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CONVERGENCE OF THE AVERAGES AND FINITENESS OF ERGODIC POWER FUNCTIONS IN WEIGHTED L^1 SPACES

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Abstract

Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $T : X \longrightarrow X$ be a measure preserving transformation and let $A_n f$ denote the average of $T^k f$, $k = 0, \ldots, n$. Given a real positive function v on X, we prove that $\{A_n f\}$ converges in the a.e. sense for every f in $L^1(v\, d\mu)$ if and only if $\inf_{i \geq 0} v(T^i x) >$ 0 a.e., and that the same condition is equivalent to the finiteness of a related ergodic power function $P_r f$ for every f in $L^1(v\, d\mu)$. We apply this result to characterize, being T null-preserving, the finite measures ν for which the sequence $\{A_n f\}$ converges a.e. for every $f \in L^1(d\nu)$ and to prove that uniform boundedness of the averages in L^1 is sufficient for finiteness a.e. of P_r.

1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a finite measure space and let $T : X \longrightarrow X$ be a measure preserving transformation. For every measurable function f on X we consider the averages

$$A_n f = (n+1)^{-1} \sum_{j=0}^{n} T^j f$$

where $T^j f(x) = f(T^j x)$, the maximal operator

$$Mf = \sup_{n \geq 0} A_n |f|$$

and the power function

$$P_r f = \left(\sum_{n=0}^{\infty} |A_{n+1} f - A_n f|^r \right)^{1/r} \qquad (r > 1).$$

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In [7], Martín Reyes and A. de la Torre characterized the positive measurable functions v such that $\{A_n f\}$ converges a.e. for all f in $L^1(v\,d\mu)$ as those functions that verify

$$\inf_{i\geq 0} v(T^i x) > 0 \text{ a.e.} \tag{1.1}$$

(see also [15] for a ratio theorem).

Section 3 of this note is devoted to give a simpler proof for this result and to prove a similar theorem for P_r. It is seen that condition (1.1) is also valid for P_r. The main tools we use are Nikishin's theorem and conditional expectations which solve the problems derived from the non-invertibility of the transformation. These technics have been recently used to solve the problem of the convergence of the averages for $p > 1$ (see [8]).

As a previous result, we have to state the weak type $(1,1)$ for P_r. This question was solved in [17] by Yoshimoto. Our approach is different, but suitable for our purposes. It was also treated in [5] and [11], but under more restrictive conditions.

Finally, in section 4 we work with a null-preserving transformation T and characterize the finite measures ν for which the sequence of the ergodic averages $\{A_n f\}$ converges a.e. for every $f \in L^1(d\nu)$ as those measures with the property: there exists a measure γ equivalent to ν such that

$$\gamma(\{x \in X / Mf(x) > \lambda\}) \leq \lambda^{-1} \int_X |f|\,d\nu$$

for every $f \in L^1(d\nu)$.

In [13], Ryll-Nardzewski characterized the finite measures ν for which the ergodic averages $\{A_n f\}$ converge a.e. to a L^1-function for every $f \in L^1(d\nu)$ as those measures that verify Hartman's condition: there exists a constant K such that

$$\limsup_n n^{-1} \sum_{i=0}^{n-1} \nu(T^{-i}E) \leq K\nu(E)$$

for every set E.

Our result is different from the Ryll-Nardzevsky's one, because we allow the limit function not to be in $L^1(d\nu)$. This situation is possible as Dowker's example shows (see [1] and, for a two-dimensional version, see [12]) and, therefore, our condition is strictly weaker than Hartman's condition.

As a corollary, we prove that uniform boundedness of the averages is a sufficient condition for finiteness of P_r for every $f \in L^1$. This result is a L^1 version of theorem 3.1 in [10]. Other references about P_r are [14] and [16].

2. Previous results

We will need two lemmas and several results about the operators P_r, q_r and Q_r, where q_r is defined on functions on $\mathbf{N}$, the set of the natural numbers, by

$$q_r a(i) = \left(\sum_{k=0}^{\infty} |a(i+k)|^r (k+1)^{-r} \right)^{1/r} \qquad (i \in N)$$

and Q_r on functions on X by

$$Q_r f(x) = \left(\sum_{k=0}^{\infty} |f(T^k x)|^r (k+1)^{-r} \right)^{1/r} .$$

Lemma 1. *Let k be a natural number. Then, there exists a countable family $\{B_i : i \in \mathbf{N}\}$ of measurable sets such that*

i) $X = \cup_i B_i$

ii) $B_i \cap B_j = \phi$ *if* $i \neq j$

iii) *For every i, there exists a natural number $s(i)$ with $0 \leq s(i) \leq k$ such that the sets $\{T^{-j} B_i : 0 \leq j \leq s(i)\}$ are pairwise disjoint and such that if $s(i) < k$ then $T^{-1-s(i)} A = A$ for every subset A of B_i. Consequently, for every subset A of B_i*

$$\sum_{j=0}^{k} \chi_{T^{-j}A} \leq C(i) \sum_{j=0}^{s(i)} \chi_{T^{-j}A} \leq 2 \sum_{j=0}^{k} \chi_{T^{-j}A}$$

where $C(i)$ is the least integer bigger than or equal to $(k+1)(1+s(i))^{-1}$.

For the proof see lemma (2.10) in [9] changing T^h by T^{-h}.

Lemma 2. *Let $(X, \mathcal{F}, \mu)$ be a finite measure space and let $\{\mathcal{F}_n\}$ be a decreasing sequence of sub-σ-algebras. Let $\mathcal{F}_\infty = \cap_n \mathcal{F}_n$ and denote by E_n the conditional expectation with respect to $\mathcal{F}_n$. If $\{f_n\}$ is an a.e. convergent sequence of functions such that $|f_n| \leq C$ a.e. and f is the a.e. limit of f_n then $E_\infty f$ is the a.e. limit of $E_n f_n$.*

This lemma follows from theorem 7.6 in [6] and the decreasing martingale theorem.

Theorem 1. *q_r is of weak type $(1,1)$ with respect to the counting measure on* $\mathbf{N}$.

Proof: The proof is the same as the one of theorem (3.8) in [10] with obvious changes derived from the facts that we are working in $\mathbf{N}$ and that lemma (3.2) (in [10]) is not necessary. ■

Theorem 2. *Q_r is of weak type $(1,1)$.*

Proof: It follows from theorem 1 and transference arguments (see [**11**]). ■

Theorem 3. *P_r is of weak type $(1,1)$ and, as a consequence, the series*

$$\sum_{k=0}^{\infty} |A_{k+1}f - A_k f|^r$$

is a.e. convergent for every f in $L^1(d\mu)$.

Proof: It follows inmediately by theorem 2, the ergodic theorem and the well-known inequality

$$P_r f \leq CMf + Q_r f. \quad ■$$

Remark. Note that theorems 2 and 3 do not need finiteness of the measure space.

3. Main result

Theorem 4. *Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $T : X \longrightarrow X$ be a measure preserving transformation. Let v be a positive measurable function on X. The following are equivalent:*

a) *The sequence $\{A_n f\}$ converges a.e. for all f in $L^1(v\,d\mu)$.*
b) *$\sum_{k=0}^{\infty} |A_{k+1}f - A_k f|^r < \infty$ in the a.e. sense for all f in $L^1(v\,d\mu)$.*
c) *$\sum_{k=0}^{\infty} (k+1)^{-r} |T^k f|^r < \infty$ in the a.e. sense for all f in $L^1(v\,d\mu)$.*
d) *$Mf < \infty$ a.e. for all f in $L^1(v\,d\mu)$.*
e) *There exists a positive measurable function u such that $\int_{\{x: Mf(x) > \lambda\}} u\,d\mu \leq \lambda^{-1} \int_X |f| v\,d\mu$ for all $\lambda > 0$ and all f in $L^1(v\,d\mu)$.*
f) *There exists a positive measurable function u such that $\sup_{k\geq 0} \int_{\{x: A_k f(x) > \lambda\}} u\,d\mu \leq \lambda^{-1} \int_X |f| v\,d\mu$ for all $\lambda > 0$ and all f in $L^1(v\,d\mu)$.*
g) *There exists a positive measurable function u such that $\int_{\{x: P_r f(x) > \lambda\}} u\,d\mu \leq \lambda^{-1} \int_X |f| v\,d\mu$ for all $\lambda > 0$ and all f in $L^1(v\,d\mu)$.*
h) *There exists a positive measurable function u such that $\int_{\{x: Q_r f(x) > \lambda\}} u\,d\mu \leq \lambda^{-1} \int_X |f| v\,d\mu$ for all $\lambda > 0$ and all f in $L^1(v\,d\mu)$.*
i) *$\inf_{i\geq 0} v(T^i x) > 0$ a.e.*

Proof: Implications a) ⇒ d) and e) ⇒ f) are clear. d) implies e), b) implies g) and c) implies h) by Nikishin's theorem (see [**2**] pages 536–537 and [**3**]). Nikishin's theorem needs the continuity in measure of the operators M, P_r and Q_r from $L^1(v\,d\mu)$ to $L^0(d\mu)$. This condition follows by theorem 1.1.1 in [**4**], page 10.

f) $\Rightarrow$ i) We may assume $u \leq 1$. Let k be a nonnegative integer. Let $\{B_i\}$ be the sequence of sets associated to k by lemma 1. Fix i and let A be a measurable subset of B_i. Let $R = \cup_{0\leq j\leq k} T^{-j}A = \cup_{0\leq j\leq s(i)} T^{-j}A$. It is clear that R is contained in $\{x : A_k(\chi_A)(x) \geq C(i)(k+1)^{-1}\}$. Then f), lemma 1 and the fact that T is m.p.t. give

$$\int_{T^{-k}A} \sum_{j=0}^{k} u(T^j x)\, d\mu = \sum_{j=0}^{k} \int_{T^{-j}A} u\, d\mu \leq C(i) \sum_{j=0}^{s(i)} \int_{T^{-j}A} u\, d\mu = C(i) \int_R u\, d\mu$$
$$\leq (k+1) \int_A v\, d\mu = (k+1) \int_{T^{-k}A} v(T^k x)\, d\mu.$$

The above inequality has been proved for a measurable subset A of B_i. Since $X = \cup_i B_i$, it is clear that the inequality is true for every measurable subset A of X and therefore if E_k is the conditional expectation with respect to the sub-σ-algebra $T^{-k}\mathcal{F}$ we have

$$E_k\left((k+1)^{-1}\sum_{j=0}^{k} T^j u\right)(x) \leq T^k v(x) \text{ a.e. } x \in X.$$

Taking lim inf when k tends to infinity, Birkhoff's theorem and lemma 2 give

$$Eu(x) \leq \lim_{k\longrightarrow\infty}\inf T^k v(x) \text{ a.e. } x \in X,$$

where Eu is the conditional expectation of u with respect to the sub-σ-algebra of the invariant sets.

Since Eu is positive a.e., we obtain $\inf_{k\geq 0} v(T^k x) > 0$ a.e. .

g) $\Rightarrow$ i) We may assume $u \leq 1$. Let k be a natural number and $\{B_i\}$ be the sequence of sets given by lemma 1. Fix i with $s(i) > 0$ and let A be contained in B_i. Let $R = \cup_{0\leq j\leq k} T^{-j}A = \cup_{0\leq j\leq s(i)} T^{-j}A$. Let's see that $R - A$ is contained in $\{x : P_r(\chi_A)(x) \geq (1+s(i))^{-1}\}$.

Let $y \in R - A$. There exists one and only one h with $0 < h \leq s(i)$ such that $T^h y \in A$. Then

$$P_r(\chi_A)(y) \geq \left|(h+1)^{-1}\sum_{j=0}^{h} \chi_A(T^j y) - h^{-1}\sum_{j=0}^{h-1} \chi_A(T^j y)\right| \geq (1+s(i))^{-1}.$$

Therefore g) gives

$$\int_{R-A} u\, d\mu \leq (1+s(i)) \int_A v\, d\mu.$$

Since $u \leq v$ we have

$$\int_R u\, d\mu \leq (2 + s(i)) \int_A v\, d\mu.$$

Recall that we have been working with $s(i) > 0$. But if $s(i) = 0$ the last inequality is trivial. Then

$$\int_{T^{-k}A} \sum_{j=0}^{k} u(T^j x)\, d\mu \leq C(i) \int_R u\, d\mu \leq C(i)(s(i)+2) \int_A v\, d\mu \leq 4(k+1) \int_A v\, d\mu.$$

Now, the same argument used in the above implication gives i).

h) $\Rightarrow$ i) Let k, $\{B_i\}$, A and R as in f) $\Rightarrow$ i). It is easy to see that R is contained in $\{x : Q_r(\chi_A)(x) > (1 + s(i))^{-1}\}$. Then, the argument follows as in f) $\Rightarrow$ i).

i) $\Rightarrow$ a) The proof of this implication can be seen in [7]. We include it for this section to be selfcontained.

Let $B_k = \{x : \inf_{i \geq 0} v(T^i x) < 2^{-k}\}$. B_k and $X - B_k$ are invariant under T and since $v(x) \geq 2^{-k}$ on $X - B_k$ we have that $L^1(X - B_k, v\, d\mu)$ is contained in $L^1(X - B_k, d\mu)$. Then Birkhoff's theorem shows that $\{A_n f\}$ converges a.e. on $X - B_k$ for every $f \in L^1(X - B_k, v\, d\mu)$. Since $\lim_k \mu(B_k) = 0$ by (i), we obtain (a).

Finally, i) $\Rightarrow$ b) and i) $\Rightarrow$ c) by the same argument that the above but using theorems 3 and 2 respectively in place of Birkhoff's theorem. ■

4. Convergence of the averages and finiteness of P_r in the general case

Theorem 5. *Let $(X, \mathcal{F}, \nu)$ be a finite measure space and let $T : X \longrightarrow X$ be a null-preserving transformation. The following statements are equivalent:*

a) There exists a measure γ equivalent to ν such that

$$\gamma(\{x \in X / Mf(x) > \lambda\}) \leq \lambda^{-1} \int_X |f|\, d\nu$$

for every $f \in L^1(d\nu)$.

b) There exists a measure γ equivalent to ν such that

$$\sup_{n \geq 0} \gamma(\{x \in X / A_n |f|(x) > \lambda\}) \leq \lambda^{-1} \int_X |f|\, d\nu$$

for every $f \in L^1(d\nu)$.

c) $\{A_n f\}$ converges a.e. for every $f \in L^1(d\nu)$.

d) $Mf(x) < \infty$ a.e. for every $f \in L^1(d\nu)$.

Moreover, if one of the above conditions holds, then $Q_r f$ and $P_r f$ are finite a.e. for every $f \in L^1(d\nu)$.

Proof: Implications a) $\Rightarrow$ b) and c) $\Rightarrow$ d) are obvious. On the other hand, a) follows from d) by Nikishin's theorem. We only have to prove b) $\Rightarrow$ c). Simultaneously, we will see that b) implies finiteness a.e. for Q_r and P_r.

From b) and Marcinkiewicz's interpolation theorem we have

$$\sup_{k\geq 0} \int_X |A_k f|^2 \, d\gamma \leq C \int_X |f|^2 \, d\nu \text{ for every } f \in L^2(d\nu). \tag{4.1}$$

Let L be a Banach's limit (for instance see [**6**]) and define

$$\mu(E) = L\left(\left\{\int_X A_k \chi_E \, d\gamma\right\}\right) \qquad (E \in \mathcal{F}).$$

μ is well defined by (4.1). μ is an invariant measure and it is absolutely continuous with respect to ν. Let v be the Radon-Nikodym derivative $d\mu/d\nu$, $D = \{x : v(x) \neq 0\}$ and $Y = \cap_{n\geq 0} T^{-n} D$. It is clear that $\mu(X - Y) = 0$ and $T|_Y$ applies Y in Y. Therefore we have that $\nu|_Y$ is equivalent to the invariant measure $\mu|_Y$. Then it follows by theorem 4 that the averages $\{A_k f\}$ converge and Mf, $Q_r f$ and $P_r f$ are finite a.e. (ν) in Y for every $f \in L^1(d\nu)$.

To prove the a.e. (ν) convergence of $\{A_k f\}$ and the finiteness of Mf, $Q_r f$ and $P_r f$ on $X - Y$ we shall first state that for almost all x (ν) in X there exists n such that $T^n x \in Y$. If this property is not true, then there exists B with $\nu(B) > 0$ such that for every i, B is contained in $T^{-i}(X - Y)$. Then for every k

$$\gamma(B) \leq (k+1)^{-1} \sum_{i=0}^{k} \gamma(T^{-i}(X - Y)) = \int_X A_k \chi_{X-Y} \, d\gamma$$

and the properties of Banach's limits give

$$\gamma(B) \leq L\left(\left\{\int_X A_n \chi_{X-Y} \, d\gamma\right\}\right) = \mu(X - Y) = 0,$$

which goes against $\nu(B) > 0$ since γ and ν are equivalent.

Let x be in $X - Y$ and let n be an integer verifying $T^n x \in Y$. Let $k \geq n$. Then

$$A_k f(x) = (k+1)^{-1} \left(\sum_{i=0}^{n-1} f(T^i x)\right) + (k-n+1)(k+1)^{-1}(k-n+1)^{-1} \sum_{i=n}^{k} f(T^i x)$$

$$\text{and } \sum_{j=0}^{k} (j+1)^{-r} |f(T^j x)|^r \leq \sum_{j=0}^{n-1} (j+1)^{-r} |f(T^j x)|^r$$

$$+ \sum_{j=n}^{k} (j-n+1)^{-r} |f(T^j x)|^r.$$

Since $T^n x \in Y$ and T applies Y in Y, when k tends to infinity we obtain finite limits. Therefore, we have proved that $\{A_k f(x)\}$ converges a.e. and that $Mf(x)$ and $Q_r f(x)$ are finite a.e. for every f in $L^1(d\nu)$. Then, since $P_r f \le CMf + Q_r f$ we obtain the finiteness of P_r. ■

Corollary. *Let $(X, \mathcal{F}, \nu)$ be a finite measure space and let $T : X \longrightarrow X$ be a null-preserving transformation. If $\sup_{k \ge 0} \|A_k\|_1 < \infty$ then $\{A_k f\}$ converges a.e. and Mf, $Q_r f$ and $P_r f$ are finite a.e. for every $f \in L^1(d\nu)$.*

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