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WREATH PRODUCTS AND FITTING CLASSES OF $\mathfrak{S}_1$-GROUPS

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Abstract

A Fitting class $\mathfrak{X}$ of $\mathfrak{S}_1$-groups is *normal* if $G_{\mathfrak{X}}$ is the unique $\mathfrak{X}$-injector of G, for each $G \in \mathfrak{S}_1$; $\mathfrak{X}$ is *abelian normal* if $G_{\mathfrak{X}} \geq G'$ for each $G \in \mathfrak{S}_1$. It is a well known result of Blessenohl and Gaschütz that the corresponding concepts coincide for finite soluble groups. Here we consider the *wreath product property* (*wpp*) : $\mathfrak{X}$ satisfies *wpp* if whenever $G \in \mathfrak{X}$ and p is a prime, there is an integer n such that $G^n \wr C_p \in \mathfrak{X}$. An abelian normal Fitting class satisfies *wpp* but a nonabelian normal Fitting class may not. Embedding theorems related to those of Blessenohl and Gaschütz show further distinctions between abelian and nonabelian normal Fitting classes. For example, if $\mathfrak{X}$ is an abelian normal Fitting class, then $s\mathfrak{X} = \mathfrak{S}_1$; this is false for nonabelian normal Fitting classes.

1. Introduction

In [1], we introduced the concept of a Fitting class $\mathfrak{X}$ of $\mathfrak{K}$-groups, for certain subclasses $\mathfrak{K}$ of $\mathfrak{S}_1$, the class of soluble groups in which each abelian section has finite total rank. We obtained a sufficient condition for the existence and conjugacy of $\mathfrak{X}$-injectors. Menegazzo and Newell [9] proved a form of converse so that we have the following result: Let $\mathfrak{X}$ be a Fitting class of $\mathfrak{K}$-groups. Then every $\mathfrak{K}$-group has $\mathfrak{X}$-injectors if and only if, for each $G \in \mathfrak{K}$, there is a normal subgroup M of G such that (i) every $\mathfrak{X}$-subgroup of G containing the $\mathfrak{X}$-radical $G_{\mathfrak{X}}$ is contained in M and (ii) $M/G_{\mathfrak{X}}$ is finite. If $\mathfrak{X}$ is such a $\mathfrak{K}$-Fitting class then the $\mathfrak{X}$-injectors of G are necessarily conjugate.

In [3], [4] it was observed that in all the known examples, the subgroup M above can be chosen to be the $\mathfrak{Y}$-radical of G for some normal $\mathfrak{K}$-Fitting class $\mathfrak{Y}$. A $\mathfrak{K}$-Fitting class $\mathfrak{Y}$ is *normal* if $G_{\mathfrak{Y}}$ is the unique $\mathfrak{Y}$-injector of G for each $G \in \mathfrak{K}$. We say that $\mathfrak{Y}$ is an *abelian normal* $\mathfrak{K}$-Fitting class if $G_{\mathfrak{Y}} \geq G'$, for each $G \in \mathfrak{K}$. It is a well known result of

Blessenohl and Gaschütz [5] that for finite soluble groups every normal Fitting class is abelian normal. This is not the case for $\mathfrak{S}_1$-Fitting classes.

One of the most important constructions used in the finite case for investigating normal Fitting classes is the wreath product and our aim here is to see how this can be used in the infinite case, where it is seen to be of most value in considering abelian normal Fitting classes and shows further distinctions between the abelian and nonabelian normal Fitting classes.

We say that a $\mathfrak{K}$-Fitting class $\mathfrak{X}$ satisfies the *wreath product property* if, whenever $G \in \mathfrak{X}$ and p is a prime, there is a positive integer $n = n(G,p)$ such that $G^n \wr C_p \in \mathfrak{X}$, where G^n denotes the direct product of n copies of G.

Fitting classes of finite soluble groups with the wreath product property were first studied by Makan [8]. He showed that such classes are normal (and hence abelian normal by Satz 5.3 of [5]). Hauck [7] also investigated Fitting classes of finite soluble groups which satisfied certain properties of wreath products. The wreath product property seems to highlight some of the important features of the proofs of our results and also illustrates more clearly the methods we are using here.

Our first result (Lemma 3.1) shows that an abelian normal $\mathfrak{K}$-Fitting class satisfies the wreath product property. This result was one of the steps in the proof by Blessenohl and Gaschütz [5] that a normal Fitting class of finite soluble groups is abelian normal. Their proof can be extended to normal Fitting classes of Černikov groups (Theorem 3.6). However, examples show that this result does not hold for normal $\mathfrak{K}$-Fitting classes if the class $\mathfrak{K}$ contains nonperiodic groups even if $G/G_{\mathfrak{X}}$ is always finite or if $\mathfrak{K}$ consists of abelian-by-finite groups.

The final section is devoted to embedding theorems related to further results of Blessenohl and Gaschütz [5]. We show in Theorem 4.2 that if $\mathfrak{X}$ is an abelian normal $\mathfrak{K}$-Fitting class then $s\mathfrak{X} = \mathfrak{K}$. This result fails for nonabelian normal Fitting classes; for example, we saw in [3] that the class of Černikov-by-nilpotent groups is an s-closed Fitting class of $\mathfrak{S}_1$-groups.

The second embedding theorem of Blessenohl and Gaschütz [5] considered here is the embedding of finite soluble groups in $G/G_{\mathfrak{X}}$ if $\mathfrak{X}$ is a non-normal Fitting class of finite soluble groups. The results in the infinite case are rather more complicated as there are obviously many $\mathfrak{K}$-Fitting classes such that $\mathfrak{K}$-groups can not be embedded in $G/G_{\mathfrak{X}}$. There are $\mathfrak{K}$-Fitting classes $\mathfrak{X}$ such that $G/G_{\mathfrak{X}}$ is always finite and, for any $\mathfrak{X}$ which contains all hypercentral $\mathfrak{K}$-groups, $G/G_{\mathfrak{X}}$ is finitely generated abelian-by-finite [1, Theorem 3.1]. Because of this we prove two results here. Theorem 4.5 asserts that if $\mathfrak{X}$ is a $\mathfrak{K}$-Fitting class which is not abelian normal and if H is a finite soluble group then there is a $\mathfrak{K}$-group G such that H is isomorphic to a subgroup of $G/G_{\mathfrak{X}}$.

Our second embedding result (Theorem 4.6) requires the assumption that there is a $\mathfrak{K}$-group L such that $L/L_{\mathfrak{X}}$ is infinite nonabelian and some further fairly weak technical restrictions but then asserts that if H is a finitely generated abelian-by-finite group then there is a $\mathfrak{K}$-group G such that H is isomorphic to a subgroup of $G/G_{\mathfrak{X}}$.

The proof of Theorem 4.5 is close to that for the finite case, again making considerable use of wreath products. These results can be interpreted as saying that abelian normal $\mathfrak{K}$-Fitting classes are large ($s\mathfrak{X} = \mathfrak{K}$) whereas any other Fitting class is rather small since all finite soluble groups can appear above the $\mathfrak{X}$-radical. This provides a further illustration of the difference between abelian and nonabelian normal Fitting classes. These results also indicate some of the limitations of the wreath product in investigating nonabelian normal Fitting classes.

2. Notation

An $\mathfrak{S}_1$-group is a group with a finite normal series whose factors are abelian groups of finite rank and whose torsion subgroups are Černikov groups [**9**, Part 2, p. 137]. The following subclasses of $\mathfrak{S}_1$ will be referred to here.

$\mathfrak{H}$, the class of locally nilpotent (or hypercentral) $\mathfrak{S}_1$-groups

$\mathfrak{N}$, the class of nilpotent $\mathfrak{S}_1$-groups

$\mathfrak{F}$, the class of finite soluble groups

$\mathfrak{C}$, the class of soluble Černikov groups (or extremal groups)

$\mathfrak{P}$, the class of polycyclic groups

$\mathfrak{M}$, the class of soluble minimax groups

$\mathfrak{X}\,\mathfrak{Y} = \{G \in \mathfrak{S}_1 : G/G_{\mathfrak{X}} \in \mathfrak{Y}\}$, where $\mathfrak{X}$ and $\mathfrak{Y}$ are $\mathfrak{S}_1$-Fitting classes

$\mathfrak{C}\,\mathfrak{N}$, the class of all Černikov-by-nilpotent $\mathfrak{S}_1$-groups.

Throughout, $\mathfrak{K}$ denotes an $\{s, D_0\}$-closed subclass of $\mathfrak{S}_1$ which is closed under finite soluble extensions. Examples of such $\mathfrak{K}$ include $\mathfrak{F}$, $\mathfrak{C}$, $\mathfrak{P}$, $\mathfrak{M}$ and $\mathfrak{S}_1$ itself. Also the classes of abelian-by-finite groups in any of these classes can also be taken for $\mathfrak{K}$.

A $\mathfrak{K}$-Fitting class $\mathfrak{X}$ is a subclass of $\mathfrak{K}$ which is closed under ascendant subgroups and such that every $\mathfrak{K}$-group which is a join of ascendant $\mathfrak{X}$-subgroups is an $\mathfrak{X}$-group. Let p be a prime, two of the less obvious examples described in [**1**] that will be referred to are

$$\mathfrak{C}(p) = \{G \in \mathfrak{S}_1 : \mathrm{Soc}_p(G) \leq Z(G)\}$$
$$\mathfrak{Z}(p) = \{G \in \mathfrak{P} : G/C_G(O_p(G)) \text{ is a } p\text{-group}\}.$$

We recall that a $\mathfrak{K}$-Fitting class $\mathfrak{X}$ is called a *Lockett class* provided that $\mathfrak{X}^* = \mathfrak{X}$. Properties of Lockett classes and the Lockett *-construction are given in detail in [**2**]. If $\mathfrak{X}$ is a $\mathfrak{K}$-Fitting class the *Lockett section* of $\mathfrak{X}$ consists of all $\mathfrak{K}$-Fitting classes $\mathfrak{Y}$ such that $\mathfrak{Y}^* = \mathfrak{X}^*$. This is denoted by $\mathrm{Locksec}(\mathfrak{X})$.

3. Wreath Product Property

Our first two results provide examples of certain types of Fitting class with the wreath product property.

Lemma 3.1. *Let $\mathfrak{X}$ be an abelian normal $\mathfrak{K}$-Fitting class. Then $\mathfrak{X}$ satisfies the wreath product property.*

Proof: Let $G \in \mathfrak{X}$ and let p be a prime. Let q be a prime distinct from p and let M be a faithful irreducible $\mathbb{Z}_pC_q$-module. Let $Y = MC_q$ be the semidirect product of M by C_q; then $Y' = M$. Let $W = G \wr Y$ and let B be the base group. Since $\mathfrak{K}$ is closed under finite direct products and finite extensions, W is a $\mathfrak{K}$-group. Also $B \in \mathfrak{X}$ and so $W_{\mathfrak{X}} \geq B$. Since $\mathfrak{X}$ is abelian normal, $W_{\mathfrak{X}} \geq BY'$ and so $BY' \in \mathfrak{X}$. Let C be a subgroup of M of order p. Then $C \triangleleft M$ and so $BC \triangleleft BY'$ and $BC \in \mathfrak{X}$. But $BC \cong G^n \wr C_p$, where $n = |Y : C|$. Therefore, $\mathfrak{X}$ satisfies the wreath product property. ■

Lemma 3.2. *Let $\mathfrak{X}$ be a normal $\mathfrak{C}$-Fitting class. Let $G \in \mathfrak{X}$, let p be a prime and Z a cyclic group of order p. Then there is a positive integer $m = m(G,p)$ such that $G^{nm} \wr Z \in \mathfrak{X}$ for all positive integers n. In particular, $\mathfrak{X}$ satisfies the wreath product property.*

Proof: Assume that the lemma is false. Since we are only considering Černikov groups we can choose a counterexample (G,p) such that G is minimal; that is, the lemma holds for all proper subgroups of G. By Lemma 3.1 of [**4**], $\mathfrak{X}$ contains all hypercentral $\mathfrak{C}$-groups. In particular $Z \in \mathfrak{X}$ and so $G \neq 1$. Thus G has a proper normal subgroup G_1 such that G/G_1 is a q-group for some prime q.

If $q \neq p$, let Y be the group constructed in Lemma 3.1 with unique minimal normal subgroup M of order p^α. If $q = p$, let $Y = C_q$. In either case, let Z_1 be a subgroup of Y of order q.

Since $G_1 < G$, there is an integer m_1 such that $G_1^{nm_1} \wr Z_1 \in \mathfrak{X}$, for all positive integers n. Now for an arbitrary positive integer n, consider $W = G^{nm_1} \wr Y$. Let $D = (G^{nm_1})^Y$ be the base group of W and let $D_1 = (G_1^{nm_1})^Y \leq D$. As in the proof of Lemma 5.1 by Blessenohl and Gaschütz [**5**], $D_1 Z_1 \cong G_1^{nm_1 p^\alpha} \wr Z_1 \in \mathfrak{X}$ and DZ_1/D_1 is a q-group. Therefore $D_1 Z_1$ is an ascendant subgroup of DZ_1 and so $D_1 Z_1 \leq (DZ_1)_{\mathfrak{X}}$. But D is a normal $\mathfrak{X}$-subgroup of DZ_1 and so $(DZ_1)_{\mathfrak{X}} \geq DD_1 Z_1 = DZ_1$; that is, $DZ_1 \in \mathfrak{X}$.

If $q = p$, then $G^{nm_1} \wr Z \cong DZ_1 \in \mathfrak{X}$ and we can take $m = m_1$. Suppose therefore that $q \neq p$. In this case $W_{\mathfrak{X}} \geq D$ and $W_{\mathfrak{X}}$ is a maximal $\underline{\underline{X}}$-subgroup of W. But D is properly contained in the $\mathfrak{X}$-subgroup $D\overline{Z_1}$. Therefore $D < W_{\mathfrak{X}}$. Since W/D has a unique minimal normal subgroup DM/D, we have $DM \leq W_{\mathfrak{X}}$. There is a subgroup Z of M having order p and $DZ \lhd DM \lhd W_{\mathfrak{X}}$ so that $DZ \in \mathfrak{X}$. But $DZ \cong G^{nm_1 p^{\alpha-1} q} \wr Z$, since $|Y : Z| = p^{\alpha-1} q$, and we can take $m = m_1 p^{\alpha-1} q$. This completes the proof. ■

The next result, which generalizes a result of J. Cossey [**6**, Lemma 2.2], indicates how radicals of Lockett classes behave in wreath products.

Lemma 3.3. *Let $\mathfrak{X}$ be a $\mathfrak{K}$-Lockett class and let $G \in \mathfrak{K} \setminus \mathfrak{X}$. If H is a finite soluble group and n a positive integer, then $(G^n \wr H)_{\mathfrak{X}} = B_{\mathfrak{X}}$, where B is the base group of $G^n \wr H$.*

Proof: Let $W = G^n \wr H$; then the base group of W is $B = G^{nh}$, where $h = |H|$. By Theorem 2.9 of [**2**], $B_{\mathfrak{X}} = (G_{\mathfrak{X}})^{nh}$.

Note that $W/B_{\mathfrak{X}} \cong (G/G_{\mathfrak{X}})^n \wr H$ and, under this isomorphism, $B/B_{\mathfrak{X}}$ corresponds to the base group. Therefore, the centralizer of $B/B_{\mathfrak{X}}$ in $W/B_{\mathfrak{X}}$ is contained in $B/B_{\mathfrak{X}}$. But $[W_{\mathfrak{X}}, B] \leq W_{\mathfrak{X}} \cap B = B_{\mathfrak{X}}$ and so $W_{\mathfrak{X}}$ centralizes $B/B_{\mathfrak{X}}$. Therefore $W_{\mathfrak{X}} \leq B$ and so $W_{\mathfrak{X}} = B_{\mathfrak{X}}$, as required. ■

Theorem 3.4. *Let $\mathfrak{X}$ be a $\mathfrak{K}$-Lockett class satisfying the wreath product property.*

(i) *If $\mathfrak{Y} \in$ Locksec(X), then $\mathfrak{Y}$ satisfies the wreath product property.*
(ii) *$\mathfrak{X}\mathfrak{F} = \mathfrak{X}$ (see Satz 4.1 of [**7**]).*

Proof: (i) This follows in exactly the same way as the result is established for $\mathfrak{F}$-Lockett classes in Lemma 5.6 of [**7**].

(ii) Suppose that $\mathfrak{X}\mathfrak{F} \neq \mathfrak{X}$ and let $G \in \mathfrak{X}\mathfrak{F} \setminus \mathfrak{X}$. Then $G/G_{\mathfrak{X}}$ is a finite soluble group and hence contains a subnormal subgroup $H/G_{\mathfrak{X}}$ of prime order p. Thus $H_{\mathfrak{X}} = G_{\mathfrak{X}}$ and $H/H_{\mathfrak{X}} \cong C_p$.

By hypothesis, there is a positive integer n such that $(H_{\mathfrak{X}})^n \wr C_p \in \mathfrak{X}$. Let $W = H^n \wr C_p$ and let $B = H^{np}$ be the base group of W. By Lemma

3.3, $W_{\mathfrak{X}} = B_{\mathfrak{X}}$. But $W/B_{\mathfrak{X}}$ is a finite p-group and so $(H_{\mathfrak{X}})^n \wr C_p$ sn W contrary to $W_{\mathfrak{X}} = B_{\mathfrak{X}} = (H_{\mathfrak{X}})^{np}$. ■

We now consider the converse of Lemma 3.1; when is a Fitting class with the wreath product property an abelian normal Fitting class?

Lemma 3.5. *Let $\mathfrak{K} \subseteq \mathfrak{H}\,\mathfrak{F}$ and let $\mathfrak{X}$ be a $\mathfrak{K}$-Fitting class such that $\mathfrak{X} \supseteq \mathfrak{H} \cap \mathfrak{K}$. If $\mathfrak{X}$ satisfies the wreath product property then $\mathfrak{X}$ is an abelian normal $\mathfrak{K}$-Fitting class.*

Proof: We show that $\mathfrak{X}^* = \mathfrak{K}$ and then it follows from Theorem 2.1 of [**4**] that $\mathfrak{X}$ is abelian normal. Suppose then that $\mathfrak{X}^* \neq \mathfrak{K}$ and let $G \in \mathfrak{K} \setminus \mathfrak{X}^*$. Then $G/G_{\mathfrak{X}^*}$ contains a subnormal subgroup $H/G_{\mathfrak{X}^*}$ which is cyclic of prime order p. Note $H_{\mathfrak{X}^*} = G_{\mathfrak{X}^*}$. By Theorem 2.3(d) of [**2**], $H_{\mathfrak{X}^*}/H_{\mathfrak{X}}$ is central in H and so $H/H_{\mathfrak{X}}$ is a finite nilpotent group. Let $P/H_{\mathfrak{X}}$ be the Sylow p-group of $H/H_{\mathfrak{X}}$. Then $P/P_{\mathfrak{X}^*} \cong C_p$ and $P/P_{\mathfrak{X}}$ is a finite p-group. Since $\mathfrak{X}$ satisfies the wreath product property, there is a positive integer n such that $(P_{\mathfrak{X}})^n \wr C_p \in \mathfrak{X}$. Let $W = P^n \wr C_p$ and note that $W/(P_{\mathfrak{X}})^{np}$ is a p-group. Thus $(P_{\mathfrak{X}})^n \wr C_p$ sn W and so $(P_{\mathfrak{X}})^n \wr C_p \leq W_{\mathfrak{X}} \leq W_{\mathfrak{X}^*}$. By Lemma 3.3, $W_{\mathfrak{X}^*} = (P_{\mathfrak{X}^*})^{np}$, which is a contradiction. Hence $X^* = \mathfrak{K}$, as required. ■

It should be noted that the conditions on $\mathfrak{X}$ and $\mathfrak{K}$ in Lemma 3.5 are necessary. Firstly, it is possible to have a Fitting class $\mathfrak{X}$ satisfying the wreath product property but not containing all hypercentral $\mathfrak{K}$-groups and in this case $\mathfrak{X}$ need not be a normal Fitting class. For example, we could take $\mathfrak{X} = \mathfrak{C}$ and $\mathfrak{K} = \mathfrak{H}\,\mathfrak{F}$. If we omit the condition that $\mathfrak{K} \subseteq \mathfrak{H}\,\mathfrak{F}$ then we can not say that $H/H_{\mathfrak{X}}$ is finite and there will be no similar results. For example, let $\mathfrak{K} = \mathfrak{P}$ and $\mathfrak{X} = \mathfrak{N}\,\mathfrak{F}$. Then $\mathfrak{X}$ is s-closed and so is even a Lockett class. If $G \in \mathfrak{N}\,\mathfrak{F}$, then $G \wr C_p \in \mathfrak{N}\,\mathfrak{F}$ and so $\mathfrak{N}\,\mathfrak{F}$ satisfies the wreath product property. But $\mathfrak{N}\,\mathfrak{F}$ is not a normal $\mathfrak{P}$-Fitting class.

The following theorem shows that a normal $\mathfrak{C}$-Fitting class is abelian normal. This generalizes Satz 5.3 of [**5**]. Note also that it generalizes Theorem 3.2 of [**7**].

Theorem 3.6. *Let $\mathfrak{X}$ be an $\mathfrak{C}$-Fitting class. Then the following are equivalent:*

(a) *$\mathfrak{X}$ is a normal $\mathfrak{C}$-Fitting class.*
(b) *$\mathfrak{X}$ satisfies the wreath product property.*
(c) *$\mathfrak{X}$ is an abelian normal $\mathfrak{C}$-Fitting class.*

Proof: (a) implies (b) is Lemma 3.2. (b) implies (c) follows from Lemma 3.5 since $\mathfrak{C} \subseteq \mathfrak{H}\,\mathfrak{F}$ and any Fitting class $\mathfrak{X}$ which satisfies the wreath product property contains all cyclic groups of prime order and so contains all hypercentral $\mathfrak{C}$-groups. It is clear that (c) implies (a). ■

One might expect that Theorem 3.6 would extend to further classes $\mathfrak{K}$ or perhaps hold with finiteness conditions on $G/G_{\mathfrak{X}}$. However, if we take $\mathfrak{K}$ to be the class of abelian-by-finite polycyclic groups then $\mathfrak{X} = \mathfrak{C}\mathfrak{N} \cap \mathfrak{K}$ is a normal $\mathfrak{K}$-Fitting class by the main theorem of [3] and $G/G_{\mathfrak{X}}$ is finite for each $G \in \mathfrak{K}$. But $G = C_\infty \wr S_3$ has $G_{\mathfrak{C}\mathfrak{N}} = G_{\mathfrak{N}}$ and $G/G_{\mathfrak{N}} \cong S_3$ is nonabelian.

4. Embedding theorems

This section is devoted to obtaining appropriate generalizations of Satz 5.3 and Satz 6.3 of [5] for Fitting classes of $\mathfrak{S}_1$-groups. The following simple lemma will be useful in dealing with infinite cyclic factors.

Lemma 4.1. *Let the group G be an extension of the group H by an infinite cyclic group $\langle x\rangle$. If $K = (H \times H)\langle(x, x^{-1})\rangle \leq G \times G$, then G is isomorphic to a subgroup of K.*

Proof: The mapping $\phi : G \to K$ defined by $(hx^n)\phi = (hx^n, x^{-n})$ is a monomorphism. ■

Theorem 4.2. *If $\mathfrak{X}$ is an abelian normal $\mathfrak{K}$-Fitting class then $s\mathfrak{X} = \mathfrak{K}$.*

Proof: Let $G \in \mathfrak{K}$; then $G/G_{\mathfrak{X}}$ is abelian. By Lemma 3.1 of [4], $\mathfrak{H} \cap \mathfrak{K} \subseteq \mathfrak{X}$ and so $G/G_{\mathfrak{X}}$ is finitely generated [1, Theorem 3.1]. We prove by induction on the torsion-free rank r of $G/G_{\mathfrak{X}}$ that $G \in s\mathfrak{X}$.

Case 1. $r = 0$.

In this case G contains an $s\mathfrak{X}$-subgroup $L = G_{\mathfrak{X}}$ of finite index and we use induction on $|G/L|$. Since G/L is finite abelian it has a maximal normal subgroup M/L such that G/M is cyclic of prime order p. By induction, $M \in s\mathfrak{X}$ and so there is an $\mathfrak{X}$-group X and a subgroup X_0 of X such that $X_0 \cong M$. Now G is isomorphic to a subgroup G_0 of $X_0 \wr C_p$. By Lemma 3.1 there is a positive integer n such that $X^n \wr C_p \in \mathfrak{X}$. Therefore $G \cong G_0 \leq X_0 \wr C_p \leq X \wr C_p \leq X^n \wr C_p$ and so $G \in s\mathfrak{X}$.

Case 2. $r \geq 1$.

In this case G has a normal subgroup A such that G/A is infinite cyclic and $A/G_{\mathfrak{X}}$ has torsion-free rank $r - 1$. Let $T = \langle t\rangle$ be a cyclic group of order 2 and let the wreath product $W = G \wr T$ have base group $B = G \times G$. Let $G = A\langle x\rangle$ and consider $K = (A \times A)\langle(x, x^{-1}), t\rangle \leq W$. Then $K_{\mathfrak{X}} \geq (G_{\mathfrak{X}} \times G_{\mathfrak{X}})K'$. But, for each $a \in A$, K' contains the element $(a, a^{-1}) = [(1, a), t]$. Also K' contains $(x, x^{-1})^2 = [(x^{-1}, x), t]$ and so $K/K_{\mathfrak{X}}$ has torsion-free rank $r - 1$. By induction, $K \in s\mathfrak{X}$. By Lemma 4.1, G is isomorphic to a subgroup of $(A \times A)\langle(x, x^{-1})\rangle \leq K$ and so $G \in s\mathfrak{X}$. ■

The following two lemmas, which are used to establish Theorems 4.5 and 4.6, are generalizations of Lemmas 5.2 and 5.3 of [**5**]. The proofs of these results are the same as in [**5**] and hence are omitted.

Lemma 4.3. *Let $\mathfrak{X}$ be a $\mathfrak{K}$-Fitting class and let $G \in \mathfrak{K}$. Let $G = N_1 N_2 \cdots N_r$, where $N_i \lhd G$, $1 \leq i \leq r$. Then $G_{\mathfrak{X}}/\prod_{i=1}^{r}(N_i)_{\mathfrak{X}}$ is contained in the centre of $G/\prod_{i=1}^{r}(N_i)_{\mathfrak{X}}$.*

Lemma 4.4. *Let X and Y be groups and let $G = X \wr Y$. Let N be an abelian normal subgroup of G which is contained in the base group B of G. If $C_G(B/N)$ is not contained in B, then X is abelian.*

Theorem 4.5. *Let $\mathfrak{X}$ be a $\mathfrak{K}$-Fitting class which is not abelian normal. If $H \in \mathfrak{F}$, then there is a $\mathfrak{K}$-group G such that H is isomorphic to a subgroup of $G/G_{\mathfrak{X}}$.*

Proof: Since $\mathfrak{X}$ is not an abelian normal Fitting class, there is a group $L \in \mathfrak{K}$ such that $L/L_{\mathfrak{X}}$ is nonabelian. Let $H \in \mathfrak{F}$ and let $G = L \wr H$. Let B be the base group of G so that $B = L^m$, where $m = |H|$. Then $B_{\mathfrak{X}} \leq G_{\mathfrak{X}}$ and $B_{\mathfrak{X}}/(L_{\mathfrak{X}})^m$ is abelian, by Lemma 4.3.

Suppose that $G_{\mathfrak{X}}$ is not contained in B and let $W = (L/L_{\mathfrak{X}}) \wr H$; then $B_{\mathfrak{X}}/(L_{\mathfrak{X}})^m$ is an abelian normal subgroup of W. Since $[G_{\mathfrak{X}}, B] \leq G_{\mathfrak{X}} \cap B = B_{\mathfrak{X}}$, it follows that $G_{\mathfrak{X}}/(L_{\mathfrak{X}})^m$ centralizes $(L/L_{\mathfrak{X}})^m/(B_{\mathfrak{X}}/(L_{\mathfrak{X}})^m)$. It follows from Lemma 4.4 that $L/L_{\mathfrak{X}}$ is abelian, contrary to our choice of L. Therefore, $G_{\mathfrak{X}} \leq B$ and so $G/G_{\mathfrak{X}} \geq HG_{\mathfrak{X}}/G_{\mathfrak{X}} \cong H$. ■

Let $\mathfrak{X}$ be any one of the following $\mathfrak{S}_1$-Fitting classes: $\mathfrak{H}, \mathfrak{H}^2, \mathfrak{C}\mathfrak{N}, \mathfrak{C}(p)$ or $\mathfrak{H}\mathfrak{F}(p)$. Then $\mathfrak{X}$ is not abelian normal, and Theorem 4.5 shows that if H is a finite soluble group then there is an $\mathfrak{S}_1$-group G such that $G/G_{\mathfrak{X}}$ contains a subgroup isomorphic to H. It should be noted that this result applies to nonabelian normal Fitting classes (e.g. $\mathfrak{C}\mathfrak{N}$). If $\mathfrak{X}$ is one of the classes $\mathfrak{H}^2, \mathfrak{C}(p)$ or $\mathfrak{H}\mathfrak{F}(p)$ then $G/G_{\mathfrak{X}}$ is always a finite soluble group and so there is no possibility of embedding an arbitrary $\mathfrak{S}_1$-group in $G/G_{\mathfrak{X}}$. In order to extend Theorem 4.5 therefore it is necessary to consider Fitting classes such that $G/G_{\mathfrak{X}}$ is not always finite. Most of the $\mathfrak{K}$-Fitting classes in which we are interested contain $\mathfrak{H} \cap \mathfrak{K}$; this is the case for normal $\mathfrak{K}$-Fitting classes and, if $\mathfrak{K} \supseteq \mathfrak{P}$, it is true for all $\mathfrak{K}$-Fitting classes such that every $\mathfrak{K}$-group has $\mathfrak{X}$-injectors. If $\mathfrak{X} \supseteq \mathfrak{H} \cap \mathfrak{K}$ then $G/G_{\mathfrak{X}}$ is finitely generated abelian-by-finite and so we consider an extension of Theorem 4.5 in which finitely generated abelian-by-finite groups are embedded in $G/G_{\mathfrak{X}}$.

Theorem 4.6. *Let $\mathfrak{X}$ be a $\mathfrak{K}$-Fitting class such that $\mathfrak{H} \cap \mathfrak{K} \subseteq \mathfrak{X}$ and there is a $\mathfrak{K}$-group L such that $L/L_{\mathfrak{X}}$ is infinite nonabelian (in particular, $\mathfrak{X}$ is not an abelian normal Fitting class). Suppose also that $\mathfrak{X}$ satisfies one of the two conditions:*

(a) $\mathfrak{X}^* = \mathfrak{X}$;
(b) *there is a $\mathfrak{K}$-group T such that $T/T_{\mathfrak{X}}$ is infinite nonabelian and has finite centre.*

If H is a polycyclic abelian-by-finite group, then there is a group $G \in \mathfrak{K}$ such that H is isomorphic to a subgroup of $G/G_{\mathfrak{X}}$.

Proof: The polycyclic abelian-by-finite group H contains a free abelian normal subgroup M of finite rank r, say, such that H/M is finite. We will show first that there is a $\mathfrak{K}$-group B such that $B/B_{\mathfrak{X}}$ is nonabelian and $B/B_{\mathfrak{X}}$ has a free abelian normal subgroup of rank at least r.

Let L be a $\mathfrak{K}$-group such that $L/L_{\mathfrak{X}}$ is infinite and nonabelian; then by Theorem 3.1 of [**1**], $L/L_{\mathfrak{X}}$ has a free abelian normal subgroup of rank s, say. Choose a positive integer m such that $ms \geq r$ and let $B = L^m$.

If $\mathfrak{X}$ satisfies condition (a) $\mathfrak{X}^* = \mathfrak{X}$, then $B_{\mathfrak{X}} = (L_{\mathfrak{X}})^m$ [**2**, Theorem 2.9] and so $B/B_{\mathfrak{X}}$ has a free abelian normal subgroup of rank $ms \geq r$. Also $B/B_{\mathfrak{X}}$ is nonabelian. So suppose that $\mathfrak{X}$ satisfies condition (b); then we may suppose that $L/L_{\mathfrak{X}}$ has finite centre. Therefore $Z(B/(L_{\mathfrak{X}})^m) \cong (Z(L/L_{\mathfrak{X}}))^m$ is also finite. By Lemma 4.3, $B_{\mathfrak{X}}/(L_{\mathfrak{X}})^m \leq Z(B/(L_{\mathfrak{X}})^m)$ and so $B_{\mathfrak{X}}/(L_{\mathfrak{X}})^m$ is finite. Now $B/(L_{\mathfrak{X}})^m$ has a free abelian normal subgroup $A/(L_{\mathfrak{X}})^m$ of rank $ms \geq r$. Since $B_{\mathfrak{X}}/(L_{\mathfrak{X}})^m$ is finite, $A/(L_{\mathfrak{X}})^m \cong AB_{\mathfrak{X}}/B_{\mathfrak{X}}$ and so $AB_{\mathfrak{X}}/B_{\mathfrak{X}}$ is a free abelian normal subgroup of $B/B_{\mathfrak{X}}$ of rank at least r. Also, since $L/L_{\mathfrak{X}}$ is nonabelian, $B/B_{\mathfrak{X}}$ must be nonabelian.

Therefore, using either (a) or (b), we have shown that there is a $\mathfrak{K}$-group B such that $B/B_{\mathfrak{X}}$ is nonabelian and has a free abelian normal subgroup $A/B_{\mathfrak{X}}$ of rank at least r.

Now let $F = H/M$ have order n and let $G = B \wr F$. As in the proof of Theorem 4.5 we have $(B_{\mathfrak{X}})^n \leq G_{\mathfrak{X}} \leq B^n$, the base group of G. Thus $G_{\mathfrak{X}} = (B^n)_{\mathfrak{X}}$. In case (a), $(B_{\mathfrak{X}})^n = (B^n)_{\mathfrak{X}} = G_{\mathfrak{X}}$ while in case (b), the same argument as for $B_{\mathfrak{X}}/(L_{\mathfrak{X}})^m$ shows that $(B^n)_{\mathfrak{X}}/(B_{\mathfrak{X}})^n$ is finite. Now $A^n/(B_{\mathfrak{X}})^n$ is a free abelian normal subgroup of $G/(B_{\mathfrak{X}})^n$

and $A^n \cap G_{\mathfrak{X}} \le A^n \cap (B^n)_{\mathfrak{X}}$ so that $A^n \cap G_{\mathfrak{X}} = (B_{\mathfrak{X}})^n$. Therefore, we have

$$\begin{aligned} FA^n \cap G_{\mathfrak{X}} &= FA^n \cap (B^n \cap G_{\mathfrak{X}}) \\ &= A^n(F \cap B^n) \cap G_{\mathfrak{X}} \\ &= A^n \cap G_{\mathfrak{X}} \\ &= (B_{\mathfrak{X}})^n. \end{aligned}$$

Thus $FA^nG_{\mathfrak{X}}/G_{\mathfrak{X}} \cong FA^n/(B_{\mathfrak{X}})^n \cong (A/B_{\mathfrak{X}}) \wr F$. But $A/B_{\mathfrak{X}}$ contains a free abelian subgroup of rank r, say $A_0/B_{\mathfrak{X}} \cong M$. Thus $W = (A/B_{\mathfrak{X}}) \wr F$ contains a subgroup isomorphic to $M \wr F$. This in turn contains a subgroup isomorphic to H. Hence H is isomorphic to a subgroup of $FA^nG_{\mathfrak{X}}/G_{\mathfrak{X}}$ and so to a subgroup of $G/G_{\mathfrak{X}}$. ∎

The conditions (a) and (b) in this theorem are fairly weak conditions but some restriction is necessary. For example, let $\mathfrak{Y}$ be an abelian normal $\mathfrak{S}_1$-Fitting class such that there is an $\mathfrak{S}_1$-group G with $G/G_{\mathfrak{Y}}$ infinite cyclic. Such an $\mathfrak{S}_1$-Fitting class $\mathfrak{Y}$ and $\mathfrak{S}_1$-group G are constructed in [4]. Consider $\mathfrak{X} = H^2 \cap \mathfrak{Y}$; $\mathfrak{X}^* = \mathfrak{H}^2$ and so $\mathfrak{X}$ is not abelian normal and $\mathfrak{X}$ is not a Lockett class. If L is any $\mathfrak{S}_1$-group, then $L/L_{\mathfrak{X}}$ is centre-by-finite since $L_{\mathfrak{X}^*}/L_{\mathfrak{X}}$ is central [**2**, Lemma 2.2 (c)]. Thus $\mathfrak{X}$ does not satisfy condition (b) and clearly there are many polycyclic abelian-by-finite groups which can not be embedded in an $L/L_{\mathfrak{X}}$.

We note that Theorem 4.6 can be applied to the nonabelian normal $\mathfrak{S}_1$-Fitting class $\mathfrak{C}\mathfrak{N}$ and also to the class $\mathfrak{H}$ of hypercentral $\mathfrak{S}_1$-groups. Both of these classes are s-closed and so are Lockett classes [**2**, Lemma 2.8].

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