POINTWISE CONVERGENCE OF THE FOURIER TRANSFORM ON LOCALLY COMPACT ABELIAN GROUPS

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Abstract

We extend to locally compact abelian groups, Fejer's theorem on pointwise convergence of the Fourier transform. We prove that \( \lim_{\mu_U} f(y) = f(y) \) almost everywhere for any function \( f \) in the space \( (L^p, L^q)(G) \) (hence in \( L^p(G) \)), \( 2 \leq p \leq \infty \), where \( \{\mu_U\} \) is Simon's generalization to locally compact abelian groups of the summability Fejer Kernel. Using this result, we extend to locally compact abelian groups a theorem of F. Holland on the Fourier transform of unbounded measures of type \( q \).

1. Notation and Preliminary Results

Throughout, \( G \) is a locally compact abelian group, with dual group \( \Gamma \), and Haar measure \( m \). By the structure theorem, \( G \) is represented by \( \mathbb{R}^a \times G_1 \), where \( a \) is a nonnegative integer and \( G_1 \) is a group which contains an open compact subgroup \( H \). The set of basic neighbourhoods of \( x \in G \) is denoted by \( N_x(G) \). We write \( C_c(G), C_0(G) \) for the spaces of functions on \( G \) that are continuous, with compact support and vanish at infinity, respectively. We consider the amalgam spaces \( (L^p, L^q)(G), (C_0, L^q)(G) \) \( (1 \leq p, q \leq \infty) \) as defined in [S]. The Fourier transform (inverse Fourier transform) of a measure \( \mu \) is denoted by \( \hat{\mu} \) (\( \check{\mu} \)). We let \( A_c(G) \) be the set of all functions \( f \) in \( C_c(G) \) such that \( \hat{f} \in L^1(\Gamma) \). The characteristic function of a subset \( E \) of \( G \) is denoted by \( \chi_E \). The conjugate \( p' \) of a number \( p \) is such that \( 1/p + 1/p' = 1 \). For each \( U \in N_0(G) \), A.B. Simon [Si] defined a function \( \varphi_U \) as the product of two functions \( \alpha_U \) and \( \beta_U \) defined on \( \mathbb{R}^a \) and on \( G_1 \), respectively.

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The function $\beta_U$ is continuous, nonnegative, with $L^1(G)$-norm equal to 1, and

\begin{equation}
\sup_G |\beta_U(x)| = B_U \leq 2m(U)/1 - 2m(U) \quad \text{finite.}
\end{equation}

Hence $B_U \to 0$ as $U \to 0$.

The function $\alpha_U$ is defined as follows.

Let $(-\delta_1, \delta_1) \times \cdots \times (-\delta_a, \delta_a) \times U_H$ be a product neighbourhood contained in $U$, where $\delta_i > 0 \ (i = 1, \ldots, a)$, and $U_H$ is an element of $N_0(G)$ included in $H$. For $i = 1, \ldots, a$ we set $U_i = (-\delta_i, \delta_i), N_i = 1/\delta_i$, and define the function $\alpha_{U_i}$ on $\mathbb{R}$ by

$$\alpha_{U_i}(t) = \frac{1 - \cos(N_i t)}{\pi N_i t^2}$$

For $t = (t_1, \ldots, t_a)$ in $\mathbb{R}^a$, the function $\alpha_U$ is given by $\alpha_U(t) = \Pi_{i=1}^a \alpha_{U_i}(t_i)$. Clearly $\alpha_U$ is continuous, nonnegative, and its $L^1(\mathbb{R}^a)$-norm is equal to 1. Each $\varphi_U$ has the following properties. For a proof see [Si].

1.1) $\varphi_U$ is continuous, nonnegative and bounded
1.2) $\varphi_U$ is integrable and $\|\varphi_U\|_1 = 1$
1.3) $\varphi_U \in C_c(\Gamma)$ and $\|\varphi_U\|_\infty \leq 1$
1.4) $\varphi_U(x) = \int_\Gamma \varphi(\gamma) \gamma(x) d\gamma$ by 1.3)
1.5) For $\varepsilon > 0$ and $U \in N_0(G)$ given, we can find a $V$ such that if $V' \leq V$ and $x \notin U$, then $\varphi_V(x) < \varepsilon$ and $\int_{G-V} \varphi_{V'}(x) dx < \varepsilon$.
1.6) $\lim_{U} \varphi_U(\gamma) = 1$.
1.7) the family $\{\varphi_U | U \in N_0(G)\}$ is an approximate identity in $L^1(G)$.

We add to this list the fact that each $\varphi_U$ belongs to the Wiener algebra $(C_0, l^1)(G)$ [W].

**Proposition 1.1.** For each $U$ in $N_0(U)$, the function $\alpha_U$ belongs to $(C_0, l^1)(\mathbb{R}^a)$.

**Proof:** Since $\alpha_{U_i}(i = 1, \ldots, a)$ is an even function we have for $n$ in $\mathbb{Z} - \{0, -1\}$ that

$$\sup_{t \in [0,1]} \alpha_{U_i}(t + n) = \sup_{t \in [0,1]} \alpha_{U_i}(t - (1 + n)) \leq \frac{2}{N_i \pi} \frac{1}{n^2}.$$

If $n \in \{0, -1\}$, then there exists a constant $C_i$ such that

$$\sup_{t \in [0,1]} \alpha_{U_i}(t + n) \leq \frac{N_i}{\pi} C_i$$
because the limit
\[
\lim_{t \to -n} \frac{1 - \cos N_i(n + t)}{(N_i(n + t))^2}
\]
exists.

Therefore for all \( i = 1, \ldots, a \) and all integer \( n \) we have that
\[
\sup_{t \in [0, 1]} |\alpha_{U_i}(t + n)| \leq c \alpha_n
\]
where
\[
c = \max_{1 \leq i \leq a} \left( \frac{2}{N_i \pi}, N_i C_i / \pi \right)
\]
and \( \alpha_n \) is equal to \( 1/n^2 \) if \( n \in \mathbb{Z} - \{0, -1\} \), and to \( 1 \) if \( n \in \{0, -1\} \).

Finally, for \( i = 1, \ldots, a \) we have that
\[
\|\alpha_{U_i}\|_{\infty 1} = \sum_{Z} \sup_{t \in [n, n+1]} |\alpha_{U_i}(t)| = \sum_{Z} \sup_{t \in [0, 1]} |\alpha_{U_i}(t + n)| \leq C \sum_{Z} \alpha_n < \infty.
\]

From the definition of the norm \( \| \|_{\infty 1} \), it is easy to see that
\[
\|\alpha_U\|_{\infty 1} = \Pi_{i=1}^a \|\alpha_{U_i}\|_{\infty 1}.
\]

**Corollary 1.2.** For each \( U \) in \( \mathcal{N}_0(G) \), the function \( \varphi_U \) belongs to \( (C_0, l^1)(G) \).

**Proof:** By (1) we have for all \( (t, s) \) in \( G \) that
\[
\varphi_U(t, s) = \alpha_U(t) \beta_U(s) \leq B_U \alpha_U(t),
\]

hence
\[
\|\varphi_U\|_{\infty 1} \leq B_U \sum_{n \in \mathbb{Z}^a} \sup_{t \in [0, 1]^a} |\alpha_U(t)| = B_U \|\alpha_U\|_{\infty 1}.
\]

For the rest of this paper \( \varphi_U, \alpha_U, \) and \( \beta_U \) are as indicated in this section.
2. Main Theorem

In this second section we want to prove that

\[ \lim_{U \to 0} (\phi_u * f)(y) = f(y) \quad \text{almost everywhere} \]

for all \( f \) in \( (L^p, l^\infty)(G)(2 < p \leq \infty) \).

First, we prove two lemmas.

**Lemma 2.1.** Let \( V \) and \( K \) be two elements of \( N_0(G) \) of the form

\[ V = (-\delta_1, \delta_1) \times \cdots \times (-\delta_a, \delta_a) \times \mathcal{V}_H \]

and \( K = [-\gamma_1, \gamma_1] \times \cdots \times [-\gamma_a, \gamma_a] \times K_H \), where \( \delta_i > 0, \gamma_i > 0 \) (\( i = 1, \ldots, a \)), \( \mathcal{V}_H \) and \( K_H \) are elements of \( N_0(G) \) contained in \( H \), and \( K_H \) is compact.

For \( 1 \leq p < \infty \), we define \( \eta_i = \min(\delta_i^{2p}, \gamma_i) \) (\( i = 1, \ldots, a \)), and we let \( \mathcal{W}_H \) be the interior of \( K_H \). Then the set \( W = [-\eta_1, \eta_1] \times \cdots \times [-\eta_a, \eta_a] \times \mathcal{W}_H \) belongs to \( N_0(G) \) and for a fixed \( y = (y_0, s_0) = (y_1, \ldots, y_a, s_0) \) in \( G \), the element \( W_y = y + W \) of \( N_y(G) \) has the following properties:

1. \( W_y \subseteq y + K_H \)
2. \( \Pi_a = [-\eta_1 + y_1, \eta_1 + y_1] \times \cdots \times [-\eta_a + y_a, \eta_a + y_a] \), then

\[ \left[ \int_{\Pi_a} a_U(y_0 - x)^p dx \right]^{1/p} = O(\Pi_a^a \delta_i) \]

3. \( \mathbb{R}^a - \Pi_a \subseteq \cup \mathbb{I}_n \), where \( \{\mathbb{I}_n\} \) is a countable family of compact subsets of \( \mathbb{R}^a \), and

\[ \sum_{\mathcal{N}} \left[ \int_{\mathbb{I}_n} a_U(y_0 - x)^p dx \right]^{1/p} = O(\Pi_{i=1}^a \delta_i). \]

4. There exists a constant \( C \) such that \( \sup_{\mathcal{N}} C(I_n) \leq C \), where \( C(I_n) \) is the cardinality of the set

\[ \{ j \in \mathbb{Z}^a \mid (j + [0, 1]^a) \cap I_n \neq \emptyset \}. \]

**Proof:** Several constants will appear during the proof and since their specific value is irrelevant for our needs we just write \( C_1, C_2, \ldots, C_q \). Part
2.1) is clear. Set \( J_i = [-\eta_i + y_i, \eta_i + y_i] \) \((i = 1, \ldots, a)\). Part 2.2) follows from the continuity of \( \alpha U_i \) because
\[
\left[ \int_{-\eta_i}^{\eta_i} \alpha U_i(y_i - x)^p \, dx \right]^{1/p} = \left[ \int_{-\eta_i}^{\eta_i} \alpha U_i(x)^p \, dx \right]^{1/p} \leq C_1 N_i \eta_i^{1/p} \leq C_2 \delta_i.
\]
Now, for each \( i = 1, \ldots, a \), let \( L(n, i) \) and \( R(n, i) \) \((n \in \mathbb{N})\) be the intervals
\[
[-n - 1 - \eta_i + y_i, -n - \eta_i + y_i] \text{ and } [n + \eta_i + y_i, n + 1 + \eta_i + y_i]
\]
respectively. Then
\[
R - J_i = (-\infty, -\eta_i + y_i) \cup (\eta_i + y_i, \infty) \subseteq \bigcup_{n} L(i, n) \cup \bigcup_{n} R(i, n),
\]
and
\[
\int_{L(n, i)} \alpha U_i(y_i - x)^p \, dx \leq C_3 \delta_i^2 a_n
\]
where
\[
a_n = \frac{1}{(n_i + n)^{2p-1}} \left( \frac{1}{(n_i + n + 1)^{2p-1}} \right).
\]
Since \( \sum a_n^{1/p} \) converges we conclude that
\[
\sum_{n} \left[ \int_{L(n, i)} \alpha U_i(y_i - x)^p \, dx \right]^{1/p} \leq C_4 \delta_i.
\]
Similarly
\[
\sum_{n} \left[ \int_{R(n, i)} \alpha U_i(y_i - x)^p \, dx \right]^{1/p} = C_5 \delta_i.
\]
Clearly \( \sup_{n} C(L(n, i)) \) and \( \sup_{n} C(R(n, i)) \) are less than or equal to 2, hence for \( i = 1, \ldots, a \), the set \( R - J_i \) is equal to \( \bigcup I_n \), where each \( I_n \) is compact, \( \sup C(I_n) \leq 2 \) and
\[
(4) \quad \sum_{n} \left[ \int_{I_n} \alpha U_i(y_i - x)^p \, dx \right]^{1/p} = O(\delta_i).
\]
Since \( R = (R - J_a) \cup J_a \), and \( J_a \) is compact, by (3) and (4) we see that \( R = \bigcup K_n \), with each \( K_n \) compact, \( \sup C(K_n) \leq C_6 \), and
\[
(5) \quad \sum_{n} \left[ \int_{K_n} \alpha U_a(y_a - x)^p \, dx \right]^{1/p} = O(\delta_a).
\]
We prove properties 2.3) and 2.4) by induction on $a$. The case $a = 1$ follows from (3). Suppose that 2.3) and 2.4) hold for $a - 1$. That is, $\mathbb{R}^{a-1} - \Pi(a-1) \subseteq \bigcup I_n$, each $I_n$ a compact subset of $\mathbb{R}^{a-1}$, sup $C(I_n) \leq C_7$, and

$$
\sum \left[ \int_{I_n} \Pi_{i=1}^{a-1} \alpha U_i (y_i - x_i)^p \, dx \right]^{1/p} = O(\Pi_{i=1}^{a-1} \delta_i).
$$

By (4) with $i = a$, we have that $\mathbb{R} - J_a \subseteq \bigcup I_j$, each $I_j$ a compact subset of $\mathbb{R}$, sup $C(I_j) \leq 2$ and

$$
\sum \left[ \int_{I_j} \alpha U_a (y_a - x)^p \, dx \right]^{1/p} = O(\delta_a).
$$

Then

$$
\mathbb{R}^a - \Pi a = (\mathbb{R}^{a-1} \times \mathbb{R}) - (\Pi(a-1) \times J_a)
$$

$$
= (\mathbb{R}^{a-1} - \Pi(a-1)) \times (\mathbb{R} \cup \Pi(a-1)) \times (\mathbb{R} - J_a)
$$

$$
\leq \bigcup_{n,m} (I_n \times K_m) \bigcup_{i=1}^{a} \Pi(a-1) \times I_j.
$$

The sets $I_n \times K_m$ and $\Pi(a-1) \times I_j$ are compact subsets of $\mathbb{R}$, for all $n, m, j$. Hence sup $C(I_n \times K_m) \leq C_8$ and sup $C(\Pi(a-1) \times I_j) \leq C_9$. Therefore 2.4) holds with $C = \max(C_8, C_9)$. Finally, by (5) and (6) we have that

$$
\sum \left[ \int_{I_n \times K_m} \alpha U(y_0 - x)^p \, dx \right]^{1/p} =
$$

$$
= \sum \left[ \int_{I_n} \Pi_{i=1}^{a-1} \alpha U_i (y_i - x_i)^p \, dx \right]^{1/p} \sum \left[ \int_{K_m} \alpha U_a (y_a - x)^p \, dx \right]^{1/p} =
$$

$$
= O(\Pi_{i=1}^{a-1} \delta_i).
$$

We conclude from (3) and (7) that

$$
\sum \left[ \int_{\Pi(a-1) \times I_j} \alpha U(y_0 - x)^p \, dx \right]^{1/p} =
$$

$$
= \Pi_{i=1}^{a-1} \left[ \int_{I_i} \alpha U_i (y_i - x)^p \, dx \right]^{1/p} \sum \left[ \int_{I_j} \alpha U_a (y_a - x)^p \, dx \right]^{1/p} =
$$

$$
= O(\Pi_{i=1}^{a-1} \delta_i).
$$
**Lemma 2.2.** For each $V_y$ in $\mathcal{N}_y(G)$ ($y \in G$)

$$\lim_{u \to 0} \int_{G-V_y} \varphi_U(y-x)f(x)dx = 0$$

for all $f$ in $(L^p, l^\infty)(G)(1 < p \leq \infty)$.

**Proof.** Let $y = (y_1, \ldots, y_a, s_0) = (y_0, s_0)$ be an element of $\mathbb{R}^a \times G_1$. We choose two elements $V$ and $K$ of $\mathcal{N}_0(G)$ with the same form as in Lemma 2.2, such that $y + K \subseteq V_y$ and $V \subseteq U$.

Following the notation of Lemma 2.2, we set $\eta_i = \min(\epsilon_i^2, \gamma_i)(i = 1, \ldots, a)$, and $W_H$ the interior of $K_H$. Then the set $W = [-\eta_1, \eta_1] \times \cdots \times [-\eta_a, \eta_a] \times W_H$ satisfies the properties listed in Lemma 2.2. Hence by property 2.1) it is enough to prove that

$$\lim_{U \to 0} \int_{G-W_y} \varphi_U(y-x)f(x)dx = 0.$$

Since

$$G - W_y = (\mathbb{R}^a - \Pi_a) \times G_1 \cup \Pi_a \times (G_1 - (s_0 + W_H)),$$

we have by the definition of the function $\varphi_U$, that

$$\varphi_U(y-x) = \alpha_U(y_0 - t) \beta_U (s_0 - s) = 0$$

if $s_0 - s \notin H$, and $x = (t, s)$ in $G$. Hence

$$\int_{G-W_y} \varphi_U(y-x)f(x)dx = \int_{(\mathbb{R}^a - \Pi_a) \times (s_0 + H)} \varphi_U(y-x)f(x)dx$$

$$+ \int_{\Pi_a \times (s_0 + (H-W_H))} \varphi_U(y-x)f(x)dx.$$

Let $\{I_n\}$ be the countable family of sets given by property 2.3). For each $I_n$ we have by the Hölder inequality and (1)

$$\int_{I_n \times (s_0 + H)} |\varphi_U(y-x)f(x)|dx \leq \|f|x_{I_n \times (s_0 + H)}\|_{pB_U} \leq$$

$$\leq \left[\int_{I_n} \alpha_U(y_0 - x)^p dx\right]^{1/p'}$$
By property 2.4) sup_N |S(I_n \times (s_0 + H))| \leq C, where C is a constant, and |S(I_n \times (s_0 + H))| is the number of $K_\alpha$'s (as defined in [S]) such that $I_n \times (s_0 + H) \cap K_\alpha \neq \emptyset$. This implies that for all $n \in \mathbb{N}$

$$||f \chi_{I_n \times (s_0 + H)}||_p \leq |S(I_n \times (s_0 + H))| ||f||_\infty \leq C ||f||_\infty.$$ 

Thus, we conclude from 2.2) that

$$\int_{(\mathbb{R}^n - \Pi_\alpha) \times G} \varphi_U(y - x) |f(x)| \, dx \leq \left( C ||f||_\infty B_U \sum_{n} \int_{I_n} \alpha_U(y_0 - x) \, dx \right)^{1/p'} = O(\Pi_{i=1}^\alpha \delta_i B_U).$$

Applying Hölder's inequality we get

$$\int_{\Pi_\alpha \times (s_0 + (H - W_H))} \varphi_U(y - x) |f(x)| \, dx \leq B_U ||f||_\infty |S(\Pi_\alpha \times (s_0 + (H - W_H)))| \left( \int_{\Pi_\alpha} \alpha_U(y_0 - x) \, dx \right)^{p'}. $$

Note that $\Pi_\alpha \times (s_0 + (H - W_H))$ is compact ($H$ is compact and $H - W_H$ is closed), and because $B_U \to 0$ as $U \to 0$

Now, since $\Pi_\alpha \to y$ as $U \to 0$ and $s_0 + (H - W_H) \leq s_0 + H$ is independent of $U$, we have that $|S(\Pi_\alpha \times (s_0 + (H - W_H))| \to 0$ as $U \to 0$. Therefore by property 2.2)

$$\int_{\Pi_\alpha \times (s_0 + (H - W_H))} \varphi_U(y - x) |f(x)| \, dx \to 0 \quad \text{as} \quad U \to 0.$$ 

The result follows from (8), (9), and (10).

**Theorem 2.3.** For all $f$ in $L^p(G)$, $2 \leq p \leq \infty$,

$$\lim_{U \to 0} \int_G \varphi_U(y - x) f(x) \, dx = f(y)$$

almost everywhere.

**Proof:** Let $V_y$ be in $N_y$ compact. We have to show, by Lemma 2.2, that

$$\lim_{V \to 0} \int_{V_y} \varphi_U(y - x) f(x) \, dx$$
Further, convergence of the Fourier transform on groups converges to \( f(x) \) almost everywhere. Since the function \( f \) belongs to \((L^p, l^\infty) \subseteq (L^2, l^\infty)\), the function \( g = f\chi_G \) belongs to \( L^2(G) \), and by Corollary 1.2, 1.1, and 1.4) each \( \varphi_U \) also belongs to \( L^2(G) \). Hence by the Parseval identity, we have that

\[
\int_{V_v} \varphi_U(y - x)f(x)dx = \int_G \varphi_U(y - x)g(x)dx \\
= \int_\Gamma \varphi_U(\vec{x})g(-\vec{x})|y, \vec{x}|d\vec{x}
\]

By the Lebesgue Dominated Convergence theorem (see properties 1.3 and 1.6) we have that

\[
\lim_{U \to 0} \int_{V_v} \varphi_U(y - x)f(x)dx = \lim_{U \to 0} \int_\Gamma \varphi_U(\vec{x})g(-\vec{x})|y, \vec{x}|d\vec{x} \\
= \int_\Gamma g(-\vec{x})|y, \vec{x}|d\vec{x} = g(y)
\]

almost anywhere.

3. Fourier Transform of Unbounded Measures

The space \( M_q(G)(1 \leq p < \infty) \) of unbounded measures of type \( q \) [S], consists of Radon measures \( \mu \) with finite norm \( ||\mu||_q \) given by \( (\sum |\mu|(K_\alpha)^q)^{1/q} \). If \( G = \mathbb{R} \), then the family \( \{K_\alpha\} \) can be taken as \( \{[n, n+1]|n \in \mathbb{Z}\} \).

In this section we generalize to locally compact abelian groups, the following theorem due to F. Holland [H].

**Theorem 3.1.** Let \( 1 \leq q \leq 2 \) and \( \mu \in M_q(\mathbb{R}) \). Then as \( N \to \infty \)

\[
\frac{1}{\sqrt{2\pi}} \int_{-N}^N e^{-ixt}d\mu(t)
\]

converges in the norm of \((L^q', l^\infty)\) to a function \( \tilde{\mu} \) and

\[
\int h(x)\tilde{\mu}(x)dx = \int h(x)d\mu(x) \quad (h \in L^q, l^1)(\mathbb{R})).
\]

Further

\[
\sqrt{2\pi} \tilde{\mu}(x) = (C.1) \int e^{-ixt}d\mu(t)
\]
almost everywhere.

(C.1) means that the integral on the right is summable by the Cesáro method of order 1 to the value \( \sqrt{2\pi} \hat{\mu}(x) \).

It is easy to see, that for any measure \( \mu \) in \( M_q \) \((1 \leq q \leq 2)\), there is a net \( \mu_\alpha \) of bounded measures such that \( \lim ||\mu_\alpha - \mu||_q = 0 \), and therefore by [S. Theorem 4.2] \( \lim ||\hat{\mu}_\alpha - \hat{\mu}||_{q,\infty} = 0 \). This generalizes the first part of the theorem.

**Theorem 3.2.** Let \( \mu \) be an element of \( M_q \) \((1 \leq q \leq 2)\)

\( i) \int_G \hat{f}(\gamma) \hat{\mu}(\gamma) d\gamma = \int_G \overline{f}(x) d\mu(x) \) for all \( f \) in \( (L^q, 1^1)(\Gamma) \).

\( ii) \) \( \int_G \gamma(x) d\mu(x) := \lim_{U \to 0} \int_G \phi_U(x) \gamma(x) d\mu(x) = \hat{\mu}(\gamma) \) almost everywhere.

(C.1) means that the integral on the right is summable by the Cesáro method of order 1 to the value \( \hat{\mu}(\gamma) \).

**Proof:** Let \( \mu_\alpha \) be the net of bounded measures related to \( \mu \), as mentioned above.

Since \((L^q, 1^1)\) is a subspace of \( L^1 \) [S.(3,4)], we have by the Extended Parseval Formula [S. Lemma 4.1] that for any \( f \) in \((L^q, 1^1)(\Gamma)\),

\[
\int_{\Gamma} \hat{f}(\gamma) \hat{\mu}_\alpha(\gamma) d\gamma = \int_G \overline{f}(x) d\mu_\alpha(x)
\]

By the Hölder inequality

\[
\int_{\Gamma} |f(\gamma)| |\hat{\mu}_\alpha(\gamma) - \hat{\mu}(\gamma)| d\gamma \leq \\
\leq \sum_j \left[ \int_{K_\alpha} |f(\gamma)|^q \right]^{1/q} \left[ \int_{K_\alpha} |\hat{\mu}_\alpha(\gamma) - \hat{\mu}(\gamma)|^{q'} d\gamma \right]^{1/q'} \leq \|f\|_q \|\mu_\alpha - \mu\|_{q,\infty}.
\]

Similarly

\[
\int_G |\overline{f}(x)| d|\mu_\alpha - \mu|(x) \leq \||\overline{f}\|_{q,\infty} \|\mu_\alpha - \mu\|_{q,\infty}.
\]

Therefore the left side of (ii) converges to \( \int_{\Gamma} \hat{f}(\gamma) d\mu(\gamma) \), and the right side to \( \int_G \overline{f}(x) d\mu(x) \). This proves \( i) \).

By proposition 1.1 and [S.(3.1)], each \( \varphi_U \) belongs to \((L^q, 1^1)(\Gamma)\), so from \( i) \)

\[
\int_{\Gamma} \varphi_U(y - \gamma) \hat{\mu}(\gamma) d\gamma = \int_G y(x) \overline{\varphi_U(x)} d\mu(x).
\]

Hence, part \( ii) \) follows from Theorem 2.3.
Convergence of the Fourier Transform on Groups

References


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