2}{kdv_2}\in B_2$, we have $i\in\sigma^+(B_2\backslash P)$ and $v_{p_i}(v_2)<n_i-r_i$, i.e., we have condition γ_2. Similarly, if we had assumed $p_i\mid u_2$ then we could conclude that $p_i\mid u_1$ or γ_1 is true.

Let p_i be a prime so that $p_i\mid u_1$ or γ_1 holds and $p_i\mid u_2$ or γ_2 holds. If $p_i\mid u_1$ and $p_i\mid u_2$ then $u_1=p_iu_1'$ and $u_2=p_iu_2'$. Now $x=(\frac{u_1'}{p_i^{r_i}v_1},\frac{u_2'}{p_i^{r_i}v_2})\in B_1\oplus B_2$ since we assume $\frac{u_1}{v_1},\frac{u_2}{v_2}$ are reduced and $r_i\le m_i$, n_i for all i. Note that $x\notin\operatorname{im}P$ in $B_1\oplus B_2$ but p_ix is the image of $\frac{1}{p_i^{r_i}}$ in $B_1\oplus B_2$. Thus $[x]$ is a nontrivial torsion element in Q.

Now assume $p_i \mid u_1$ and γ_2 holds and again let $u_1 = p_i u_1'$. Observe that $x = (\frac{u_1'}{p_i^{r_i} v_1}, \frac{u_2}{p_i^{r_i+1} v_2}) \in B_1 \oplus B_2$ for as above $\frac{u_1'}{p_i^{r_i} v_1} \in B_1$ and γ_2 ensures that $\frac{u_2}{p_i^{r_i+1} v_2} \in B_2$. Also as above $[x]$ is an element of order p_i in Q. A symmetric argument shows that Q has torsion if we assume condition γ_1 and that $p_i \mid u_2$. Since $\sigma^+(B_1 \backslash P) \cap \sigma^+(B_2 \backslash P)$ is empty the case where both γ_1 and γ_2 hold cannot occur. ■

4. Torsionfree Quotients.

We are almost ready to describe when $\mathrm{Ext}(\bar{P}, P)$ contains nontrivial completely decomposable extensions. Recall that this means that we seek the existence of groups of pseudo-integers B_1 and B_2 ($P \not\cong B_1$ or $\bar{P} \not\cong B_2$ and $P \not\cong B_2$ or $\bar{P} \not\cong B_1$) so that P embeds into $B_1 \oplus B_2$ with quotient isomorphic to $\bar{P}$. Hence, given that P embeds into $B_1 \oplus B_2$, we now characterize the torsionfree quotient of $Q = \frac{B_1 \oplus B_2}{P}$, denoted Q/TQ, with respect to any embedding. This allows us, in the particular case when the torsion subgroup TQ is trivial, to decide if Q is isomorphic to $\bar{P}$. Again we represent our groups of pseudo-integers by $P = \langle \frac{1}{p_i^{r_i}} \rangle$, $B_1 = \langle \frac{1}{p_i^{m_i}} \rangle$, and $B_2 = \langle \frac{1}{p_i^{n_i}} \rangle$.

Theorem 4.1. *Suppose that P, B_1, and B_2 are groups of pseudo-integers so that P embeds into $B_1 \oplus B_2$ with quotient Q. Then the torsionfree quotient of Q, denoted Q/TQ, with respect to any embedding of P into $B_1 \oplus B_2$ is isomorphic to $\langle \frac{1}{p_i^{s_i}} \rangle$ where $s_i = \max(m_i, n_i)$.*

Proof: Since P embeds into $B_1 \oplus B_2$, we know that $r_i \le m_i$, n_i for almost all i by Theorem 2.2. Our argument considers the two cases that $r_i \le m_i$, n_i for all i and $r_i \not\le m_i$, n_i for all i. We suppose first that $r_i \le m_i, n_i$ for all i so that $1 \mapsto (1,1)$ determines an embedding of P into $B_1 \oplus B_2$ (Theorem 2.1). By the proof of Theorem 3.1, we know that TQ with respect to this embedding is $\frac{\{(\frac{a}{b}, \frac{a}{b}) \mid \frac{a}{b} \in B_1 \cap B_2\}}{P}$ and hence $Q/TQ = \frac{\frac{B_1 \oplus B_2}{P}}{\frac{\{(\frac{a}{b}, \frac{a}{b}) \mid \frac{a}{b} \in B_1 \cap B_2\}}{P}} \simeq \frac{B_1 \oplus B_2}{\{(\frac{a}{b}, \frac{a}{b}) \mid \frac{a}{b} \in B_1 \cap B_2\}}$. Now consider the following sequence

$$\left\{ \left(\frac{a}{b}, \frac{a}{b} \right) \mid \frac{a}{b} \in B_1 \cap B_2 \right\} \overset{i}{\rightarrowtail} B_1 \oplus B_2 \overset{\pi}{\twoheadrightarrow} \left\langle \frac{1}{p_i^{s_i}} \right\rangle$$

where i is inclusion, π is defined by $\pi(\frac{a_1}{b_1}, \frac{a_2}{b_2}) = \frac{a_1}{b_1} - \frac{a_2}{b_2}$, and $s_i = \max(m_i, n_i)$. We see that the sequence is exact. First we note that im $i \subseteq \ker \pi$. Now suppose $(\frac{a_1}{b_1}, \frac{a_2}{b_2}) \in \ker \pi$. Then $\frac{a_1}{b_1} - \frac{a_2}{b_2} = 0$ which implies $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and hence that $(\frac{a_1}{b_1}, \frac{a_2}{b_2}) \in$ im i. Finally, we show that π

is onto. Let $\frac{1}{p_i^{s_i}}$ be an arbitrary generator of $\langle\frac{1}{p_i^{s_i}}\rangle$. Now $s_i = \max(m_i, n_i)$ and so if $s_i = m_i$ then $\pi(\frac{1}{p_i^{m_i}}, 0) = \frac{1}{p_i^{s_i}}$. Otherwise $(0, -\frac{1}{p_i^{n_i}}) = \frac{1}{p_i^{s_i}}$. Our sequence is thus short exact and $Q/TQ \cong \frac{B_1 \oplus B_2}{\{(\frac{a}{b}, \frac{a}{b}) \mid \frac{a}{b} \in B_1 \cap B_2\}} \cong \langle\frac{1}{p_i^{s_i}}\rangle$.

We now assume that $r_i \not\leq m_i, n_i$ for all i and we let $1 \mapsto (\frac{u_1}{v_1}, \frac{u_2}{v_2})$ determine an arbitrary embedding of P into $B_1 \oplus B_2$. Consider the following commutative diagram

$$\begin{array}{lllll}
1 & \mapsto & (\frac{u_1}{v_1}, \frac{u_2}{v_2}) & & \\
P & \rightarrowtail & B_1 \oplus B_2 & \twoheadrightarrow & Q \\
\| & & \downarrow \psi = \frac{v_1}{u_1} \oplus \frac{v_2}{u_2} & & \downarrow \bar{\psi} \\
P & \rightarrowtail & B_1' \oplus B_2' & \twoheadrightarrow & Q' \\
1 & \mapsto & (1, 1) & &
\end{array}$$

where $B_1' = \langle\frac{1}{p_i^{m_i'}}\rangle$ with $m_i' = m_i + v_{p_i}(\frac{u_1}{v_1})$, $B_2' = \langle\frac{1}{p_i^{n_i'}}\rangle$ with $n_i' = n_i + v_{p_i}(\frac{u_2}{v_2})$, $Q = \frac{B_1 \oplus B_2}{P}$, $Q' = \frac{B_1' \oplus B_2'}{P}$, ψ is the isomorphism defined by multiplication by $\frac{v_1}{u_1}$ in the first coordinate and by $\frac{v_2}{u_2}$ in the second, and $\bar{\psi}$ is the isomorphism induced by ψ on the quotients. We note that $1 \mapsto (1, 1)$ is an embedding of P into $B_1' \oplus B_2'$. For, since $1 \mapsto (\frac{u_1}{v_1}, \frac{u_2}{v_2})$ is an embedding of P into $B_1 \oplus B_2$, $r_i - v_{p_i}(\frac{u_1}{v_1}) \leq m_i$ and $r_i - v_{p_i}(\frac{u_2}{v_2}) \leq n_i$ for all i by Theorem 2.1. Hence $r_i \leq m_i + v_{p_i}(\frac{u_1}{v_1}) = m_i'$ and $r_i \leq n_i + v_{p_i}(\frac{u_2}{v_2}) = n_i'$ for all i and again by Theorem 2.1, $1 \mapsto (1, 1)$ is an embedding of P into $B_1' \oplus B_2'$.

Since $\bar{\psi}$ is an isomorphism from Q to Q', $\bar{\psi}$ restricts to $\psi' : TQ \cong TQ'$. Thus we have the following commutative diagram

$$\begin{array}{ccccc}
TQ & \rightarrowtail & Q & \twoheadrightarrow & Q/TQ \\
\downarrow \psi' & & \downarrow \bar{\psi} & & \downarrow \hat{\psi} \\
TQ' & \rightarrowtail & Q' & \twoheadrightarrow & Q'/TQ'
\end{array}$$

where $\hat{\psi}$ is the isomorphism induced by $\bar{\psi}$ on the torsionfree quotients. Now P embeds into $B_1' \oplus B_2'$ by $1 \mapsto (1, 1)$ so by an argument identical to the above $Q'/TQ' \cong \langle\frac{1}{p_i^{s_i'}}\rangle$ where $s_i' = \max(m_i', n_i') = \max(m_i + v_{p_i}(\frac{u_1}{v_1}), n_1 + v_{p_i}(\frac{u_2}{v_2}))$. Let $s_i = \max(m_i, n_i)$ and note that $s_i = s_i'$ almost everywhere. Hence $\langle\frac{1}{p_i^{s_i'}}\rangle \cong \langle\frac{1}{p_i^{s_i}}\rangle$ and thus $Q/TQ \cong Q'/TQ' \cong \langle\frac{1}{p_i^{s_i'}}\rangle \cong \langle\frac{1}{p_i^{s_i}}\rangle$ where $s_i = \max(m_i, n_i)$. We remark that if P embeds into $B_1 \oplus B_2$ with torsionfree quotient then $Q \cong \langle\frac{1}{p_i^{s_i}}\rangle$. ∎

All the theorems in Sections 2 and 3 generalize in the obvious ways to the case where P is embedded into $\bigoplus_{j=1}^{k} B_j$, $B_j = \langle \frac{1}{p_i^{m_{ij}}} \rangle$, the direct sum of k groups of pseudo-integers for $k > 2$. However, we see that Theorem 4.1 does not generalize. For, we might expect that if P embeds into $\bigoplus_{j=1}^{k} B_j$ with quotient $Q = \bigoplus_{j=1}^{k} B_j/P$ then the torsionfree quotient Q/TQ would be isomorphic to $\bigoplus_{j=1}^{k-1} B'_j$ where $B'_j = \langle \frac{1}{p_i^{s_{ij}}} \rangle$, $s_{ij} = \max(m_{ij}, m_{i_{j+1}})$. But this is not always true. It is certainly the case that $TQ \cong \frac{\{(\frac{a}{b}, \frac{a}{b}, \cdots, \frac{a}{b}) \mid \frac{a}{b} \in \bigcap_{j=1}^{k} B_j\}}{B}$ and hence $Q/TQ \cong \frac{\bigoplus_{j=1}^{k} B_j}{\{(\frac{a}{b}, \frac{a}{b}, \cdots, \frac{a}{b}) \mid \frac{a}{b} \in \bigcap_{j=1}^{k} B_j\}}$.

Further , we can show that the following sequence is exact:

$$\left\{ \left(\frac{a}{b}, \frac{a}{b}, \cdots, \frac{a}{b} \right) \mid \frac{a}{b} \in \bigcap_{j=1}^{k} B_j \right\} \overset{i}{\rightarrowtail} \bigoplus_{j=1}^{k} B_j \overset{\pi}{\to} \bigoplus_{j=1}^{k-1} B'_j$$

where i is inclusion map and $\pi : \bigoplus_{j=1}^{k} B_j \to \bigoplus_{j=1}^{k-1} B'_j$ is defined by $\pi(b_1, b_2, \cdots, b_k) = (b_1 - b_2, b_2 - b_3, \cdots, b_{k-1} - b_k)$. However, it is not in general true that π is onto. For suppose for some p_i, $m_{i_1} = 1$, $m_{i_2} = 2$, $m_{i_3} = 1$ so that $\max(m_{i_1}, m_{i_2}) = 2$ and $\max(m_{i_2}, m_{i_3}) = 2$. It is not possible to find an element in $\bigoplus_{j=1}^{k} B_j$ which goes to $(\frac{1}{p_i^2}, \frac{1}{p_i^2}, 0, \cdots, 0)$ under π. So we know only that $Q/TQ \cong \operatorname{im} \pi = \langle (b_1 - b_2, b_2 - b_3, \cdots, b_{k-1} - b_k) \mid b_i \in B_i \rangle$ in $\bigoplus_{j=1}^{k-1} B'_j$.

We now determine when $\operatorname{Ext}(\bar{P}, P)$ contains nontrivial completely decomposable extensions.

Theorem 4.2. *Let $\bar{P} = \langle \frac{1}{p_i^{\bar{r}_i}} \rangle$ and $P = \langle \frac{1}{p_i^{r_i}} \rangle$ be groups of pseudo-integers. Then $Ext(\bar{P}, P)$ contains nontrivial completely decomposable extensions if and only if $r_i \leq \bar{r}_i$ almost everywhere and $r_i < \bar{r}_i$ for an infinite number of primes.*

Proof: Suppose $\mathrm{Ext}(\bar{P}, P)$ contains nontrivial completely decomposable extensions. Let $P \rightarrowtail B_1 \oplus B_2 \twoheadrightarrow \bar{P}$ be such an extension where $B_1 = \langle \frac{1}{p_i^{m_i}} \rangle$, $B_2 = \langle \frac{1}{p_i^{n_i}} \rangle$, and $1 \mapsto (\frac{u_1}{v_1}, \frac{u_2}{v_2})$ determines the embedding of P into $B_1 \oplus B_2$. Since $Q = \frac{B_1 \oplus B_2}{P}$ is isomorphic to $\bar{P}$, Q is torsionfree. By Theorem 3.2, $TQ \cong \bigoplus_{p_i} \mathbb{Z}/p_i^{l_i - r_i}\mathbb{Z}$ where $l_i = \min(m_i + v_{p_i}(\frac{u_1}{v_1}), n_i + v_{p_i}(\frac{u_2}{v_2}))$. Hence $l_i = r_i$ for all i. Let $S_1 = \{i \mid v_{p_i}(\frac{u_1}{v_1}) \neq 0 \text{ or } v_{p_i}(\frac{u_2}{v_2}) \neq 0\}$ and note that S_1 is finite. For $i \notin S_1$, $l_i = \min(m_i, n_i)$. Now by Theorem 4.1, $Q/TQ \cong Q \cong \langle \frac{1}{p_i^{s_i}} \rangle$ where $s_i = \max(m_i, n_i)$. So $\bar{P} \cong \langle \frac{1}{p_i^{s_i}} \rangle$ which implies that $\bar{r}_i = s_i$ for almost all i. Let $S_2 = \{i \mid \bar{r}_i \neq s_i\}$. For $i \notin S_1 \cup S_2$ (a finite set), $r_i = \min(m_i, n_i) \leq \max(m_i, n_i) = s_i = \bar{r}_i$. Suppose $r_i < \bar{r}_i$ for only a finite number of primes and let $S_3 = \{i \mid r_i < \bar{r}_i\}$. For $i \notin S_1 \cup S_2 \cup S_3$ (a finite set), $r_i = \min(m_i, n_i) = \max(m_i, n_i) = \bar{r}_i$. Hence $m_i = n_i = r_i = \bar{r}_i$ and $P \cong \bar{P} \cong B_1 \cong B_2$ which contradicts our assumption of nontriviality.

Now suppose $r_i \leq \bar{r}_i$ almost everywhere and $r_i < \bar{r}_i$ for an infinite number of primes. We may assume without loss of generality that $r_i \leq \bar{r}_i$ everywhere. For, since $r_i \leq \bar{r}_i$ for all but a finite number of primes, we could replace P with an isomorphic group $P' = \langle \frac{1}{p_i^{r'_i}} \rangle$ so that $r'_i \leq r_i$ for all primes. Now let $S = \{i \mid r_i < \bar{r}_i\}$ and $\bar{S} = \{i \mid r_i = \bar{r}_i\}$. Divide S into two disjoint infinite subsets S_1 and S_2. Let $B_1 = \langle \frac{1}{p_i^{m_i}} \rangle$ where $m_i = \begin{cases} r_i & i \in S_1 \\ \bar{r}_i & i \in S_2 \cup \bar{S} \end{cases}$ and $B_2 = \langle \frac{1}{p_i^{n_i}} \rangle$ where $n_i = \begin{cases} r_i & i \in S_2 \cup \bar{S} \\ \bar{r}_i & i \in S_1 \end{cases}$. Now $1 \mapsto (1,1)$ determines an embedding of P into $B_1 \oplus B_2$ since $r_i \leq m_i, n_i$ for all i. By Theorem 3.1, $TQ \cong \bigoplus_{p_i} \mathbb{Z}/p_i^{l_i - r_i}$ where $l_i = \min(m_i, n_i)$. But $\min(m_i, n_i) = \min(r_i, \bar{r}_i) = r_i$ since $r_i \leq \bar{r}_i$ for all i and thus TQ is trivial. According to Theorem 4.1, $Q/TQ \cong Q \cong \langle \frac{1}{p_i^{s_i}} \rangle$ where $s_i = \max(m_i, n_i)$. But $\max(m_i, n_i) = \max(r_i, \bar{r}_i) = \bar{r}_i$. Hence $Q \cong \langle \frac{1}{p_i^{s_i}} \rangle = \bar{P}$. Now $P \ncong B_1$, $P \ncong B_2$, $\bar{P} \ncong B_1$, and $\bar{P} \ncong B_2$ since S_1 and S_2 are infinite sets. Thus $P \rightarrowtail B_1 \oplus B_2 \twoheadrightarrow \bar{P}$ with the embedding $1 \mapsto (1,1)$ is a nontrivial completely decomposable extension of P by $\bar{P}$. ∎

We can now prove the following theorem which will allow us to simplify our notion of nontrivial extension. We again assume $P = \langle \frac{1}{p_i^{r_i}} \rangle$, $\bar{P} = \langle \frac{1}{p_i^{\bar{r}_i}} \rangle$, $B_1 = \langle \frac{1}{p_i^{m_i}} \rangle$, and $B_2 = \langle \frac{1}{p_i^{n_i}} \rangle$ are groups of pseudo-integers.

Theorem 4.3. *Suppose $P \rightarrowtail B_1 \oplus B_2 \twoheadrightarrow \bar{P}$ is exact. Then $P \cong B_1$ if and only if $\bar{P} \cong B_2$.*

Proof: We see, from the Proof of Theorem 4.2, that $r_i = \min(m_i, n_i)$ almost everywhere. By Theorem 4.1, $\bar{r}_i = \max(m_i, n_i)$ almost everywhere. Hence $r_i = m_i$ almost everywhere if and only if $m_i \le n_i$ almost everywhere if and only if $\bar{r}_i = n_i$ almost everywhere. ∎

We could just as easily prove that if $P \rightarrowtail B_1 \oplus B_2 \twoheadrightarrow \bar{P}$ is exact then $P \cong B_2$ if and only if $\bar{P} \cong B_1$. These results do allow us to make our definition of nontrivial extension more concise. We had previously defined an extension $P \rightarrowtail B_1 \oplus B_2 \twoheadrightarrow \bar{P}$ to be trivial if $P \cong B_1$, $\bar{P} \cong B_2$ or $P \cong B_2$, $\bar{P} \cong B_1$ and nontrivial otherwise. With regard to the above, such an extension is now trivial if $P \cong B_1$ or $P \cong B_2$ and hence nontrivial if $P \ncong B_1$ and $P \ncong B_2$. We note that we could just as easily use $\bar{P}$ in place of P in our new definition.

In [4] it is shown that if $\bar{P} = \langle \frac{1}{p_i^{\bar{r}_i}} \rangle$ and $P = \langle \frac{1}{p_i^{r_i}} \rangle$ are groups of pseudo-integers so that $r_i < \bar{r}_i$ for an infinite number of i then $\operatorname{Ext}(\bar{P}, P)$ contains extensions $P \rightarrowtail E \twoheadrightarrow \bar{P}$ where E is indecomposable as a pseudofree group of rank 2. So, referring to Theorem 4.2, we see that if $\operatorname{Ext}(\bar{P}, P)$ contains extensions which are completely decomposable it must also contain those which are not. Further, we can use the theorems in Section 1 to describe the abelian group structures of $\operatorname{Hom}(\bar{P}, P)$ and $\operatorname{Ext}(\bar{P}, P)$ for the case when $\operatorname{Ext}(\bar{P}, P)$ contains completely decomposable extensions, i.e. when $r_i \le \bar{r}_i$ almost everywhere and $r_i < \bar{r}_i$ for an infinite number of i. Since r_i must be less than $\bar{r}_i$ for infinitely many i, we see that $\operatorname{Hom}(\bar{P}, P)$ must be trivial and hence $\operatorname{Ext}(\bar{P}, P) \cong V \oplus (\mathbb{Q}/\mathbb{Z})$ where V is a $\mathbb{Q}$-vector space of rank c. It seems that we should be able to determine where the completely decomposable extensions are to be found in this structure. Do they correspond to torsion elements, infinite elements, mixed elements, or some combination of these? We may also ask a similar question for the indecomposable extensions.

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