MAXIMAL QUOTIENT RINGS AND ESSENTIAL RIGHT IDEALS IN GROUP RINGS OF LOCALLY FINITE GROUPS

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Dedicated to the memory of Pere Menal

Abstract

Let \( k \) be a commutative field. Let \( G \) be a locally finite group without elements of order \( p \) in case \( \text{char } k = p > 0 \). In this paper it is proved that the type \( I_\infty \) part of the maximal right quotient ring of the group algebra \( kG \) is zero.

1. Introduction

Let \( k \) be a commutative field, \( G \) a group, and suppose that the group ring \( kG \) is regular in the sense of von Neumann. By [7, p.69], this means precisely that \( G \) is locally finite with no elements of order equal to the characteristic of \( k \). Then the maximal right quotient ring \( Q'(kG) \) of \( kG \) is a regular right self-injective ring [3, Corollaries 1.2 and 1.24], and as such, is uniquely a direct product of rings of types \( I_f, I_\infty, II_f, II_\infty \), and \( III \). We refer to [3] for general background on regular rings. The main theorem of this paper is the following.

Theorem. With the above notation, the type \( I_\infty \) part of \( Q'(kG) \) is zero.

The first author has obtained this result under various supplementary hypotheses [2], and in particular when \( G \) is \( \Delta \)-hypercentral. These results will be used in the proof of the general case. He has also shown that the type \( I_f \) part of \( Q'(kG) \) is non-zero if and only if \( |G : \Delta(G)| < \infty \) and \( \Delta(G)' \) is finite, where as usual, \( \Delta(G) \) is the subgroup of \( G \) consisting of the elements with finitely many conjugates [1].

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In 1983, Menal proposed the study of the maximal quotient ring of regular group algebras to the first author. One of the problems proposed was the characterization of the type I part of $Q'(kG)$ for $kG$ regular. Goursaud and Valette [4] had a partial result, namely they had characterized this part when $k$ has positive characteristic or contains all roots of unity. Finally, with [1, Theorem 2.3] and the Theorem of this paper, the problem has been solved.

The remainder of the paper began as an attempt to answer the following question, raised for example in [6].

**Question.** Let $G$ be a locally finite group, $k$ a field and $H$ an infinite subgroup of $G$. Is it true that if $J$ is an essential right ideal of $kH$, then the right ideal $JkG$ of $kG$ generated by $J$ is essential in $kG$?

We are only able to answer this question in a very special case, namely the case when $J$ is the augmentation ideal of $kH$. However the method we use involves attaching a certain numerical invariant to each right (or left) ideal of $kG$, and possibly this may be of some interest in its own right. It is closely related to the invariant $d$ discussed in [5], but is not quite the same. Our hope was that this invariant would distinguish between essential right ideals and the rest, but we give an example indicating the contrary. A result giving a positive answer to the question when $kG$ is regular and $H$ has finite index is proved in [2, Lemma 1.4]. The proof there is rather indirect and it would be interesting to have a direct proof in that case.

**Definition 1.1.** Let $G$ be a locally finite group, $k$ be a field, and $J$ be a right ideal of $kG$. For each finite subgroup $F$ of $G$, let $\alpha(J, F) = \dim(J \cap kF)/|F|$. Let $\alpha(J) = \sup \alpha(J, F)$, where $F$ ranges over all finite subgroups of $G$.

Of course, the same definition can be made for left ideals of $kG$. Clearly $\alpha(J) \leq 1$. The main properties of $\alpha(J)$ are the following.

**Lemma 1.2.** Let $G, k$ be as above and let $H$ be a subgroup of $G$.

(i) If $J$ is a right ideal of $kH$, then $\alpha(J) \leq \alpha(JkG)$.

(ii) If $\alpha(J) = 1$, then $J$ is an essential right ideal of $kG$.

(iii) If $\omega(kG)$ denotes the augmentation ideal of $kG$ and $G$ is infinite, then $\alpha(\omega(kG)) = 1$.

(iv) If $J$ and $L$ are right ideals of $kG$ and $J$ is isomorphic to a submodule of $L$, then $\alpha(J) \leq \alpha(L)$.

These facts clearly imply the following.
Corollary 1.3. With the above notation, if $H$ is an infinite subgroup of $G$, then $\omega(kH)kG$ is an essential right ideal of $kG$.

This was obtained somewhat less generally in [6, p.250] (see below). It is unfortunate that the converse of part (ii) of Lemma 1.2 is false. In fact, we have the following.

Example 1.4. Let $G$ be any countably infinite periodic abelian group. Then for any $\epsilon > 0$, there is an essential ideal $J$ of $CG$ such that $\alpha(J) < \epsilon$.

### 2. Properties of $\alpha(J)$

In this section, we shall prove Lemma 1.2, after mentioning some other basic facts. Throughout, $G$ denotes a locally finite group, and $k$ a field.

Lemma 2.1. Let $J$ be a right ideal of $kG$, and let $F_1, F_2$ be finite subgroups of $G$ with $F_1 \leq F_2$. Then $\alpha(J, F_1) \leq \alpha(J, F_2)$.

Proof. Clearly, $(J \cap kF_1)kF_2 \leq J \cap kF_2$. Therefore,

$$\dim(J \cap kF_1)[F_2]/[F_1] \leq \dim J \cap kF_2.$$ 

Dividing by $|F_2|$ gives the result. $\blacksquare$

The following is a useful consequence.

Lemma 2.2. Suppose that $G$ is countable, and let $G_1 \leq G_2 \leq \ldots$ be a tower of finite subgroups of $G$ such that $\bigcup_{i=1}^{\infty} G_i = G$. Let $J$ be a right ideal of $kG$ and $\alpha_i = \dim(J \cap kG_i)/|G_i|$. Then $\alpha(J) = \lim_{i \to \infty} \alpha_i$.

Proof. Let $\beta = \lim_{i \to \infty} \alpha_i$. Clearly, $\beta \leq \alpha(J)$. On the other hand, if $F$ is any finite subgroup of $G$, then $F \leq G_i$ for some $i \geq 1$, and then Lemma 2.1 shows that $\alpha(J, F) \leq \alpha_i \leq \beta$. Hence $\alpha(J) \leq \beta$, and the two are equal. $\blacksquare$

Proof of Lemma 1.2: (i) Let $F$ be a finite subgroup of $H$. Then clearly $J \cap kF \leq JkG \cap kF$. Hence $\dim(J \cap kF)/|F| \leq \dim JkG \cap kF)/|F| \leq \alpha(JkG)$. Since $F$ is an arbitrary finite subgroup of $H$, this gives $\alpha(J) \leq \alpha(JkG)$.

(ii) Let $J$ be a right ideal of $kG$ with $\alpha(J) = 1$, and suppose if possible that $L$ is a non-zero right ideal of $kG$ with $J \cap L = 0$. Fix a finite subgroup $E$ of $G$ such that $L \cap kE \neq 0$, and let $F$ be any finite subgroup of $G$ containing $E$. Then $(L \cap kE)kF \cap (J \cap kF) = 0$, and so $\dim(L \cap kE)kF)/|E| \leq \dim(J \cap kF) \leq |F|$. Hence $\alpha(J, F) \leq 1 - \alpha(L, E) < 1$. 
Now if $F_1$ is any finite subgroup of $G$ and we take $F = \langle E, F_1 \rangle$, then we deduce from Lemma 2.1 that $\alpha(J, F_1) \leq 1 - \alpha(L, E) < 1$, whence $\alpha(J) < 1$, a contradiction.

(iii) This is trivial.

(iv) Let $F_1$ be any finite subgroup of $G$, and let $\phi : J \rightarrow L$ be a right $G$-monomorphism. Then $\phi$ maps $J \cap kF_1$ into $L \cap kF_2$, for some finite subgroup $F_2$ of $G$. Let $F = \langle F_1, F_2 \rangle$. Then $\phi$ embeds $(J \cap kF_1)kF$ into $L \cap kF$. Therefore, $\dim(J \cap kF_1) F/|F_1| < \dim(L \cap kF)$. Therefore $\alpha(J, F_1) < \alpha(L, F) < \alpha(L)$, and since $F_1$ is arbitrary, the result follows.

We note that if $G$ is a finite elementary abelian $2$-group of order $2^n$ and $k$ is a field of characteristic $2$, then $\xi^2 = 0$ for all $\xi \in \omega(kG)$. Hence $kG\xi$ annihilates $\xi$, and so $\dim \xi kG \leq 2^{n-1}$. It follows from this that if $H$ is an infinite elementary abelian $2$-group, then $\alpha(J) \leq 1/2$ for each principal ideal $J \leq \omega(kG)$. This may be compared with [6, p.250].

3. The Example

Write $G = \bigcup_{i=1}^{\infty} G_i$, where the $G_i$ form a strictly increasing tower of finite subgroups of $G$. We construct the ideal $J$ to satisfy the following condition:

$\ast$. For each $i$ and primitive idempotent $e \in CG_i$, there exists $\alpha \in CG$ such that $0 \neq e\alpha \in J$.

Since each non-zero ideal of $CG$ contains such an element $e$, we see that $\ast$ implies that $J$ is essential in $CG$.

Now we construct $J$ as the union of a tower $J_1 \leq J_2 \leq \ldots$, where $J_i$ is an ideal of $CG_i$. We begin with any minimal ideal of $CG_1$ as $J_1$. Thus,

\begin{equation}
\dim J_1 = 1.
\end{equation}

We also let $e_1, e_2, \ldots$ be a sequence formed by taking first the (finitely many) primitive idempotents in $CG_1$, then those in $CG_2$, and so on.

Suppose we have $J_i$, a proper ideal of $CG_i$. Then there is a primitive idempotent $e$ of $CG_i$ not in $J_i$, and if we write $\overline{J}_i = J_i CG_{i+1}$, we have the direct sum $eCG_{i+1} \oplus \overline{J}_i$ of ideals of $CG_{i+1}$. Choose the first $j$ such that we have a direct sum $e_j CG_{i+1} \oplus \overline{J}_i$. Thus
For each $l < j$, there exists $\alpha \in CG_{i+1}$ such that $0 \neq e_i \alpha \in \overline{J}$. It will also be clear from the next step that $j \geq i - 1$. Let $f$ be a primitive idempotent in $e_j CG_{i+1}$, and put

$$J_{i+1} = \overline{J}_i \oplus Cf,$$

an ideal of $CG_{i+1}$. Clearly, $J_i \leq J_{i+1}$, and

$$\dim J_{i+1} = 1 + (\dim J_i)|G_{i+1}|/|G_i|$$

(2)

Also, we claim that

$$J_{i+1} \cap CG_i = J_i,$$

(3)

from which it follows in particular that $J_{i+1}$ is a proper ideal of $CG_{i+1}$. To verify the above claim, it suffices to note that if $J_{i+1} \cap CG_i = L_i > J_i$, then $\dim J_{i+1} \geq (\dim L_i)|G_{i+1}|/|G_i| \geq \dim J_{i+1} + |G_{i+1}|/|G_i| - 1$, a contradiction, since the sequence $(G_i)$ is strictly increasing. Putting $J = \cup_{i=1}^\infty J_i$, we have ($\ast$), and so $J$ is essential in $CG$.

Let $\alpha_i = \dim J \cap CG_i/|G_i|$. Then from (2) and (3), we have $\alpha_{i+1} = \alpha_i + 1/|G_{i+1}|$, so from (1),

$$\alpha_i = \sum_{j=1}^i \frac{1}{|G_j|}.$$ 

Now $|G_{i+1}| \geq 2|G_i|$, so

$$\alpha_i \leq \frac{1}{|G_1|} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{2^{i-1}} \right) \leq \frac{2}{|G_1|}.$$ 

Choosing $G_1$ suitably, we obtain what we want.

4. Proof of the Theorem

The theorem follows from [2, Theorem 1.3] and Lemma 4.1 below.

Lemma 4.1. Suppose that $kG$ is regular and the type I part of $Q^*(kG)$ is non-zero. Then $G$ is $\Delta$-hypercentral.

Before beginning the proof we recall some notation and terminology. Let $\Delta(G) = \{ g \in G : |G : CG(g)| < \infty \}$. We define the transfinite upper $\Delta$-series of $G$ by the rules

$$\Delta_0(G) = 1,$$

$$\Delta_{\rho+1}(G)/\Delta_\rho(G) = \Delta(G/\Delta_\rho(G)),$$

$$\Delta_\beta(G) = \cup_{\alpha < \beta} \Delta_\alpha(G),$$
for ordinals $\rho$ and limit ordinals $\beta$. The last term in this series is denoted by $\Delta_\infty(G)$ and called the $\Delta$-hypercentre of $G$. We say that $G$ is $\Delta$-hypercentral, if $G = \Delta_\infty(G)$.

We also write $\pi(G)$ for the set of primes $p$ such that $G$ has an element of order $p$.

We use implicitly the following fact, which is well known. Since we have not found an explicit reference in the literature, we give a proof for completeness. Recall that an idempotent $e$ in a regular ring $R$ is called abelian, if every idempotent in $eR$ is central.

**Lemma 4.2.** Let $R$ be a regular ring with maximal right quotient ring $Q$. Then $R$ contains a non-zero abelian idempotent if and only if $Q$ does.

**Proof:** Let $e$ be an abelian idempotent in $Q$. Then $eQ \cap R \neq 0$, so $eQ \cap R$ contains a non-zero idempotent $f$. Now the map $\phi: fQf \rightarrow feQfe$ defined by $\phi(x) = xe = exe$ is a ring isomorphism with inverse $\phi^{-1}(y) = yf$. Since $feQfe \subseteq eQe$, it follows that $fQf$ is abelian. Therefore so is $fRf$.

Conversely, let $e$ be an abelian idempotent in $R$. We claim that $eRe$ is essential as a right $eRe$-submodule of $eQe$, whence [3, Theorem 3.2 and Corollary 7.4] show that $eQe$ is abelian. Now $eR = eRe \oplus eR(1-e)$, and since the second summand can contain no non-zero idempotent, it can contain no non-zero right ideal of $R$. Let $x$ be a non-zero element of $eQe$. Then $xR \cap R$ is a non-zero right ideal of $R$ contained in $eR$, and hence it is not contained in $eR(1-e)$. Therefore there exists an element $r \in R$ such that $0 \neq xre \in eRe$, and clearly we can replace $r$ by ere here, as required.

**Proof of Lemma 4.1:** Since the type I part of $Q'(kG)$ is non-zero, $kG$ contains a non-zero abelian idempotent $e$. Let $H$ be the subgroup generated by the support of $e$, and $G_1 = \langle \Delta_\infty(G), H \rangle$. Then $e$ is an abelian idempotent of $kG_1$, and so the type I part of $Q = Q'(kG_1)$ is non-zero. Since $G_1$ is $\Delta$-hypercentral, [2, Theorem 1.3] tells us that the type $I_\infty$ part of $Q$ is zero, and therefore its type $I_f$ part must be non-zero. By [1, Theorem 2.3], $|G_1 : \Delta(G_1)| < \infty$ and $\Delta(G_1)'$ is finite. Further, if $M$ is the smallest normal subgroup of $G_1$ such that $G_1/M$ is abelian-by-finite, then the type I part of $Q$ is $(\hat{M}/|M|)Q$, by the first part of the proof of Theorem 2.3 in [1]. Now $M$ is also the smallest normal subgroup of $\Delta_\infty(G)$ such that $\Delta_\infty(G)/M$ is abelian-by-finite, and as such, it is normal in $G$. Since $e \in (\hat{M}/|M|)kG \cong k[G/M]$, we see that we may assume that $\Delta_\infty(G)$ is abelian-by-finite. By [7, Lemma 12.2.2], $\Delta_\infty(G)$ has a characteristic abelian subgroup $A$ of finite index.
Now $A$ is the direct product $A = \prod A_p$ of its primary components. Let $\pi = \pi(H)$, and $A_{\pi'} = \prod_{p \in \pi'} A_p$. Then $A_{\pi'} \subseteq C G$. Consider $\overline{G} = G/A_{\pi'}$. By [3, Lemma 7.6], the image of $e$ in $k\overline{G}$ is an abelian idempotent, and clearly it is non-zero. Therefore the type I part of $Q^r(k\overline{G})$ is non-zero. Further, $\Delta_{\infty}(G) = \Delta_{\infty}(G)/A_{\pi'}$. Therefore, we may assume that $A_{\pi'} = 1$, and so $\pi(\Delta_{\infty}(G)) = \sigma$ is finite.

Let $p$ be a prime such that $\alpha(e_kG) > \frac{1}{p}$, and $p \notin \pi \cup \sigma$. We shall see that $p \notin \pi(G)$. Suppose on the contrary that $G$ contains an element $g$ of order $p$. Let $\tilde{g} = 1 + g + \ldots + g^{p-1}$. Since the powers of $g$ lie in distinct cosets of $G_1$, we see that $\tilde{g}e \neq 0$. Since $kG$ is regular, there exists $\beta \in kG$ such that

\begin{equation}
\tilde{g}e = \tilde{g}e\beta e.
\end{equation}

By squaring it, we see that $e\beta e$ is an idempotent in $ekG$. By [2, Lemma 2.1], we see that $e\beta e \in kG_1$. Using the fact that we have a direct sum \(\sum_{i=0}^{\infty} g^i kG_1\), we deduce from (4) that $e = e\beta e$. It follows from this that the map $f : ekG \rightarrow \tilde{g}e\beta kG$ defined by $f(\gamma) = \tilde{g}\gamma$ is an isomorphism, $(f^{-1}(e) = e\beta e)$, and so by Lemma 1.2 (iv), $\alpha(ekG) \leq \alpha(\tilde{g}kG)$. But an easy calculation shows that $\alpha(\tilde{g}kG) = \frac{1}{p}$. This contradicts the choice of $p$. Hence $\pi(G)$ is finite. By [1, Proposition 1.2], we find that $G$ is hypercentral, and the proof is complete. \[\blacksquare\]

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