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AN INTEGRAL FORMULA ON SUBMANIFOLDS OF DOMAINS OF $\mathbb{C}^n$

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Abstract

A Bochner-Martinelli-Koppelman type integral formula on submanifolds of pseudoconvex domains in $\mathbb{C}^n$ is derived; the result gives, in particular, integral formulas on Stein manifolds.

1. Introduction

The method of integral representations in several complex variables has been proved to be quite efficient in constructing holomorphic functions and more general analytic objects (differential forms solving the $\overline{\partial}$-equation, sections of holomorphic vector bundles etc); see, for example, Henkin and Leiterer [4], [5], Henkin and Polyakov [6] and Range [10]. This method revolves about Bochner-Martinelli-Koppelman's integral formula: if $D \subset \mathbb{C}^n$ is a bounded domain in $\mathbb{C}^n$ with smooth boundary ∂D then

$$u = \int_{\partial D} u \wedge K_q - \int_D \overline{\partial} u \wedge K_q + \overline{\partial}(\int_D u \wedge K_{q-1}) \tag{1.1}$$

for every $(0, q)$-form u with C^1-coefficients on $\overline{D}$, where K_q are appropriately constructed kernels (see Ovrelid [9]). The integral formula (1.1) can be modified to produce a variety of other formulas with which various problems (such as the $\overline{\partial}$-equation and interpolation problems) can be solved; thus (1.1) is the first main step in several constructions.

The purpose of this paper is to construct an analogue of (1.1) if D is replaced by a submanifold M of D. More precisely let D be a bounded domain in $\mathbb{C}^n$ and $\widetilde{D}$ a pseudoconvex one with $\overline{D} \subset \widetilde{D}$; let $\widetilde{M}$ be a (closed and complex) submanifold of $\widetilde{D}$ and let $M =: D \cap \widetilde{M}$. Assume that $\partial M = (\partial D) \cap \widetilde{M}$ is smooth. Then we will construct kernels $K_q(\zeta, z)$ defined for $(\zeta, z) \in \widetilde{M} \times \widetilde{M}$ with $\zeta \neq z$ so that

$$u(z) = \int_{\zeta \in \partial M} u(\zeta) \wedge K_q(\zeta, z) - \int_{\zeta \in M} \overline{\partial} u(\zeta) \wedge K_q(\zeta, z) + \overline{\partial}_z(\int_{\zeta \in M} u(\zeta) \wedge K_{q-1}(\zeta, z)) \tag{1.2}$$

for $u \in C^1_{(0,q)}(\overline{M})$ and $z \in M$.

In [3] we derived an integral formula like (1.2) in the case M is a complete intersection (i.e., if there exist functions $h_1, \ldots, h_p$, holomorphic on $\widetilde{D}$, so that $\widetilde{M} = \{h_1 = \cdots = h_p = 0\}$ and $dh_1 \wedge \cdots \wedge dh_p \neq 0$ on $\overline{M}$); the construction in the general case (i.e., when M is not necessarily a complete intersection) will be based on the result of [3].

A similar construction was carried out by Berndtsson [1]. As a matter of fact our kernels are equivalent to those of Berndtsson but written in a different way. We think however that our version of the construction as well as the different proof of the integral formula are of further interest.

Here is an outline of the construction. First we cover $\widetilde{M}$ by sufficiently small open sets $\{U_\sigma\}$ (open in $\mathbb{C}^n$) so that each $\widetilde{M} \cap U_\sigma$ is a complete intersection. Then the result in [3] gives kernels K_q^σ for which (1.2) holds in $\widetilde{M} \cap U_\sigma$. Such kernels are not unique in the sense that there are certain choices that can be made, in particular, K_q^σ depends on Hefer decompositions of the holomorphic functions which define $\widetilde{M} \cap U_\sigma$ (as the set of their common zeros). But using a result of Berndtsson [1] we show that such Hefer decompositions can be chosen appropriately so that

$$K_q^\sigma = K_q^\tau \text{ on } \widetilde{M} \cap U_\sigma \cap U_\tau;$$

thus we can define a global kernel K_q (by setting $K_q =: K_q^\sigma$ on $\widetilde{M} \cap U_\sigma$). Then we have to show that (1.2) holds; and this is done along the same lines as in [3]. Here is an outline of this proof in the case u is a holomorphic function on $\widetilde{M}$. In this case we have to show that

$$u(z) = \int_{\zeta \in \partial M} u(\zeta) K_0(\zeta, z), \quad z \in M \tag{1.3}$$

Fix a $z \in M$ and pick a U_σ with $z \in U_\sigma$. But, as a computation shows, $d_\zeta K_0^\tau = 0$ (for each τ and $\zeta \neq z$) and hence $d_\zeta[u(\zeta) K_0^\tau(\zeta, z)] = 0$; thus (1.3) is equivalent (via Stokes' theorem) to

$$u(z) = \int_{\zeta \in \partial U_\sigma} u(\zeta) K_0(\zeta, z)$$

which holds by the result of [3] (or [2] for that case) since $K_0 = K_0^\sigma$ in $U_\sigma \cap M$.

Our result gives, in particular, integral formulas for domains $D \subset\subset X$, if X is a Stein manifold, via the theorem that Stein manifolds admit embedding in some $\mathbb{C}^N$.

Integral formulas on Stein manifolds have been constructed previously, using different techniques, by Henkin and Leiterer [4]. Related is also the work of Berndtsson [1], Hortmann [7], Palm [9] and Stout [11].

2. Notation

Let $h = (h_1, \ldots, h_p)$ where the h_i's are holomorphic functions defined in some open set of $\mathbb{C}^n$ and suppose that $\{h_{ij}(\zeta, z), j = 1, \ldots, n\}$ is a Hefer decomposition of h_i, i.e., $h_{ij}(\zeta, z)$ is holomorphic in both ζ and z and

$$h_i(\zeta) - h_i(z) = \sum_{j=1}^{n} h_{ij}(\zeta, z)(\zeta_j - z_j), i = 1, \ldots, p.$$

We associate to these data the following differential forms.

First

$$\alpha^h(\zeta, z) = c \det[h_{1j}, \ldots, h_{pj}, \gamma_j, \overbrace{(\overline{\partial}_\zeta + \overline{\partial}_z)\gamma_j}^{n-p-1}] \tag{2.1}$$

where $\gamma = (\gamma_1, \ldots, \gamma_n)$ is some smooth function (we will be more specific about γ later) and c is normalizing constant: $c = (-1)^{\frac{m(m-1)}{2}} \frac{1}{(2\pi i)^m} \cdot \frac{1}{m!}$, where $m =: n - p$. In the determinant of (2.1), j runs from $j = 1$ up to $j = n$ forming the n rows of it; also the column of differential forms $\begin{pmatrix} (\overline{\partial}_\zeta + \overline{\partial}_z)\gamma_1 \\ \vdots \\ (\overline{\partial}_\zeta + \overline{\partial}_z)\gamma_n \end{pmatrix}$ is repeated in the determinant $(n - p - 1)$-times (as it is indicated); for properties of such determinants see [4]. Also define

$$\beta^h(\zeta) = \frac{1}{|\nabla h(\zeta)|^2} \det \left[\frac{\overline{\partial h_1}}{\partial \zeta_j}, \ldots, \frac{\overline{\partial h_p}}{\partial \zeta_j}, \overbrace{d\zeta_j}^{n-p} \right] \tag{2.2}$$

where

$$|\nabla h(\zeta)|^2 = \sum_{1 \le j_1 < \ldots < j_p \le n} \left| \frac{\partial(h_1, \ldots, h_p)}{\partial(\zeta_{j_1}, \ldots, \zeta_{j_p})}(\zeta) \right|^2;$$

of course $\beta^h(\zeta)$ is defined for ζ with $|\nabla h(\zeta)| \neq 0$. With this notation set

$$K^h(\zeta, z) = \alpha^h(\zeta, z) \wedge \beta^h(\zeta).$$

Let α_q^h be the part of α^h which is of type $(0, q)$ in z and $(0, n - p - q - 1)$ in ζ, i.e.,

$$\alpha_q^h = c \binom{n-p-1}{q} \det[h_{1j}, \ldots, h_{pj}, \gamma_j, \overbrace{\overline{\partial}_z \gamma_j}^{q}, \overbrace{\overline{\partial}_\zeta \gamma_j}^{n-p-q-1}].$$

Also let $K_q^h(\zeta, z) = \alpha_q^h(\zeta, z) \wedge \beta^h(\zeta)$.

Of course K_q^h depends on the choice of the Hefer decompositions $\{h_{ij}\}$ of h_i, although we do not indicate this in the notation; but it will be clear from the context which choice we will be using in each case. Also we have been vague about the domains in which the above functions and differential forms are defined, because here, we simply introduced notation and we will be very specific about this later.

3. Construction of the kernels

First we describe the setting. Let $D, \widetilde{D}$ be domains in $\mathbb{C}^n$ with D bounded, $\widetilde{D}$ pseudoconvex and $\overline{D} \subset \widetilde{D}$. Let $\widetilde{M}$ be a closed complex submanifold of $\widetilde{D}$ of (complex) dimension m. Set $M = D \cap \widetilde{M}$ and $\partial M = (\partial D) \cap \widetilde{M}$ and assume that ∂D is smooth and that $\widetilde{M}$ meets ∂D transversally so that ∂M is also smooth.

Let $\gamma = (\gamma_1, \ldots, \gamma_n) : \widetilde{D} \times \widetilde{D} - \{\zeta = z\} \to \mathbb{C}^n$ be a smooth map satisfying (3.1)

$$\sum_{j=1}^{n} \gamma_j(\zeta_j - z_j) = 1 \text{ and } \gamma_j = \frac{\overline{\zeta}_j - \overline{z}_j}{|\zeta - z|^2} \text{ for } 0 < |\zeta - z| < \delta \text{ for some small } \delta > 0$$

(an example of γ is given by $\gamma_j = \frac{\overline{\zeta}_j - \overline{z}_j}{|\zeta - z|^2}$ all ζ, z with $\zeta \neq z$).

In this setting we will now construct the kernels. Let $\{U_\sigma\}$ be a set of small convex open sets of $\mathbb{C}^n$ so that $\widetilde{M} \subset \bigcup_\sigma U_\sigma$ and moreover let

$$h^\sigma = (h_1^\sigma, \ldots, h_p^\sigma) : \widetilde{D} \to \mathbb{C}^p, p =: n - m,$$

be holomorphic maps so that $\widetilde{M} \cap U_\sigma = \{z \in U_\sigma : h^\sigma(z) = 0\}$ and $|\nabla h^\sigma| \neq 0$ on $\widetilde{M} \cap \overline{U}_\sigma$; that such functions h_i^σ exist follows from Cartan's Theorem A since $\widetilde{D}$ is assumed to be pseudoconvex.

Furthermore there exist $p \times p$ matrices $A_{\sigma\tau} = [(A_{\sigma\tau})_{ik}]_{1 \leq i,k \leq p}$ of functions $(A_{\sigma\tau})_{ik}$, holomorphic in $U_\sigma \cap U_\tau$, so that $A_{\sigma\tau} \begin{pmatrix} h_1^\tau \\ \vdots \\ h_p^\tau \end{pmatrix} = \begin{pmatrix} h_1^\sigma \\ \vdots \\ h_p^\sigma \end{pmatrix}$ on $U_\sigma \cap U_\tau$, i.e.,

$$h_i^\sigma = \sum_{k=1}^{p} (A_{\sigma\tau})_{ik} h_k^\tau, i = 1, \ldots, p. \tag{3.2}$$

The existence of such matrices $A_{\sigma\tau}$ follows from Cartan's Theorem B since $U_\sigma \cap U_\tau$ is convex.

Lemma 1.

$$A_{\sigma\tau} \begin{pmatrix} \frac{\partial h_1^\tau}{\partial \zeta_j} \\ \vdots \\ \frac{\partial h_p^\tau}{\partial \zeta_j} \end{pmatrix} = \begin{pmatrix} \frac{\partial h_1^\sigma}{\partial \zeta_j} \\ \vdots \\ \frac{\partial h_p^\sigma}{\partial \zeta_j} \end{pmatrix}, j = 1, \ldots, n, \ \zeta \in U_\sigma \cap U_\tau \cap \widetilde{M}.$$

Proof. Differentiating (3.2) we obtain

$$\frac{\partial h_i^\sigma}{\partial \zeta_j} = \sum_{k=1}^{p} (A_{\sigma\tau})_{ik} \frac{\partial h_k^\tau}{\partial \zeta_j} + \sum_{k=1}^{p} h_k^\tau \frac{\partial (A_{\sigma\tau})_{ik}}{\partial \zeta_j}.$$

Since $h_k^\tau = 0$ for $\zeta \in U_\sigma \cap U_\tau \cap \widetilde{M}$, the formula of the lemma follows immediately. ∎

The following lemma is proved by Berndtsson [1, p.414]; its proof is based on Cartan's Theorem B, using again the pseudoconvexity of $\widetilde{D}$.

Lemma 2. *There exist functions $h_{ij}^\sigma(\zeta, z)$, $i = 1, \ldots, p, j = 1, \ldots, n$, holomorphic in $(\zeta, z) \in (U_\sigma \cap \widetilde{M}) \times \widetilde{M}$ so that*

$$A_{\sigma\tau}\begin{pmatrix} h_{1j}^\tau \\ \vdots \\ h_{pj}^\tau \end{pmatrix} = \begin{pmatrix} h_{1j}^\sigma \\ \vdots \\ h_{pj}^\sigma \end{pmatrix} \text{ for } j = 1, \ldots, n, \zeta \in U_\sigma \cap U_\tau \cap \widetilde{M}, z \in \widetilde{M}. \tag{3.3}$$

Furthermore $h_{ij}^\sigma(\zeta, z)$ have holomorphic extensions in ζ in a neighbourhood (in $\mathbb{C}^n$) of $U_\sigma \cap \widetilde{M}$, satisfying

$$\sum_{j=1}^n h_{ij}^\sigma(\zeta, z)(\zeta_j - z_j) = h_i^\sigma(\zeta) \text{ for } z \in \widetilde{M}. \tag{3.4}$$

Lemma 3. *Let $A \in \mathbb{C}^{p\times p}$ and $B, C \in \mathbb{C}^{n\times p}$ so that*

$$A \cdot B^T = C^T \ (T \text{ denotes transpose}).$$

Then

$$\det[C, ***] = \det(A)\det[B, ***]$$

where $***$ denote appropriate differential forms (the same on both sides of the equation) so that the matrices $[B, ***]$ and $[C, ***]$ are $n \times n$.

Proof: This is a straightforward computation based on the multilinearity of the determinants (as functions of their columns) and the definition of the determinants. (This generalizes the fact that the determinant of the product of square matrices is equal to the product of the determinants of these matrices). ∎

Lemma 4. *If α^{h^σ} is the form (2.1) associated to the Hefer decompositions $\{h_{ij}^\sigma\}$ and α^{h^τ} is defined similarly, then*

$$\alpha^{h^\sigma} = \det(A_{\sigma\tau}) \cdot \alpha^{h^\tau} \text{ for } \zeta \in U_\sigma \cap U_\tau \cap \widetilde{M}, z \in \widetilde{M} \tag{3.5}$$

and, hence, $\alpha_q^{h^\sigma} = \det(A_{\sigma\tau}) \cdot \alpha_q^{h^\tau}$.

Proof: Defining

$$H^\sigma = \begin{bmatrix} h_{11}^\sigma & \ldots & h_{p1}^\sigma \\ \vdots & & \vdots \\ h_{1n}^\sigma & \ldots & h_{pn}^\sigma \end{bmatrix}$$

and similarly H^τ, we obtain from Lemma 2 that

$$A_{\sigma\tau} \cdot (H^\tau)^T = (H^\sigma)^T \text{ for } \zeta \in U_\sigma \cap U_\tau \cap \widetilde{M}, z \in \widetilde{M};$$

thus Lemma 3 applies and gives (3.5). ∎

Lemma 5. *We have*

$$\beta^{h^\sigma} = \frac{1}{\det(A_{\sigma\tau})}\beta^{h^\tau} \text{ for } \zeta \in U_\sigma \cap U_\tau \cap M. \tag{3.6}$$

Proof: By Lemma 1 we obtain

$$\left[\frac{\partial h_i^\sigma}{\partial \zeta_{j_k}}\right]_{1\le i,k\le p} = A_{\sigma\tau}\cdot\left[\frac{\partial h_i^\tau}{\partial \zeta_{j_k}}\right]_{1\le i,k\le p}.$$

Therefore

$$\frac{\partial(h_1^\sigma,\dots,h_p^\sigma)}{\partial(\zeta_{j_1},\dots,\zeta_{j_p})} = \det(A_{\sigma\tau})\cdot\frac{\partial(h_1^\tau,\dots,h_p^\tau)}{\partial(\zeta_{j_1},\dots,\zeta_{j_p})}$$

and

$$|\nabla h^\sigma|^2 = |\det(A_{\sigma\tau})|^2|\nabla h^\tau|^2 \tag{3.7}$$

(this also shows that $\det(A_{\sigma\tau}) \neq 0$ on $U_\sigma \cap U_\tau \cap \widetilde{M}$). Also setting

$$B^\sigma = \begin{bmatrix} \overline{\frac{\partial h_1^\sigma}{\partial \zeta_1}} & \cdots & \overline{\frac{\partial h_p^\sigma}{\partial \zeta_1}} \\ \vdots & & \vdots \\ \overline{\frac{\partial h_1^\sigma}{\partial \zeta_n}} & \cdots & \overline{\frac{\partial h_p}{\partial \zeta_n}} \end{bmatrix}$$

and similarly for B^τ we see that Lemma 1 gives

$$\overline{A_{\sigma\tau}}\cdot(B^\tau)^T = (B^\sigma)^T.$$

Hence, by Lemma 3,

$$\det\left[\overline{\frac{\partial h_1^\sigma}{\partial \zeta_j}},\dots,\overline{\frac{\partial h_p^\sigma}{\partial \zeta_j}},\overbrace{\widehat{d\zeta_j}}^{n-p}\right] = \overline{\det(A_{\sigma\tau})}\cdot\det\left[\overline{\frac{\partial h_1^\tau}{\partial \zeta_j}},\dots,\overline{\frac{\partial h_p^\tau}{\partial \zeta_j}},\overbrace{\widehat{d\zeta_j}}^{n-p}\right] \tag{3.8}$$

Now (3.6) follows from (2.2), (3.7) and (3.8). ∎

We are now ready to define the kernels:

$$K(\zeta,z) =: K^{h^\sigma}(\zeta,z) = \alpha^{h^\sigma}(\zeta,z)\wedge\beta^{h^\sigma}(\zeta) \text{ for } \zeta\in U_\sigma\cap\widetilde{M}, z\in\widetilde{M}, \zeta\neq z.$$

By Lemmas 4 and 5, it follows that $K(\zeta,z)$ is well-defined for $\zeta\in\widetilde{M}$, $z\in\widetilde{M}$, $\zeta\neq z$. Similarly we define $K_q = \alpha_q^{h^\sigma}\wedge\beta^{h^\sigma}$ for $q\ge 0$ and $K_{-1}=0$.

4. The integral formula

With the kernels just constructed we will prove the following theorem.

Theorem 1. *If* $u \in C^1_{(0,q)}(\overline{M})$ $(0 \leq q \leq m)$ *and* $z \in M$ *then*

$$u(z) = \int_{\zeta \in \partial M} u(\zeta) \wedge K_q(\zeta, z) - \int_{\zeta \in M} \overline{\partial} u(\zeta) \wedge K_q(\zeta, z) + \overline{\partial}_z \left[\int_{\zeta \in M} u(\zeta) \wedge K_{q-1}(\zeta, z) \right].$$

First a lemma:

Lemma 6. *We have*

$$(\overline{\partial}_\zeta + \overline{\partial}_z) K(\zeta, z) = 0.$$

Proof.: It suffices to show

$$(\overline{\partial}_\zeta + \overline{\partial}_z) K^{h^\sigma}(\zeta, z) = 0 \text{ for } \zeta \in U_\sigma \cap \widetilde{M}, z \in M, \zeta \neq z. \tag{4.1}$$

Now observe that

$$(\zeta_1 - z_1)\alpha^{h^\sigma}(\zeta, z) = \det \begin{bmatrix} 0 & \dots & 0 & 1 & \overbrace{0}^{n-p-1} \\ h^\sigma_{1j} & \dots & h^\sigma_{pj} & \gamma_j & (\overline{\partial}_\zeta + \overline{\partial}_z)\gamma_j \end{bmatrix}_{2 \leq j \leq n} \tag{4.2}$$

(in the above determinant j runs from $j = 2$ to $j = n$ forming the 2nd upto the nth row of it).

We obtained (4.2) in the following way: $(\zeta_1 - z_1)$ multiplied the first row of the determinant which defines $\alpha^{h^\sigma}(\zeta, z)$ and then we added to that first row the jth-row multiplied by $(\zeta_j - z_j)$ $(j = 2, \dots, n)$. Then (4.2) follows in view of the first of the assumptions in (3.1) and (3.4). Now (4.2) easily implies that $(\overline{\partial}_\zeta + \overline{\partial}_z)\alpha^{h^\sigma} = 0$; since, moreover, $\overline{\partial}_\zeta \beta(\zeta) = 0$ (by [**2**, Corollary 1, p. 76]), we obtain (4.1) which completes the proof of the lemma. ∎

Proof of Theorem 1:

It suffices to show that

$$\int_{z \in M} u(z) \wedge \varphi(z) = \int_{z \in M} \int_{\zeta \in \partial M} u \wedge K_q \wedge \varphi - \int_{z \in M} \int_{\zeta \in M} \overline{\partial} u \wedge K_q \wedge \varphi + \int_{z \in M} (\overline{\partial}_z (\int_{\zeta \in M} u \wedge K_{q-1})) \wedge \varphi \tag{4.3}$$

for $\varphi \in (C^1_0(M))(m, m - q)$.

Let us point out the way in which the various forms in the right-hand side of (4.3) depend on the variables ζ and z: $u = u(\zeta)$, $K_q = K_q(\zeta, z)$, $K_{q-1} = K_{q-1}(\zeta, z)$, $\varphi = \varphi(z)$ and $\overline{\partial} u = \overline{\partial} u(\zeta)$. By degree reasons, (4.3) is equivalent to

$$(4.4) \quad \int_{z \in M} u(z) \wedge \varphi(z) = \int_{\partial(M \times M)} u \wedge K \wedge \varphi - \int_{M \times M} \overline{\partial} u \wedge K \wedge \varphi + \int_M (\overline{\partial}_z (\int_M u \wedge K)) \wedge \varphi$$

(in obtaining (4.4) we used also the fact that

$$\int_{(\zeta, z) \in (\partial M) \times M} u \wedge K \wedge \varphi = \int_{\partial(M \times M)} u \wedge K \wedge \varphi$$

which holds since u has compact support in M).

By Stokes' formula

$$(4.5) \quad \int_{\partial(M \times M)} u \wedge K \wedge \varphi = \int_{M \times M - \{|\zeta - z| < \varepsilon\}} d[u \wedge K \wedge \varphi] + \int_{C_\varepsilon} u \wedge K \wedge \varphi$$

where $C_\varepsilon = \{(\zeta, z) \in M \times M : |\zeta - z| = \varepsilon\}$.

By Lemma 6 and degree reasons

$$\lim_{\varepsilon \to 0} \int_{M \times M - \{|\zeta - z| < \varepsilon\}} d[u \wedge K \wedge \varphi] = \int_{M \times M} \overline{\partial} u \wedge K \wedge \varphi - (-1)^q \int_{M \times M} u \wedge K \wedge \overline{\partial} \varphi;$$

hence (4.4) will follow from (4.5) as soon as we establish the following

$$(4.6) \qquad \lim_{\varepsilon \to 0} \int_{(\zeta, z) \in C_\varepsilon} u \wedge K \wedge \varphi = \int_{z \in M} u(z) \wedge \varphi(z).$$

But we may assume without loss of generality that $\operatorname{supp}(u) \subset U^\sigma$ f or some σ. Then for ε small enough

$$\int_{C_\varepsilon} u \wedge K \wedge \varphi = \int_{C_\varepsilon} u \wedge K^{h^\sigma} \wedge \varphi$$

and (4.6) follows from

$$(4.7) \qquad \lim_{\varepsilon \to 0} \int_{C_\varepsilon} u \wedge K^{h^\sigma} \wedge \varphi = \int_{z \in M} u(z) \wedge \varphi(z)$$

But (4.7) is exactly what is proved in [**3**, p. 339–341] (in that part of the proof, (3.4) plays an important role). This completes the proof of the theorem. ∎

5. Applications

As we mentioned in the introduction the integral formula that we proved can be used to derive various other integral formulas. Here we discuss two examples.

With notation as before assume furthermore that D is a strictly pseudoconvex domain. Then, by a classical construction of Henkin and Ramirez (see [5]), there exist functions $g_j(\zeta, z)$, $j = 1, \dots, n$ defined for $(\zeta, z) \in (\partial D) \times D$, which are smooth in ζ and holomorphic in z, so that $G(\zeta, z) =: \sum_{j=1}^{n}(\zeta_j - z_j) g_j(\zeta, z) \neq 0$. Thus if we set

$$C(\zeta, z) = \frac{c}{[G(\zeta, z)]^{n-p}} \det[h^\sigma_{1j}, \dots, h^\sigma_{pj}, g_j, \overbrace{\overline{\partial}_\zeta g_j}^{n-p-1}] \wedge \beta^{h^\sigma}(\zeta)$$

for $\zeta \in U_\sigma \cap \partial M$ and $z \in M$ then $C(\zeta, z)$ is w ell-defined for $\zeta \in \partial M$ and $z \in M$, holomorphic in z and (as it follows from theorem 1) it reproduces holomorphic functions on M, i.e., for $f \in \mathcal{O}(M) \cap C(\overline{M})$ we have

$$f(z) = \int_{\zeta \in \partial M} f(\zeta) C(\zeta, z), \quad z \in M;$$

this is an analogue of Cauchy's integral formula on M.

Continuing to assume D to be strictly pseudoconvex and the g_j's as above, let us set

$$\eta_j(\zeta, z, \lambda) = (1 - \lambda) \frac{\overline{\zeta}_j - \overline{z}_j}{|\zeta - z|^2} + \lambda \frac{g_j(\zeta, z)}{G(\zeta, z)} \text{ for } \zeta \in \partial D, z \in D, \lambda \in [0, 1]$$

and

$$L_q(\zeta, z, \lambda) = c \binom{n-p-1}{q}$$

$$\det[h^\sigma_{1j}, \dots, h^\sigma_{pj}, \eta_j, \overbrace{\overline{\partial}_z \eta_j}^{q}, \overbrace{(\overline{\partial}_\zeta + d_\lambda)\eta_j}^{n-p-q-1}] \wedge \beta^{h^\sigma}(\zeta), \zeta \in U_\sigma \cap \partial M, z \in M\,;$$

then $L_q(\zeta, z, \lambda)$ is well-defined for $\zeta \in \partial M$, $z \in M$. Also set

$$B_q(\zeta, z) =: L(\zeta, z, \lambda)|_{\lambda=0};$$

then it is clear that $B_q(\zeta, z)$ is defined for $\zeta, z \in M$, $\zeta \neq z$. Using the above kernels we define the following operators:

$$T_q u(z) = \int_{(\zeta,\lambda) \in (\partial M) \times [0,1]} u(\zeta) \wedge L_{q-1}(\zeta, z, \lambda) + \int_{\zeta \in M} u(\zeta) \wedge B_{q-1}(\zeta, z), z \in M$$

for $u \in C_{(0,q)}(\overline{M})$, $q \geq 1$.

Now, using theorem 1, we can easily derive the following Leray-Koppelman type formula:

If $u \in C^1_{(0,q)}(\overline{M})$, $q \geq 1$, then u admits the following decomposition: $u = \overline{\partial}(T_q u) + T_{q+1}(\overline{\partial} u)$ on M.

Also one can use theorem 1 to derive integral formulas (of the type of Koppelman-Leray-Norguet-Weil) for analytic polyhedra on Stein manifolds (in a way analogous to the one in Henkin-Leiterer [4, chapter 4], in which they use their construction on Stein manifolds instead of theorem 1).

Finally, since our kernels are given locally by explicit formulas, their estimation is immediate (by the corresponding results in domains of $\mathbb{C}^n$) and various estimates for the $\overline{\partial}$-equation on Stein manifold can be derived.

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