Publicacions Matemàtiques, Vol 35 (1991), 375–402.

EXPLORING W.G. DWYER'S TAME HOMOTOPY THEORY

HANS SCHEERER AND DANIEL TANRÉ

Abstract

Let $\underline{S}_r$ be the category of r-reduced simplicial sets, $r \geq 3$; let $\underline{\mathcal{L}}_{r-1}$ be the category of $(r-1)$-reduced differential graded Lie algebras over $\mathbb{Z}$. According to the fundamental work [3] of W.G. Dwyer both categories are endowed with closed model category structures such that the associated *tame* homotopy category of $\underline{S}_r$ is equivalent to the associated homotopy category of $\underline{\mathcal{L}}_{r-1}$. Here we embark on a study of this equivalence and its implications. In particular, we show how to compute homology, cohomology, homotopy with coefficients and Whitehead products (in the tame range) of a simplicial set out of the corresponding Lie algebra. Furthermore we give an application (suggested by E. Vogt) to $\pi_*(B\Gamma_3)$ where $B\Gamma_3$ denotes the classifying space of foliations of codimension 3.

0. Introduction

In [13], D. Quillen defines the structure of a closed model category on the category $\underline{\mathcal{L}}_1^{\mathbf{Q}}$ of 1-reduced differential graded Lie algebras over $\mathbf{Q}$ and constructs an equivalence of categories between the rational homotopy category of 2-reduced simplicial sets and the homotopy category of $\underline{\mathcal{L}}_1^{\mathbf{Q}}$. In [3], W.G. Dwyer proves that a similar construction can be performed with respect to systems of subrings of $\mathbf{Q}$. For $r \geq 3$, he obtains an equivalence between his *tame* homotopy theory of r-reduced simplicial sets and a corresponding homotopy theory of $(r-1)$-reduced differential graded Lie algebras over $\mathbf{Z}$. This theory is recalled below in more detail. In [13], D. Quillen also shows how to detect the various essential rational homotopy invariants of a simplicial set in the corresponding Lie algebra. This point of view has been very useful in rational homotopy theory for explicit calculations as well as for theoretical purposes. In the present paper we embark on determining the essential tame homotopy invariants of a simplicial set, e.g. homology, cohomology and cup-products, homotopy groups with coefficients and Whitehead products, from the corresponding Lie algebra. We give also an application to $\pi_*(B\Gamma_3)$ where $B\Gamma_3$ denotes the classifying space of foliations of codimension 3. This application was suggested by E. Vogt. Since the paper [3] and its sequel [4] there has been -at least to our knowledge- no further study of W.G. Dwyer's tame homotopy theory and its implications. We hope that our results render this theory more accessible. Working with it has the advantage that no finite type conditions have to be imposed on the simplicial sets. This is e.g. the case in the tame homotopy theory via differential forms developed by B. Cenkl and R. Porter [2] (A report on that theory is given in [14]).

Before we can state our results we have to recall the main result of W.G. Dwyer's tame homotopy theory.

Let r be an integer, $r \geq 3$ and let s always be $r-1$. Let R_* be a tame ring system, i.e. an increasing sequence of subrings R_j ($j \geq 0$) of $\mathbf{Q}$ such that R_j contains the inverse of each integer k with $2k-3 \leq j$.

Let $\underline{S}_r$ be the category of r-reduced simplicial sets. The tame closed model category structure on $\underline{S}_r$ is defined as follows : *cofibrations* are injective maps ; *weak equivalences* are maps $f : X \to Y$ such that the induced homomorphisms $\pi_{r+k}(X) \otimes R_k \to \pi_{r+k}(Y) \otimes R_k$ are isomorphisms for all $k \geq 0$; *fibrations* are the maps having the right lifting property (RLP) with respect to the class of trivial cofibrations, i.e. $g : X \to Y$ is a fibration if in any diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "\sim"] & X \arrow[d, two heads, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

a right lifting, indicated by the dotted arrow, exists.

Here and in the sequel, "$\rightarrowtail$" (resp. "$\twoheadrightarrow$", resp. "$\xrightarrow{\sim}$") denotes a cofibration (resp. fibration, resp. weak equivalence).

Let $\underline{Ch}_s$ be the category of s-reduced chain complexes (over $\mathbf{Z}$) with the following closed model category structure : *cofibrations* are injective maps with dimensionwise projective cokernels, *weak equivalences* are maps f such that $H_{s+k}(f) \otimes R_k$ is an isomorphism for all k ; *fibrations* are maps g which are surjective in degrees $> s$, for which H_{s+k} (kernel (g)) is a R_k-module and cokernel $H_{s+k}(g)$ has no p-torsion for p invertible in R_k, $k \geq 0$.

Let $\underline{\mathcal{L}}_s$ be the category of s-reduced differential graded Lie algebras over $\mathbf{Z}$. It is given the following closed model category structure : *fibrations* and *weak equivalences* are as in $\underline{Ch}_s$; the *cofibrations* are the morphisms having the (LLP) with respect to the class of trivial fibrations. (Note that the underlying Lie algebras of cofibrant objects are retracts of free Lie algebras on free abelian groups and hence are also free). The category of Lie algebras we use will be discussed shortly in section 1.

For $X \in \underline{S}_r$ let GX be the loop group of X ; let Laz GX be the Lazard completion of GX, (denoted by log GX in [3]), considered as a simplicial Lazard Lie algebra. Let Δ-$\underline{\mathrm{Laz}}_s$ be the category of s-reduced simplicial Lazard Lie algebras. Then there are adjoint functors (the left adjoint always appears as the upper arrow) :

$$\underline{S}_r \underset{\longleftarrow}{\overset{\text{Laz G}}{\longrightarrow}} \Delta\text{-}\underline{\text{Laz}}_s$$

There is another pair of adjoint functors :

$$\Delta\text{-}\underline{\text{Laz}}_s \underset{N}{\overset{N^*}{\rightleftarrows}} \underline{\mathcal{L}}_s$$

where N denotes the normalization of a simplicial module, (note that NL is a Lie algebra for a simplicial Lie algebra L), and where N^* is a left adjoint of N, (N^* is denoted by UN^* in [**3**]).

The category Δ-$\underline{\text{Laz}}_s$ is also endowed with a closed model category structure : *fibrations* (resp. *weak equivalences*) are maps f such that Nf is a fibration (resp. weak equivalence) in $\underline{\mathcal{L}}_s$; *cofibrations* are maps which have the (LLP) with respect to trivial fibrations.

Let $\lambda := N\text{Laz } G$; since Laz G and N carry weak equivalences to weak equivalences, so does λ. Hence it induces a functor $\overline{\lambda} : \textit{Ho-}\underline{S}_r \to \textit{Ho-}\underline{\mathcal{L}}_s$ of the associated homotopy categories. The other functors have total derived functors in the sense of [**12**] ; let $\mu : \textit{Ho-}\underline{\mathcal{L}}_s \to \textit{Ho-}\underline{S}_r$ be their composition.

The main result of [**3**] now says that one obtains equivalences of categories :

$$\textit{Ho-}\underline{S}_r \underset{\mu}{\overset{\overline{\lambda}}{\rightleftarrows}} \textit{Ho-}\underline{\mathcal{L}}_s \,.$$

For each closed model category $\underline{D}$, we denote by $(\underline{D})_c$, $(\underline{D})_f$ and $(\underline{D})_{cf}$ the full subcategories of $\underline{D}$, consisting of the cofibrant, fibrant and cofibrant-fibrant objects of $\underline{D}$ respectively.

In *section* 1, if $Y \in \underline{\mathcal{L}}_s$ and A is an abelian group, we define $\pi_{s+k}(Y;A)$ as the $(s+k)^{th}$ homology group of the complex $Y \otimes A$. We will show :

Theorem A. *If $X \in \underline{S}_r$ is a Kan complex and A is a cyclic R_k-module, then the homotopy groups with coefficients, $\pi_{r+k}(X;A)$ are isomorphic to $\overline{\overline{\pi}}_{s+k}(\lambda(X);A)$, where $\overline{\overline{\pi}}_{s+k}(-;A)$ is the derived functor of $\pi_{s+k}(-;A)$.*

Therefore, if $L_X \to \lambda(X)$ is a cofibrant model, i.e. L_X is cofibrant and $L_X \xrightarrow{\sim} \lambda(X)$ a weak equivalence, we have :

$$\pi_{r+k}(X;A) \cong \pi_{s+k}(L_X;A) \,.$$

We will also prove :

Proposition. *Every H-space X, $(r-1)$-connected with $r \geq 2$, such that $\pi_{r+k}(X)$ is a R_k-module, is of the homotopy type of a weak product of Eilenberg-MacLane spaces.*

In particular, the loop space ΩX of a tame (i.e. fibrant) space in $\underline{S}_r$ is homotopy equivalent to a weak product of Eilenberg-MacLane spaces. In fact, we will deduce the proposition from this particular case. Some version of this result has been established in [**5**], [**6**].

In *section 2* we will show how one can build cofibrant models of $L \in \underline{\mathcal{L}}_s$ in a simple way. Together with the results of section 1, this allows to construct models of $\lambda(X)$ in examples. But it is also possible to build a functorial cofibrant model, i.e. a functor $F : \underline{\mathcal{L}}_s \to (\underline{\mathcal{L}}_s)_c$, together with a natural weak equivalence $FL \xrightarrow{\simeq} L$. This construction will be useful for theoretical purposes.

In *section 3* we will study the homology of $X \in \underline{S}_r$ through $\underline{\mathcal{L}}_s$. First, if $L \in \underline{\mathcal{L}}_s$, define $abL \in \underline{Ch}_s$ as $L/\Gamma_1 L$ where $\Gamma_1 L$ is the commutator subalgebra of L, and denote by $\sigma abL \in \underline{Ch}_r$ the suspension of the chain complex abL. We will prove :

Theorem B. *If $M_X \in \underline{\mathcal{L}}_s$ is a functorial cofibrant model of $\lambda(X)$, $X \in \underline{S}_r$, then there is a canonical isomorphism :*

$$H_{r+k}(X; R_k) \cong H_{r+k}(\sigma abM_X; R_k) \text{ for } k \geq 0 .$$

In *section 4* we will display the diagonal in homology. For this purpose, if $L = (\mathrm{L}(V), \partial_L)$ is a cofibrant object in $\underline{\mathcal{L}}_s$, we define $\Delta_V : \sigma abL \to \sigma abL \otimes \sigma abL$ from the quadratic part of the differential in the universal enveloping algebra UL. The map induced in homology by Δ_V does not depend on the choice of V, we denote it by Δ_L. We will prove :

Theorem C. *If $M_X \in \underline{\mathcal{L}}_s$ is a functorial cofibrant model of $\lambda(X)$, $X \in \underline{S}_r$, then the morphism*

$$\Delta_{M_X} : H_{r+k}(\sigma abM_X; R_k) \to H_{r+k}(\sigma abM_X \otimes \sigma abM_X; R_k)$$

can be identified with the reduced diagonal

$$\tilde{H}_{r+k}(X; R_k) \to \tilde{H}_{r+k}(X \wedge X; R_k)$$

in a canonical way.

As a consequence, we get a natural isomorphism

$$\tilde{H}^{\leq r+k}(X; R_k) \cong \tilde{H}^{\leq r+k}(\sigma abM_X; R_k) ,$$

compatible with cup products. The additive isomorphism can be obtained in a more direct way, as indicated in *section 5*.

In *section 6*, if $X \in \underline{S}_r$ is a Kan complex, we compute Samelson and Whitehead products with coefficients in a cyclic R_k-module A by means of the functor λ. They correspond, modulo a suspension, to the bracket on $\overline{\overline{\pi}}_{\leq s+k}(\lambda(X); A)$ induced by the Lie structure of $\lambda(X)$.

Finally, the following application will be given in *section 7*. Let $B\Gamma_3$ be the classifying space of foliations of codimension 3 and let $y \in \pi_4(B\Gamma_3)$ be a generator. Then we get :

Theorem D. *The element $[y, y] \otimes 1 \in \pi_7(B\Gamma_3) \otimes \mathbf{Q}$ is non trivial. Nevertheless, any homomorphism from $\pi_7(B\Gamma_3)$ into a subring R of $\mathbf{Q}$, containing $\frac{1}{2}$, $\frac{1}{3}$ and different from $\mathbf{Q}$, is trivial on $[y, y]$.*

We would like to mention that the results of this paper have been announced in the notes [**16**], [**17**], [**18**].

1. Homotopy groups

We first have to introduce some notations and conventions.

Let $\underline{\mathcal{L}}$ be the category of differential graded Lie algebras over $\mathbf{Z}$. We note that the underlying $\mathbf{Z}$-module of $X \in \underline{\mathcal{L}}$ may have 2- and 3-torsion and that we require only the following identities for the Lie bracket :

(i) $[x, y] = -(-1)^{|x||y|}[y, x]$

for homogeneous x, y, ($|-|$ will always denote the degree).

(ii) $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$.

The category $\underline{\mathcal{L}}_s$ is the full subcategory of $\underline{\mathcal{L}}$ given by the s-reduced objects.

Let $\underline{Ch}$ (resp. $\underline{Ch}_s$) be the category of chain complexes which are zero in degrees < 0 (resp. s-reduced chain complexes) over $\mathbf{Z}$.

We have a functor $\mathbf{L} : \underline{Ch} \to \underline{\mathcal{L}}$ which is left adjoint to the forgetful functor $\underline{\mathcal{L}} \to \underline{Ch}$; in fact, $\mathbf{L}(V)$ is the free differential Lie algebra on V.

Let $\underline{Ab}$ denote the category of (ungraded) abelian groups.

Let $\underline{C}$ be a model category and $F : \underline{C} \to \underline{D}$ a functor (compare [**12**, I.4]). If F carries weak equivalences in $\underline{C}$ into isomorphisms in $\underline{D}$, then F induces a functor $\overline{F} : Ho\text{-}\underline{C} \to \underline{D}$, such that $\overline{F} \circ \gamma = F$, where $\gamma : \underline{C} \to Ho\text{-}\underline{C}$ is the canonical map.

If $\underline{D}$ is also a model category and F carries weak equivalences in $\underline{C}$ into weak equivalences in $\underline{D}$, we will also write $\overline{F}$ for the induced functor $\overline{F} : Ho\text{-}\underline{C} \to Ho\text{-}\underline{D}$.

If F carries weak equivalences between cofibrant objects of $\underline{C}$ into isomorphisms (resp. weak equivalences) in $\underline{D}$, we will write $\overline{\overline{F}} : Ho\text{-}\underline{C} \to \underline{D}$ (resp. $\overline{\overline{F}} : Ho\text{-}\underline{C} \to Ho\text{-}\underline{D}$) for the derived (resp. total derived) functor of F.

Similar conventions will be followed in the dual situation.

Definition 1.1. For $X \in \underline{\mathcal{L}}$ and $A \in \underline{Ab}$, we denote by $\pi_j(X;A)$ the j^{th} homology group of the complex $X \otimes A$. (Note that later on, in section 3, we will introduce $H_j(X)$!).

Lemma 1.2. *Let A be a R_k-module, then the functor $\pi_{s+k}(-;A) : \underline{\mathcal{L}}_s \to \underline{Ab}$ has a left derived functor :*

$$\overline{\overline{\pi}}_{s+k}(-;A) : Ho\text{-}\underline{\mathcal{L}}_s \to \underline{Ab}\,.$$

Proof: For $X \in \underline{Ch}_s$, we have $H_{s+l}(X;R_k) \cong H_{s+l}(X;\mathbf{Z}) \otimes R_k$, by the universal coefficient formula. Let now $f : X \to Y$ be a weak equivalence in $\underline{\mathcal{L}}_s$. If A is torsion-free, then $\pi_{r+l}(X;A) \to \pi_{r+l}(Y;A)$ is an isomorphism for $l \leq k$, again by the universal coefficient formula. But this is also true in general if X and Y are cofibrant; the lemma follows now from [**12**, (Proposition 1, § I.4)].

In particular, if A is torsion free, then $\overline{\pi}_{s+k}(-;A) : Ho\text{-}\underline{\mathcal{L}}_s \to \underline{Ab}$ exists. ∎

Recall the definition of homotopy groups with coefficients [**10**]. For a Kan complex $X \in \underline{\mathcal{S}}_r$, we let $\pi_{r+k}(X;R_k) := \pi_{r+k}(X) \otimes R_k$. If A is a cyclic R_k-module of finite order, we define $\pi_{r+k}(X;A) := [M(A, r+k-1), X]$, where $[-,-]$ denotes the set of homotopy classes of maps and $M(A,n)$ is a Moore complex with $H_n(M(A,n);\mathbf{Z}) \cong A$. In this case we always want to assume $k \geq 1$, such that $M(A, r+k-1)$ may be chosen in $\underline{\mathcal{S}}_r$. For $k = 0$ the results of [**10**] give a canonical isomorphism $\pi_r(X;A) \cong \pi_r(X) \otimes A$. Hence, for our purposes we will take this formula as a definition for $\pi_r(X;A)$.

We can now state theorem A more precisely :

Theorem 1.3. *Let A be a cyclic R_k-module. Then there is a natural isomorphism $\pi_{r+k}(X;A) \cong \overline{\overline{\pi}}_{s+k}(\lambda(X);A)$ for Kan complexes $X \in \underline{\mathcal{S}}_r$.*

In particular, if $L_X \xrightarrow{\sim} N$ Laz GX is a weak equivalence with L_X cofibrant, then $\pi_{r+k}(X;A) \cong \overline{\pi}_{s+k}(L_X;A)$.

Proof: To define $\overline{\overline{\pi}}$ we have to take a trivial fibration $L_X \overset{\sim}{\twoheadrightarrow} \lambda(X)$ with L_X cofibrant ; then the adjoint $N^*L_X \to \mathrm{Laz}\, GX$ is a weak equivalence by lemma 1.4 below.

By [**3**] we have isomorphisms

$$\begin{aligned}\pi_{r+k}(X;R_k) &\cong \pi_{s+k}(GX;R_k) \cong \pi_{s+k}(\mathrm{Laz}\, GX;R_k) \cong \pi_{s+k}(N^*L_X;R_k)\\ &\cong \pi_{s+k}(NN^*L_X;R_k) \cong \pi_{s+k}(L_X;R_k)\,,\end{aligned}$$

because $L_X \to NN^*L_X$ is a weak equivalence by [**3**, proposition 8.2].

Suppose now that A is of finite order. According to [**10**] , there is an exact sequence

$$0 \to \pi_n(Y) \otimes A \to \pi_n(Y;A) \to \mathrm{Tor}(\pi_{n-1}(Y), A) \to 0$$

which is natural in Y and which is compatible with the isomorphisms $\pi_{r+k}(X;R_k)\cong\pi_{s+k}(GX;R_k)$ and $\pi_{r+k}(X;A)\cong\pi_{s+k}(GX;A)$. By the five lemma the sequence of isomorphisms $\pi_{r+k}(X;R_k)\cong\ldots\cong\pi_{s+k}(N^*L_X;R_k)$ yields the corresponding sequence of isomorphisms $\pi_{r+k}(X;A)\cong\ldots\cong\pi_{s+k}(N^*L_X;A)$. By lemma 1.5 below we have an isomorphism $\pi_{s+k}(N^*L_X;A)\cong\overline{\overline{\pi}}_{s+k}(NN^*L_X;A)$. Finally, $\overline{\overline{\pi}}_{s+k}(NN^*L_X;A)$ is isomorphic to $\overline{\pi}_{s+k}(L_X;A)$, because $L_X\to NN^*L_X$ is a weak equivalence. ■

Lemma 1.4. *For $X\in\underline{S}_r$ let $f:L_X\to N\mathrm{Laz}\,GX$ be a weak equivalence in $\underline{\mathcal{L}}_s$ with L_X cofibrant. Then the adjoint $f^{\#}:N^*L_X\to\mathrm{Laz}\,GX$ is a weak equivalence in Δ-$\underline{\mathrm{Laz}}_s$.*

Proof: By definition, $f^{\#}$ is a weak equivalence if $Nf^{\#}$ is one. In the diagram

$$\begin{tikzcd} NN^*(L_X) \arrow[r, "N(f^{\#})"] & N\mathrm{Laz}\,GX \\ L_X \arrow[u, "(1)"] \arrow[ur, dashed, "(2)"] & \end{tikzcd}$$

the arrow (1) is a weak equivalence by [**3**, proposition 8.2] and (2) is again f, hence $N(f^{\#})$ is a weak equivalence. ■

Let the category $\underline{Ch}$ of chain complexes over $\mathbf{Z}$ be endowed with its usual structure of a closed model category (see [**12**]) : *weak equivalences* are the homology isomorphisms ; *cofibrations* are the injective maps with dimensionwise projective cokernel and *fibrations* are the maps being surjective in degrees greater than 0. Then, for $A\in\underline{Ab}$, the functor $\underline{Ch}\to\underline{Ab}$, $X\mapsto H_n(X;A)$ has a left derived functor Ho-$\underline{Ch}\to\underline{Ab}$, $X\mapsto\overline{\overline{H}}_n(X;A)$.

Lemma 1.5. *Let G be a simplicial abelian group and let A be a cyclic finite abelian group, then : $\pi_n(G;A)\cong\overline{\overline{H}}_n(NG;A)$.*

Proof: Recall that N is the normalization functor. We consider the pairs of adjoint functors

$$\begin{tikzcd} \underline{S} \arrow[r, shift left, "\mathbf{Z}"] & \Delta\text{-}\underline{Ab} \arrow[l, shift left, "i"] \arrow[r, shift right, "N"'] & \underline{Ch} \arrow[l, shift right, "N^{-1}"'] \end{tikzcd}$$

where $\underline{S}$ is the category of simplicial sets with its usual structure (see [**12**]) of a closed model category and Δ-$\underline{Ab}$ denotes the category of simplicial abelian groups.

For $X\in\underline{S}$, $\mathbf{Z}X$ is the simplicial abelian group generated by X ; it is obtained by dimensionwise application of the free abelian group functor ([**12**, II.5.3]). The functor i is an inclusion. The closed model category structure on $\underline{Ch}$ is transferred to one on Δ-$\underline{Ab}$ by the equivalence N (with quasi-inverse N^{-1}). One verifies that the functors induce adjoint functors of the associated homotopy categories. Hence :

$$\begin{aligned}\pi_n(iG;A) &= [M(A,n-1), iG]_{\underline{S}} \cong [\mathbf{Z}M(A,n-1), G]_{\Delta\text{-}\underline{Ab}} \\ &\cong [N\mathbf{Z}M(A,n-1), NG]_{\underline{Ch}} \underset{(*)}{\cong} [W, NG]_{\underline{Ch}} \underset{(**)}{\cong} [W, QNG]_{\underline{Ch}}\end{aligned}$$

where W is the complex

$$\ldots 0 \to W_n \xrightarrow{f} W_{n-1} \to 0 \ldots ,$$

with $W_n = W_{n-1} = \mathbf{Z}$ and f is the multiplication by the order of A.

The isomorphism $(*)$ is induced by a weak equivalence $W \to N\mathbf{Z}M(A,n-1)$.

Moreover, $QNG \overset{\sim}{\twoheadrightarrow} NG$ is a trivial fibration with QNG cofibrant and $(**)$ holds by definition.

Clearly, $[W, QNG]_{\underline{Ch}} \cong H_n(QNG;A) = \overline{\overline{H}}_n(NG;A)$. ■

Remark 1.6. If A is of finite order, the isomorphism

$$\pi_{r+k}(X;A) \cong \overline{\pi}_{s+k}(L_X;A)$$

of theorem 1.3 depends on a choice of a weak equivalence $W \to N\mathbf{Z}M(A, r+k-2)$.

Proposition 1.7. *Let $X \in \underline{S}_r$ be fibrant, $r \geq 3$, let ΩX be a loop space of X in $\underline{S}$. Then ΩX has the homotopy type of $\prod_{i=2}^{\infty} K(\pi_i(X), i-1)$.*

Corollary 1.8. *Let $X \in \underline{S}_s$ be an H-space, $s \geq 2$, such that $\pi_{s+k}(X)$ is a R_k-module for all $k \geq 0$. Then X is of the homotopy type of a weak product of Eilenberg-MacLane spaces.*

Proof of proposition 1.7: We may take $\Omega X = GX$. Then $GX \to \text{Laz}\, GX$ induces isomorphisms :

$$\pi_{s+k}(GX) = \pi_{s+k}(GX) \otimes R_k \xrightarrow{\cong} \pi_{s+k}(\text{Laz}\, GX) \otimes R_k \, .$$

But Laz GX is in particular an object in Δ-$\underline{Ab}$, hence homotopy equivalent to a product of Eilenberg-MacLane spaces ([**9**, theorem 24-5]). Thus the result follows. ■

Proof of corollary 1.8: If Σ denotes the suspension, then there is a retraction $\Omega\Sigma X \to X$. Because of the assumption on the homotopy groups of X, we have also a retraction $\Omega(\Sigma X)_t \to X$, where $(\Sigma X)_t$ is the tamed ΣX (or ΣX made fibrant in $\underline{S}_r$). Hence, X is a product of Eilenberg-MacLane spaces as a retract of a product of Eilenberg-MacLane spaces. To see this one may e.g. apply the theorem of Moore ([**22**, ch. IX, theorem 1.9]). ■

Remark 1.9. Versions of these results (1.7 and 1.8) have been shown in [**5**], [**6**].

2. Models

Notations:

(1) For M, $N \in \underline{\mathcal{L}}$ we denote by $M \sqcup N$ the sum (free product) of the Lie algebras M, N.

(2) For $M \in \underline{\mathcal{L}}$ and $V \in \underline{Ch}$ a free complex with differential ∂, let $\tau : V \to M$ be a chain map of degree -1 (i.e $\tau\partial = -d_M\tau$) . Then there is exactly one Lie algebra differential d_τ on $M \sqcup \mathrm{L}(V)$ with : $d_{\tau|M} = d_M$, $d_\tau(v) := \tau(v) + \partial v$, for all $v \in V$. This differential Lie algebra will be denoted by $M \sqcup_\tau \mathrm{L}(V)$.

Let now s be an integer, $s \geq 2$.

Remark 2.1. Since $\underline{\mathcal{L}}_s$ is a closed model category, any $X \in \underline{\mathcal{L}}_s$ has a cofibrant "model", i.e. there is a cofibrant $M(X)$ (free, as Lie algebra, on a free abelian group) and a weak equivalence $M(X) \to X$. For computational purposes, it might be interesting to construct cofibrant models with a "small" number of generators.

Construction 2.2: Building "small" models.

Let $X \in \underline{\mathcal{L}}_s$. We inductively construct a differential free Lie algebra $M^{(k)}$ together with a morphism $M^{(k)} \xrightarrow{f^{(k)}} X$, such that $\pi_{s+l}(f^{(k)}; R_l)$ is an isomorphism for $l \leq k$.

We may start the induction at $k = -1$, by setting $R_{-1} := \mathbf{Z}$ and $M^{(-1)} := 0$.

Suppose $M^{(k)}$, $f^{(k)}$ are constructed.

(a) Choose classes $[a_i] \in \pi_{s+k+1}(X; R_{k+1})$, $i \in I$, such that $\{[a_i]\}$ and $f^{(k)}_*(\pi_{s+k+1}(M^{(k)}; R_{k+1}))$ generate $\pi_{s+k+1}(X; R_{k+1})$ and such that $a_i \in X$. Then let W be a free $\mathbf{Z}$-module generated by w_i, $i \in I$, $|w_i| = s + k + 1$, and define a map $M^{(k)} \sqcup \mathrm{L}(W) \xrightarrow{g} X$ by :

$$g_{|M^{(k)}} = f^{(k)} \ , \ g(w_i) := a_i \ .$$

Then, obviously, $\pi_{s+l}(g; R_l)$ is an isomorphism for $l \leq k$ and a surjection for $l = k + 1$.

For the next step, let us redefine $M^{(k)} := M^{(k)} \sqcup \mathrm{L}(W)$ and $f^{(k)} := g$.

(b) Note that $\pi_{s+l}(X; R_k) \cong \pi_{s+l}(X) \otimes R_k$. Choose cycles $z_i \in Z_{s+k+1}(M^{(k)})$ such that their images in $\pi_{s+k+1}(M^{(k)}; R_{k+1})$ generate kernel $(\pi_{s+k+1}(f^{(k)}; R_{k+1}))$. Hence in particular $[f^{(k)}(z_i) \otimes 1] = 0$ in $\pi_{s+k+1}(X; R_{k+1})$, or $f^{(k)}(z_i) \otimes 1 = \partial_X(\sum_j w_{i,j} \otimes \alpha_{i,j})$ for some $w_{i,j} \in X$, $\alpha_{i,j} \in R_{k+1}$. Assuming that all denominators of $\alpha_{i,j}$ are invertible in R_{k+1}, we may multiply z_i by their product to get $\tilde{z}_i$, and it is still true that the classes of $\tilde{z}_i$ generate kernel $\left(\pi_{s+k+1}(f^{(k)}; R_{k+1})\right)$. But now, we have :

$$f^{(k)}(\tilde{z}_i) \otimes 1 = \partial_X(\tilde{w}_i) \otimes 1 \quad \text{for some} \quad \tilde{w}_i \in X \ .$$

Hence, the order of $f^{(k)}(\tilde{z}_i) - \partial_X(\tilde{w}_i)$ is invertible in R_{k+1}. Multiplying $\tilde{z}_i$ by this number, we finally have $f^{(k)}(\tilde{\tilde{z}}_i) = \partial_X(\tilde{\tilde{w}}_i)$ and the classes of $\tilde{\tilde{z}}_i$ still generate the kernel.

Let W be a free $\mathbf{Z}$-module generated by w_i, $i \in I$, with $|w_i| = s+k+2$. Then we define $M^{(k+1)} := M^{(k)} \sqcup_\tau \mathbf{L}(W)$, with $\tau(w_i) = \tilde{\bar{z}}_i$, and $M^{(k+1)} \xrightarrow{f^{(k+1)}} X$, as the extension of $f^{(k)}$ by the map $\mathrm{L}(W) \to X$ given by $w_i \mapsto \tilde{\bar{w}}_i$. Then $\pi_{s+l}(f^{(k+1)}; R_l)$ is an isomorphism for $l \le k+1$.

(c) Finally, the cofibrant "small" model for X is obtained as $M(X) := \varinjlim M^{(k)}$. ∎

Remark 2.3. The above construction also works without any restriction on s and the ring system R_*. Only the conclusion that the result is cofibrant refers to the closed model category structure of $\underline{\mathcal{L}}_s$.

Proposition 2.4. *There exists a functorial cofibrant model, i.e. a functor $\underline{\mathcal{L}}_s \xrightarrow{F} (\underline{\mathcal{L}}_s)_c$ together with a natural weak equivalence $F(X) \to X$.*

Proof: The arguments are adapted from [3]. It suffices to do the construction for the constant ring system $R_* = \mathbf{Z}$, (see remark above).

(a) Let $X \in \underline{\mathcal{L}}_s$. We denote by $\tilde{Z}_{s+l}(X)$ the free $\mathbf{Z}$-module (concentrated in degree $s+l$) generated by the set of cycles $Z_{s+l}(X)$. We define $M_X^{(0)} := \sqcup_{l=0}^{\infty} \mathbf{L}(\tilde{Z}_{s+l}(X))$. The inclusions $Z_{s+l}(X) \subset X$ define a morphism $M_X^{(0)} \to X$ which induces a surjection of π_*.

Given $f : X \to Y$, we obtain $M^{(0)}(f) : M_X^{(0)} \to M_Y^{(0)}$ as induced by the restrictions of f to $Z_{r+l}(X) \to Z_{r+l}(Y)$; i.e. the diagram

$$\begin{array}{ccc} M_X^{(0)} & \longrightarrow & X \\ \Big\downarrow{\scriptstyle M^{(0)}(f)} & & \Big\downarrow{\scriptstyle f} \\ M_Y^{(0)} & \longrightarrow & Y \end{array} \qquad \text{commutes.}$$

(b) We now define functorial extensions $M_X^{(0)} \subset M_X^{(s)} \subset \dots M_X^{(s+k)} \subset \dots$ together with a natural map $\alpha_X^{(s+k)} : M_X^{(s+k)} \to X$ which induces a surjection of π_* and isomorphisms of π_i for $i \le s+k$. Suppose $M_X^{(s+k)}$, $\alpha_X^{(s+k)}$ are defined. Let P_X be the free $\mathbf{Z}$-module generated by

$$\{(z,x) \in Z_{s+k+1}(M_X^{(s+k)}) \times X_{s+k+2} / \alpha_X^{(s+k)}(z) = \partial_X(x)\} .$$

The elements $p \in P_X$ are given the degree $|p| = s+k+2$. Define : $M_X^{(s+k+1)} := M_X^{(s+k)} \sqcup_\tau \mathbf{L}(P_X)$, with $\tau(p) = z$ for $p = (z,x)$, and define

$$M_X^{(s+k+1)} \xrightarrow{\alpha_X^{(s+k+1)}} X .$$

as the extension of $\alpha_X^{(s+k)}$ by the map $\mathbf{L}(P_X) \to X$ given by $p = (z,x) \mapsto x$. Then, $\alpha_X^{(s+k+1)}$ induces still a surjection of π_* and isomorphisms of π_i, for $i \le s+k+1$.

Given $f : X \to Y$, we have an induced map $P_X \to P_Y$, defined by

$$(z, x) \mapsto (M^{(s+k)}(f)(z), f(x))$$

which gives $M^{(s+k+1)}(f) : M_X^{(s+k+1)} \to M_Y^{(s+k+1)}$.

(c) Finally, we define $M_X := \varinjlim M_X^{(i)}$. ■

Remark 2.5. It is also possible to construct functorial fibrant models but we do not need this construction here.

Corollary 2.6. *The functor F induces a functor $\bar{F} : Ho\text{-}\underline{\mathcal{L}}_s \to Ho\text{-}(\underline{\mathcal{L}}_s)_c$ which is an equivalence of categories.*

This proof is clearly part of general constructions with model categories as follows.

Lemma 2.7. *Let $\underline{C}$ be a model category and $F : \underline{C} \to (\underline{C})_c$ a functor with a natural weak equivalence $F(X) \to X$. Then F induces a functor $\overline{F} : Ho\text{-}\underline{C} \to Ho\text{-}(\underline{C})_c$ which is an equivalence of categories.*

Proof: This is a particular case of proposition 2.3 of [**13**], nevertheless we give the proof which is really simpler.

Let $f : X \to Y$ be a weak equivalence in $\underline{C}$. From the diagram :

$$\begin{tikzcd} F(X) \arrow[r, "\sim"] \arrow[d] & X \arrow[d, "\sim"] \\ F(Y) \arrow[r, "\sim"] & Y \end{tikzcd}$$

we deduce that $F(X) \to F(Y)$ is a weak equivalence. Hence F induces $\overline{F} : Ho\text{-}\underline{C} \to Ho\text{-}(\underline{C})_c$, because the homotopy categories are localizations with respect to the classes of weak equivalences.

Similarly, the inclusion $i : (\underline{C})_c \to \underline{C}$ induces $\bar{i} : Ho\text{-}(\underline{C})_c \to Ho\text{-}\underline{C}$. We claim that $\bar{i}$ and $\overline{F}$ are quasi-inverses. This follows from the fact that, for $X \in \underline{C}$, $i \circ F(X) = F(X) \to X$ is a natural weak equivalence, hence an isomorphism in the homotopy categories. ■

3. Homology

In this section we prove that the homology $H_{r+l}(X; R_l)$ of $X \in \underline{S}_r$ is naturally isomorphic with the homology $H_{r+l}(\lambda(X); R_l)$ which has to be defined.

Definition 3.1. Let $ab : \underline{\mathcal{L}}_s \to \underline{Ch}_s$ be the *functor abelianization* i.e. $ab(X) := X/\Gamma_1 X$, where $\Gamma_1 X$ is the commutator subalgebra of X.

Let $i : \underline{Ch}_s \to \underline{\mathcal{L}}_s$ be the inclusion. (For $V \in \underline{Ch}_s$, the underlying Lie algebra of $i(V)$ is abelian).

Proposition 3.2. *The functors* $\underline{\mathcal{L}}_s \underset{i}{\overset{ab}{\rightleftarrows}} \underline{Ch}_s$ *are adjoint and their total derived functors are adjoint functors :*

$$Ho\text{-}\underline{\mathcal{L}}_s \underset{\overline{\overline{i}}}{\overset{\overline{\overline{ab}}}{\rightleftarrows}} Ho\text{-}\underline{Ch}_s \,.$$

Proof: The adjointness of (ab, i) is clear. To prove the second part, we note that, by definition, i maps fibrations to fibrations and weak equivalences to weak equivalences. Hence we can apply the proposition in the appendix to this section. ■

Remark 3.3. It can be shown more directly that ab maps cofibrations to cofibrations : if $X \to X \sqcup \mathbf{L}(W)$ is a free map, then its abelianization $ab(X) \to ab(X) \oplus W$ is a cofibration. Since any cofibration is a retract of a free map, the result follows. Similarly, it can be shown more directly that ab maps weak equivalences between cofibrant objects to weak equivalences.

We denote by $\sigma : \underline{Ch}_s \to \underline{Ch}_r$ the suspension functor : $\sigma(A)_{q+1} := A_q$; $\partial_{\sigma(A)} = -\sigma\partial_A$. Note also that the functors $H_{r+l}(-; R_l)$ induce functors

$$\overline{H}_{r+l}(-; R_l) : Ho\text{-}\underline{Ch}_r \to \underline{Ab} \,.$$

Definition 3.4. For $X \in \underline{\mathcal{L}}_s$ and A a R_k-module, we define :
$H_{r+l}(X; A) := \overline{H}_{r+l}(\sigma\overline{\overline{ab}}X; A)$.

Now we can restate theorem B as :

Theorem 3.5. *Let A be a R_k-module, the functors $Ho\text{-}\underline{S}_r \to \underline{Ab}$, $X \mapsto H_{r+k}(X; A)$ and $X \mapsto H_{r+k}(\lambda(X); A)$ are isomorphic for $k \geq 0$.*

To prove it, we first have to establish the analogue of proposition 3.2 for simplicial Lazard Lie algebras :

Definition 3.6. 1) Let $\underline{\text{Laz}}$ be the category of Lazard Lie algebras. By definition, each $X \in \underline{\text{Laz}}$ is equipped with a central series $X = \Gamma_0 X \supset \Gamma_1 X \supset \ldots$ establishing the Lazard structure.

We define $abX = X/\Gamma_1 X$. Then $ab : \underline{\text{Laz}} \to \underline{Ab}$ is a functor which has a right adjoint $j : \underline{Ab} \to \underline{\text{Laz}}$ given by $j(V) = V$ with $\Gamma_1 V = 0$, $V \in \underline{Ab}$.

2) Similarly, we have the simplicial analogues denoted by the same letters :

$$\Delta\text{-}\underline{\text{Laz}}_s \underset{\text{j}}{\overset{\text{ab}}{\rightleftarrows}} \Delta\text{-}\underline{Ab}_s \,.$$

Note that $\Delta\text{-}\underline{\text{Laz}}_s$ has the closed model category structure described in the introduction and $\Delta\text{-}\underline{Ab}_s$, the category of s-reduced simplicial abelian groups, is given the closed model category structure obtained by the equivalence of categories $N : \Delta\text{-}\underline{Ab}_s \to \underline{Ch}_s$.

Proposition 3.7. *The functors* $\Delta\text{-}\underline{Laz}_s \underset{j}{\overset{ab}{\rightleftarrows}} \Delta\text{-}\underline{Ab}_s$ *are adjoint with adjoint total derived functors :*

$$Ho\text{-}\Delta\text{-}\underline{Laz}_s \underset{\overline{\overline{j}}}{\overset{\overline{\overline{ab}}}{\rightleftarrows}} Ho\text{-}\Delta\text{-}\underline{Ab}_s$$

Proof: We again apply the proposition in the appendix to this section. By definition, j maps fibrations to fibrations and weak equivalences to weak equivalences. ■

The pairs of functors in 3.2. and 3.7. are related as follows :

Lemma 3.8. *Consider the diagram :*

$$\begin{tikzcd} \underline{\mathcal{L}}_s \arrow[r, shift left, "N^*"] \arrow[d, shift right, "ab"'] & \Delta\text{-}\underline{Laz}_s \arrow[l, shift left, "N"] \arrow[d, shift left, "ab"] \\ \underline{Ch}_s \arrow[r, shift left, "N^{-1}"] \arrow[u, shift right, "i"'] & \Delta\text{-}\underline{Ab}_s \arrow[l, shift left, "N"] \arrow[u, shift left, "j"] \end{tikzcd}$$

then $ab\,N^*$ *and* $N^{-1}ab$ *are isomorphic.*

Proof: The right adjoint functors are Nj and iN which are equal. Hence their left adjoints are isomorphic. ■

Proposition 3.9. *Suppose there is a functor* $\underline{S}_r \to (\underline{\mathcal{L}}_s)_c$, $X \mapsto L_X$, *such that there is a natural weak equivalence* $L_X \to N\mathrm{Laz}GX$ *. Then there is a natural chain map* $abL_X \to NabGX$ *which is a weak equivalence in* $\underline{Ch}_s$.

For any group G we denote by abG its abelianization ; similarly, if G is a simplicial group (like GX), then abG is its (degree-wise) abelianization.

Proof: Let $f : L_X \xrightarrow{\sim} N\mathrm{Laz}GX$ be given. Then the adjoint $f^{\#} : N^*L_X \to \mathrm{Laz}\,GX$ is a weak equivalence by lemma 1.4.

Now N^*L_X and $\mathrm{Laz}\,GX$ being cofibrant, we obtain a weak equivalence (by proposition 3.7 and lemma 3.8) :

$$N^{-1}\,ab\,L_X \cong ab\,N^*\,L_X \xrightarrow{\sim} ab\,\mathrm{Laz}\,GX \cong ab\,GX$$

and hence a weak equivalence $ab\,L_X \xrightarrow{\sim} Nab\,GX$, which is natural in X if f depends functorially on X. ■

Proof of theorem 3.5: We first recall some facts from the theory of simplicial sets ([**9**, section 26]).

Let $X \in \underline{\mathcal{S}}_r$ and $R \subset \mathbf{Q}$ a subring. We denote by $\overline{R}X$ the reduced simplicial free R-module generated by X which can also be considered as a chain complex. Denote by $\tau : X \to GX$ the canonical twisting function ; denote as well by $\tau : X \to abGX$ the composition $X \to GX \to abGX$. Then $\tau : X \to abGX$ induces a chain map of degree $-1 : \overline{\mathbf{Z}}X \to abGX$, which induces isomorphisms of homology groups. It follows that the map $\hat{\tau} : \overline{\mathbf{Z}}X \to \sigma\, abGX$, defined by $x \mapsto \sigma(-\tau x)$ is a chain map inducing an isomorphism in homology.

Consider now the following diagram where F is the functorial cofibrant model of 2.4 :

$$\begin{tikzcd}
 & \underline{\mathcal{S}}_r \arrow[r, "\mathrm{Laz}\,G"] \arrow[dl, "H_{r+k}"'] \arrow[dd] & \Delta\text{-}\underline{\mathrm{Laz}}_s \arrow[r, "N"] \arrow[dd] & \underline{\mathcal{L}}_s \arrow[r, "F"] \arrow[dd] & (\underline{\mathcal{L}}_s)_c \arrow[dr, "H_{r+k}"] \arrow[dd] & \\
\underline{Ab} & & & & & \underline{Ab} \\
 & Ho\text{-}\underline{\mathcal{S}}_r \arrow[r, "\overline{\mathrm{Laz}\,G}"] \arrow[ul, "\overline{H}_{r+k}"] & Ho\text{-}\Delta\text{-}\underline{\mathrm{Laz}}_s \arrow[r, "\overline{N}"] & Ho\text{-}\underline{\mathcal{L}}_s \arrow[r, "\overline{F}"] & Ho\text{-}(\underline{\mathcal{L}}_s)_c \arrow[ur, "\overline{H}_{r+k}"'] &
\end{tikzcd}$$

For the natural isomorphism between $H_{r+k}(-;A)FN\mathrm{Laz}\ G$ and $H_{r+k}(-;A)$, we may take the isomorphism induced by

$$N\overline{\mathbf{Z}}X \to \sigma NabGX \leftarrow \sigma abFN\mathrm{Laz}GX\ .$$

(Note that the right one is established in 3.9).

Let now $L \in Ho\text{-}\underline{\mathcal{L}}_s$; by definition, $H_{r+k}(L;A)$ is equal to $\overline{H}_{r+k}(\sigma abQL;A)$ with $QL \xrightarrow{\sim} L$ a trivial fibration and QL cofibrant. In the diagram,

$$\begin{tikzcd}
 & QL \arrow[d, two heads, "\sim"] \\
FL \arrow[ur, dashed] \arrow[r, "\sim"] & L
\end{tikzcd}$$

the dotted arrow, making the diagram commute, exists. Moreover, an arrow, making the diagram homotopy commute, is unique up to left homotopy. It follows (3.2) that $FL \to QL$ induces a unique isomorphism in homology. Additionally, this isomorphism depends functorially on morphisms $L \to K$, i.e.

$$\begin{tikzcd}
 & QL \arrow[r] \arrow[d, "\sim"] & QK \arrow[d, "\sim"] & \\
 & L \arrow[r] & K & \\
FL \arrow[uur] \arrow[ur, "\sim"] \arrow[rrr] & & & FK \arrow[uul] \arrow[ul, "\sim"']
\end{tikzcd}$$

homotopy commutes. This proves the theorem. ■

Appendix to section 3. On total derived functors of adjoint functors between closed model categories.

Let $\underline{C}$, $\underline{D}$ be model categories and $\underline{C} \underset{G}{\overset{F}{\rightleftarrows}} \underline{D}$ adjoint functors. Denote the class of cofibrations, fibrations and weak equivalences by cof, fib, we respectively. Then theorem 3 of [**12**, (Section I.4)] says the following :

Assume $F(cof) \subset cof$ and $F(f) \in we$ for any weak equivalence $f \in (\underline{C})_c$, assume $G(fib) \subset fib$ and $G(g) \in we$ for any weak equivalence $g \in (\underline{D})_f$; then the total derived functors

$$Ho\text{-}\underline{C} \underset{\overline{\overline{G}}}{\overset{\overline{\overline{F}}}{\rightleftarrows}} Ho\text{-}\underline{D}$$

exist and are adjoint.

Proposition. *Let* $\underline{C} \underset{G}{\overset{F}{\rightleftarrows}} \underline{D}$ *be adjoint functors between closed model categories such that :*

(i) $G(fib) \subset fib$,

(ii) $G(we) \subset we$.

Then F *satisfies the above conditions and hence*

$$Ho\text{-}\underline{C} \underset{\overline{G}}{\overset{\overline{\overline{F}}}{\rightleftarrows}} Ho\text{-}\underline{D}$$

exist and are adjoint.

Proof: The proof will be based on [**12**] without detailed references.

(1) We first show that $F(cof) \subset cof$ and $F(cof \cap we) \subset cof \cap we$.

In a closed model category, cof (resp. $cof \cap we$) is the class of morphisms having the left lifting property (LLP) with respect to $fib \cap we$ (resp. fib). Hence, if $f : X \rightarrowtail Y$ is a cofibration in $\underline{C}$, we have to show that in any diagram

$$\begin{tikzcd} FX \arrow[r] \arrow[d, "Ff"'] & A \arrow[d, "\sim"] \\ FY \arrow[r] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

a left lifting h exists. Applying G one obtains the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & GFX \arrow[r] \arrow[d] & GA \arrow[d, "\sim"] \\ Y \arrow[r] \arrow[urr, dashed, "\tilde{h}"] & GFY \arrow[r] & GB \end{tikzcd}$$

in which a left lifting $\bar{h}$ exists. But then its adjoint h is a left lifting for Ff.

Similarly, one proves $F(cof \cap we) \subset cof \cap we$.

(2) Let A, $B \in (\underline{C})_c$, $f : A \xrightarrow{\sim} B$. We form a diagram :

$$\begin{tikzcd} A \arrow[r, tail, "\sim"] \arrow[d, "\sim f"'] & RA \arrow[d, dashed, "g"] \\ B \arrow[r, tail, "\sim"] & RB \end{tikzcd}$$

with RA, $RB \in (\underline{C})_{cf}$. A morphism g making the diagram commutative exists. Hence g is a weak equivalence and therefore a homotopy equivalence in the ordinary sense, because RA and RB are in $(\underline{C})_{cf}$, i.e. there exists $g' : RB \to RA$ with $g' \circ g \sim id_{RA}$ and $g \circ g' \sim id_{RB}$.

We want to show that the image of Fg in Ho-$\underline{D}$ is an isomorphism. This will imply that Fg is a weak equivalence because $\underline{D}$ is a closed model category.

(3) Let $X \in (\underline{C})_c$, α, $\beta : X \rightrightarrows Y$. If α, β are left homotopic, then we claim that $F\alpha$ and $F\beta$ are left homotopic.

A left homotopy between α, β may be chosen as H in the diagram :

$$\begin{tikzcd} X \vee X \arrow[r, "\alpha+\beta"] \arrow[d] \arrow[dr, tail, "\partial_0+\partial_1"] & X \\ X & \tilde{X} \arrow[l, "\sim"', "\sigma"] \arrow[u, "H"'] \end{tikzcd}$$

Applying F we get a diagram

$$\begin{tikzcd} FX \vee FX \arrow[r, "F\alpha+F\beta"] \arrow[d] \arrow[dr, tail] & FY \\ FX & F\tilde{X} \arrow[l, "F\sigma"'] \arrow[u, "FH"'] \end{tikzcd}$$

and we need only to show that $F\sigma \in we$.

But $\partial_0 : X \to \tilde{X}$ is a weak equivalence and a cofibration, hence $F\sigma$ is a weak equivalence, because $F\partial_0$ is one and $\sigma\partial_0 = id_X$.

(4) It follows that $Fg' \circ Fg$ is left homotopic to id_{FRA} and $Fg \circ Fg'$ is left homotopic to id_{FRB}. Now, the images of left homotopic maps in the homotopy category are equal ; hence Fg becomes an isomorphism in Ho-$\underline{D}$.

The diagram

$$\begin{tikzcd} FA \arrow[r, tail, "\sim"] \arrow[d, "Ff"] & FRA \arrow[d, "\sim"', "Fg"] \\ FB \arrow[r, tail, "\sim"] & FRB \end{tikzcd}$$

shows that Ff is a weak equivalence. ∎

4. The homology of the reduced diagonal

Let $L \in \underline{\mathcal{L}}$ such that the underlying Lie algebra is free on a free $\mathbf{Z}$-module V. For the sequel we fix V. We denote the underlying Lie algebra by $\mathbf{L}(V)$ and the space of commutators of length i by $\mathbf{L}^{[i]}(V)$. The differential ∂_L can be written as $\partial_L = \sum_{i\geq 0}\partial_i$, where $\partial_i(V) \subset \mathbf{L}^{[i+1]}(V)$. For instance, ∂_0 and ∂_1 are the linear and the quadratic part of ∂_L respectively.

Let σabL be the suspension of $abL \cong V$; $\sigma\otimes\sigma : abL\otimes abL \to \sigma abL\otimes\sigma abL$, $x\otimes y \mapsto (-1)^{|x|}\sigma x\otimes\sigma y$, is a chain map of degree 2.

One verifies that the universal enveloping algebra of $\mathbf{L}(V)$ is $T(V)$. The canonical map of $\mathbf{L}(V)$ into $T(V)$ is denoted by $u : \mathbf{L}(V) \to T(V)$

Definition 4.1. We define $\Delta_V : \sigma abL \to \sigma abL\otimes\sigma abL$ as
$\Delta_V := (\sigma\otimes\sigma)\circ u\circ\partial_1\circ\sigma^{-1}$.
(Note that Δ_V is of degree 0).

Lemma 4.2. *(1) With these definitions Δ_V is cocommutative and a map of chain complexes.*

(2) Let $f : L = (\mathbf{L}(V),\partial_L) \to L' = (\mathbf{L}(V'),\partial_{L'}) \in \underline{\mathcal{L}}$, then f induces $\sigma abf : \sigma abL \to \sigma abL'$ and $\Delta_{V'}\circ\sigma abf$ is chain homotopic to $(\sigma abf\otimes\sigma abf)\circ\Delta_V$.

Proof: Both parts follow by a straightforward calculation. For (2), we note that a homotopy is defined by $\sigma v \mapsto (\sigma\otimes\sigma)(f_1 v)$, where f_1 is the quadratic part of f. ■

Remark 4.3. As a consequence, we note that the map induced by Δ_V in homology does not depend on the choice of V ; we denote it by
$\Delta_L : H(\sigma abL) \to H(\sigma abL\otimes\sigma abL)$.

Then theorem C can be restated as :

Theorem 4.4. *Let $X \in \underline{S}_r$, $L = (\mathbf{L}(V),\partial_L) \in (\underline{\mathcal{L}}_s)_c$ and $L \xrightarrow{\sim} N\mathrm{Laz}GX$ be a weak equivalence. Let $\Delta : \overline{\mathbf{Q}}_2X \to \overline{\mathbf{Q}}_2X\otimes\overline{\mathbf{Q}}_2X$ be induced by the diagonal $X \to X\times X$.*

Then the diagram

$$\begin{tikzcd}
\sigma abL\otimes\mathbf{Q}_2 \arrow[r] \arrow[d, "\Delta_V"] & \sigma NabGX\otimes\mathbf{Q}_2 & N\overline{\mathbf{Q}}_2X \arrow[l] \arrow[d, "N\Delta"] \\
\sigma abL\otimes\sigma abL\otimes\mathbf{Q}_2 \arrow[r] & \sigma NabGX\otimes\sigma NabGX\otimes\mathbf{Q}_2 & N\overline{\mathbf{Q}}_2X\otimes N\overline{\mathbf{Q}}_2X \arrow[l]
\end{tikzcd}$$

commutes in homology $H_{r+l}(-;R_l)$ for $l > 0$, where $\mathbf{Q}_2 := \mathbf{Z}[\frac{1}{2}]$. (Note that the horizontal arrows induce isomorphisms).

The proof will be modelled on proposition 6.5 in [**13**].

Denote the canonical twist by $\tau : X \to GX$. Let $\hat{G} := \mathrm{Laz}GX$ for sake of simplicity and let $\hat{\Gamma}_0 = \hat{G} \supset \hat{\Gamma}_1 \supset \dots$ the series defining the Lazard structure on $\hat{G}$. Let τ also denote the composition $X \to GX \to \hat{G}$ and denote by τ_1 and τ_2 the compositions $X \to \hat{G} \to \hat{\Gamma}_0/\hat{\Gamma}_1 \otimes \mathbf{Q}_2$, $X \to \hat{G} \to \hat{\Gamma}_0/\hat{\Gamma}_2 \otimes \mathbf{Q}_2$, respectively. Note that $\hat{\Gamma}_0/\hat{\Gamma}_2 \otimes \mathbf{Q}_2$ denotes the localization of the 2-nilpotent group $\hat{\Gamma}_0/\hat{\Gamma}_2$ away from 2.

Lemma 4.5. *The diagram*

$$\begin{array}{ccccc}
\pi_n\overline{\mathbf{Q}}_2X & & \xrightarrow{\quad(-\tau_1)_*\quad} & & \pi_{n-1}(\hat{\Gamma}_0/\hat{\Gamma}_1 \otimes \mathbf{Q}_2) \\
\Big\downarrow{\scriptstyle \pi_n\Delta} & & & & \Big\downarrow{\scriptstyle \partial} \\
\pi_n(\overline{\mathbf{Q}}_2X \otimes \overline{\mathbf{Q}}_2X) & \xrightarrow{(\tau'\otimes\tau'')_*} & \pi_{n-2}(\hat{\Gamma}_0/\hat{\Gamma}_1 \otimes \hat{\Gamma}_0/\hat{\Gamma}_1 \otimes \mathbf{Q}_2) & \xrightarrow{(\frac{1}{2}\langle,\rangle)_*} & \pi_{n-2}(\hat{\Gamma}_1/\hat{\Gamma}_2 \otimes \mathbf{Q}_2)
\end{array}$$

commutes, where :

(i) $\overline{\mathbf{Q}}_2X \otimes \overline{\mathbf{Q}}_2X$ *and* $\hat{\Gamma}_0/\hat{\Gamma}_1 \otimes \hat{\Gamma}_0/\hat{\Gamma}_1$ *are dimensionwise tensor products of simplicial groups.*

(ii) $\tau' \otimes \tau''(x \otimes y) := \tau_1 d_0 x \otimes d_0\tau_1 y$.

(iii) $\langle\ ,\ \rangle : \hat{\Gamma}_0/\hat{\Gamma}_1 \otimes \hat{\Gamma}_0/\hat{\Gamma}_1 \otimes \mathbf{Q}_2 \to \hat{\Gamma}_1/\hat{\Gamma}_2 \otimes \mathbf{Q}_2$ *is induced by the commutator :* $\langle x, y\rangle = xyx^{-1}y^{-1}$.

(iv) ∂ *is the boundary homorphism of the sequence*

$$* \to \hat{\Gamma}_1/\hat{\Gamma}_2 \otimes \mathbf{Q}_2 \to \hat{\Gamma}_0/\hat{\Gamma}_2 \otimes \mathbf{Q}_2 \to \hat{\Gamma}_0/\hat{\Gamma}_1 \otimes \mathbf{Q}_2 \to * \,.$$

Note that $\frac{1}{2}\langle\ ,\ \rangle$ defines a bijection of the subgroup of skew-symmetric tensors of $\hat{\Gamma}_0/\hat{\Gamma}_1 \otimes \hat{\Gamma}_0/\hat{\Gamma}_1 \otimes \mathbf{Q}_2$ onto $\hat{\Gamma}_1/\hat{\Gamma}_2 \otimes \mathbf{Q}_2$.

Proof: Let $\alpha \in \pi_n(\overline{\mathbf{Q}}_2X)$ be represented by $z = \sum a_x x \in N(\overline{\mathbf{Q}}_2X)$; i.e. $d_i z = 0$ for all $i = 0, \dots, n$. It follows that, for all $i = 0, \dots, n$:

$$\sum_{d_i x = y} a_x = 0 \quad \text{for any} \quad y \neq * \,. \tag{4.6}$$

(In the computations, we will use 6.1. of [13]).

On the lower path z will go to $\frac{1}{2}\Sigma a_x\langle \tau_1 d_0 x\ ,\ d_0\tau_1 x\rangle$.

On the upper path we have to calculate $\partial[\Sigma a_x \tau_1 x]$. To this end we have to choose some element $z' \in N(\hat{\Gamma}_0/\hat{\Gamma}_2 \otimes \mathbf{Q}_2)$ being mapped to $\Sigma a_x\tau_1(x)$. We may view $\hat{\Gamma}_0/\hat{\Gamma}_2 \otimes \mathbf{Q}_2$ as a Lie algebra in a canonical way since it is a nilpotent group of class 2 which is local at the set of primes different from 2 (compare [21]). Therefore, the equation $\Sigma a_x\tau_2 x$ makes sense ; clearly $\Sigma a_x\tau_2 x$ maps to $\Sigma a_x \tau_1 x$. Moreover, for $j > 0$, we have $d_j(\sum a_x\tau_2 x) = \sum a_x d_j \tau_2 x = \sum a_x \tau_2 d_{j+1} x$, which is zero by equation (4.6) and the fact that $\tau_2 * = 0$.

Hence $\partial[\sum a_x \tau_1 x]$ will be represented by $d_0(\sum a_x \tau_2 x)$. Now, $d_0(\sum a_x \tau_2 x) = \sum a_x d_0 \tau_2 x = \sum a_x (\tau_2 d_0 x)^{-1} \cdot (\tau_2 d_1 x)$, where the dot is multiplication in the group $\hat{\Gamma}_0/\hat{\Gamma}_2 \otimes \mathbf{Q}_2$. We can express the multiplication by the Baker-Campbell-Hausdorff formula :

$$d_0(\sum a_x \tau_2 x) = \sum a_x (-\tau_2 d_0 x + \tau_2 d_1 x - \frac{1}{2}\langle \tau_2 d_0 x, \tau_2 d_1 x\rangle) .$$

We note several facts :

(i) the sums $\sum a_x \tau_2 d_0 x$, $\sum a_x \tau_2 d_1 x$ vanish because of (4.6).

(ii) the commutator $\langle \tau_2 d_0 x, \tau_2 d_1 x\rangle$ only depends on the classes of the corresponding elements in $\hat{\Gamma}_0/\hat{\Gamma}_1$, hence :

$$\langle \tau_2 d_0 x \ , \ \tau_2 d_1 x\rangle = \langle \tau_1 d_0 x \ , \ \tau_1 d_1 x\rangle .$$

(iii) $\tau_1 d_1 x = \tau_1 d_0 x + d_0 \tau_1 x$.

Finally, we get :

$$d_0(\sum a_x \tau_2 x) = -\frac{1}{2}\sum a_x \langle \tau_1 d_0 x \ , \ d_0 \tau_1 x\rangle \quad \blacksquare$$

Proof of theorem 4.4: We first need some notations. Let A be a chain complex over $\mathbf{Q}_2$. Then we denote by sym $(A \otimes A)$ the symmetric tensors in $A \otimes A$, by ssym $(A \otimes A)$ the skew symmetric ones. We have canonical retractions :

$$\rho : A \otimes A \to \mathrm{sym}(A \otimes A), a \otimes b \to \frac{1}{2}(a \otimes b + (-1)^{|a||b|} b \otimes a) ,$$

similarly $\rho' : A \otimes A \to \mathrm{ssym}(A \otimes A)$.

Furthermore, we note that up to homotopy a "diagonal" Δ factors as in the diagram :

$$\begin{tikzcd} N\overline{\mathbf{Q}}_2 X \arrow[r, "N\Delta"] \arrow[dr] & N\overline{\mathbf{Q}}_2 X \otimes N\overline{\mathbf{Q}}_2 X \\ & \mathrm{sym}(N\overline{\mathbf{Q}}_2 X \otimes N\overline{\mathbf{Q}}_2 X) \arrow[u] \end{tikzcd}$$

This follows from the fact that the interchange map fixes the elements of image $(N\Delta)_*$.

For any Lie algebra X we denote the descending central series by $\Gamma_0 X := X$, $\Gamma_1 X := [X, X], \ldots$. If $L \in \underline{\mathcal{L}}$, we denote by $\Gamma_i L/\Gamma_j L$, $j > i$, the induced chain complex.

Let now $L = (\mathbf{L}(V), \partial_L) \xrightarrow{\sim} N\mathrm{Laz}\, GX$ be as in the statement. Let

$$\partial_1' : H_i(\Gamma_0 L/\Gamma_1 L \otimes \mathbf{Q}_2) \to H_{i-1}(\Gamma_1 L/\Gamma_2 L \otimes \mathbf{Q}_2)$$

be the boundary homomorphism defined by the exact sequence

$$0 \to \Gamma_1 L/\Gamma_2 L \otimes \mathbf{Q}_2 \to \Gamma_0 L/\Gamma_2 L \otimes \mathbf{Q}_2 \to \Gamma_0 L/\Gamma_1 L \otimes \mathbf{Q}_2 \to 0\,.$$

Note that ∂_1' is induced by ∂_1, the quadratic part of the differential ∂_L on $\mathbf{L}(V)$.

Clearly, $abL \otimes \mathbf{Q}_2 \cong \Gamma_0 L/\Gamma_1 L \otimes \mathbf{Q}_2$ and

$$\mathrm{ssym}(abL \otimes abL \otimes \mathbf{Q}_2) \cong \Gamma_1 L/\Gamma_2 L \otimes \mathbf{Q}_2\,.$$

Next, we note :

1) $N(\hat\Gamma_i/\hat\Gamma_{i+1}) \cong N\hat\Gamma_i/N\hat\Gamma_{i+1}$,

2) $\Gamma_i(N\mathrm{Laz}GX) \subset N\hat\Gamma_i$.

1) is obvious ; one proves 2) by induction on i :
It is trivial for $i=0$. From the commutator map $\hat\Gamma_0 \otimes \hat\Gamma_i \to \hat\Gamma_{i+1}$, we derive $N(\hat\Gamma_0) \otimes N(\hat\Gamma_i) \to N(\hat\Gamma_0 \otimes \hat\Gamma_i) \to N(\hat\Gamma_{i+1})$, and hence $\Gamma_{i+1}(N\hat\Gamma_0) \subset N(\hat\Gamma_{i+1})$. (Note the different meanings of "$\otimes$" here !).

Recall that $L \to N\hat\Gamma_0 = N\mathrm{Laz}\ GX$ gives rise to map $abL \to N(\hat\Gamma_0/\hat\Gamma_1)$, which is identical with $\Gamma_0 L/\Gamma_1 L \to N\hat\Gamma_0/N\hat\Gamma_1$.

Thus, the left side of the following diagram commutes. Note that $\tilde\Delta_L : H(abL) \to H(abL \otimes abL)$ is obtained from Δ_L by composing with the appropriate suspensions.

$$\begin{tikzcd}
H_{i-1}(abL \otimes \mathbf{Q}_2) \arrow[r] \arrow[d, "\partial_1'"] \arrow[ddd, bend right=60, "\tilde\Delta_L"'] & H_{i-1}(N(\hat\Gamma_0/\hat\Gamma_1) \otimes \mathbf{Q}_2) \arrow[d, "\partial"] & H_i(N\overline{\mathbf{Q}}_2X) \arrow[l, "(-\tilde\tau)_*"'] \arrow[dd, "(N\Delta)_*"] \\
H_{i-2}(\Gamma_1 L/\Gamma_2 L \otimes \mathbf{Q}_2) \arrow[d, equal] & H_{i-2}(N(\hat\Gamma_1/\hat\Gamma_2) \otimes \mathbf{Q}_2) \arrow[dd, dashed, bend right=20] & \\
H_{i-2}(\mathrm{ssym}(abL \otimes abL \otimes \mathbf{Q}_2)) \arrow[r] \arrow[d, shift left, hook] & H_{i-2}(\mathrm{ssym}(N(\hat\Gamma_0/\hat\Gamma_1) \otimes N(\hat\Gamma_0/\hat\Gamma_1) \otimes \mathbf{Q}_2) \arrow[u, "(-,-)_*"'] \arrow[d, shift left, hook] & H_i(\mathrm{sym}(N\overline{\mathbf{Q}}_2X \otimes N\overline{\mathbf{Q}}_2X)) \arrow[l, "S_*"'] \arrow[d, hook] \\
H_{i-2}(abL \otimes abL \otimes \mathbf{Q}_2) \arrow[r] \arrow[u, shift left, "\rho_*'"] & H_{i-2}(N(\hat\Gamma_0/\hat\Gamma_1) \otimes N(\hat\Gamma_0/\hat\Gamma_1) \otimes \mathbf{Q}_2) \arrow[u, shift left, "\rho_*'"] \arrow[uu, bend left=60, "(-,-)_*"] & H_i(N\overline{\mathbf{Q}}_2X \otimes N\overline{\mathbf{Q}}_2X) \arrow[l, "S_*"']
\end{tikzcd}$$

Let $\tilde\tau : N\overline{\mathbf{Q}}_2X \to N(\hat\Gamma_0/\hat\Gamma_1) \otimes \mathbf{Q}_2$ be induced by τ_1 (see the conventions before lemma 4.5). Then S is given by the formula $S(a \otimes b) = (-1)^{|a|}\tilde\tau(a) \otimes \tilde\tau(b)$ by [**13**, proposition 6.4].

The right half also commutes by lemma 4.5. Hence, taking the appropriate suspensions proves the theorem. ∎

5. Cohomology

Definition 5.1. Let A be a R_k-module (in the ungraded sense) and $L \in \underline{\mathcal{L}}_s$; then we set :

$$H^{r+k}(L; A) := H^{r+k}(\sigma\overline{\overline{ab}}L; A)\,.$$

Remark 5.2. If M is a cofibrant model of L, then :

$$H^{r+k}(L;A) \cong H^{r+k}(\sigma abM;A)\,.$$

Theorem 5.3. *a) The functors $H^{r+k}(-;A)\circ\lambda$ and $H^{r+k}(-;A)$ from $\underline{S}_r$ to $\underline{Ab}$ are isomorphic.*

b) Let A be R_k or a quotient ring of R_k. Then the cup product structure on $H^{\leq r+k}(-;A)$ corresponds to the cup product structure on $H^{\leq r+k}(-;A)\circ\lambda$ derived from 4.1.

Proof: This follows from the results of sections 3 and 4. ■

We would like to mention a different proof of the first part of 5.3. This gives us the occasion to introduce a path object for abelian fibrant objects of $\underline{\mathcal{L}}_s$.

Definition 5.4. Let $\Gamma(t,dt)$ be the free commutative algebra with divided powers, with $|t|=0$ and $|dt|=-1$. Let $L\in\underline{\mathcal{L}}_s$, then we set :

$$(L^I)_p := (\Gamma(t,dt)\otimes L)_p \quad \text{for} \quad p>s,$$
$$(L^I)_s := Z_s(\Gamma(t,dt)\otimes L)\,.$$

Then L^I is a differential Lie algebra with Lie bracket given by

$$[a\otimes x, b\otimes y] := (-1)^{|x||b|}ab\otimes[x,y]\,.$$

Direct computations give :

Lemma 5.5. *(i) The canonical inclusion $L\to L^I$ is a homology isomorphism (over $\mathbf{Z}$).*

(ii) If L is abelian, then the evaluation maps $L^I\to L$ given by $t=0$, $t=1$ are Lie algebra morphisms.

(iii) If L is abelian fibrant, then L^I is a path object for L, i.e. there is a commutative diagram :

$$\begin{tikzcd} L \arrow[r, "\Delta"] \arrow[dr, "\sim"'] & L\times L \\ & L^I \arrow[u, "{(t=0,t=1)}"'] \end{tikzcd}$$

Lemma 5.6. *Let A be a R_k-module. Let $A^{(s+k)}$ be the graded module which is A in degree $(s+k)$ and zero in other degrees ; we consider it as an abelian differential Lie algebra. Let $X\in\underline{S}_r$ and L_X a cofibrant model of $\lambda(X)$; then there is an isomorphism $[X,\mu(A^{(s+k)})]_{\underline{S}_r}\cong H^{s+k}(abL_X;A)$, where μ : $Ho\text{-}\underline{\mathcal{L}}_s\to Ho\text{-}\underline{S}_r$ is the equivalence of categories given in* [**3**].

Proof: Since $A^{(s+k)}$ is abelian, the addition $A^{(s+k)}\times A^{(s+k)}\to A^{(s+k)}$ is a morphism in $\underline{\mathcal{L}}_s$. This defines a multiplication on $\mu(A^{(s+k)})$ such that we have the following sequence of isomorphisms :

$$[X, \mu(A^{(s+k)})]_{\underline{S}_r} \cong [\overline{\lambda}(X), A^{(s+k)}]_{\underline{\mathcal{L}}_s} \cong [L_X, A^{(s+k)}]_{\underline{\mathcal{L}}_s} \underset{(*)}{\cong} [abL_X, A^{(s+k)}]_{\underline{\mathcal{L}}_s}$$
$$\underset{(**)}{\cong} [abL_X, A^{(s+k)}]_{\underline{Ch}_s} \cong H^{s+k}(abL_X; A) .$$

Part $(*)$ is due to the commutativity of the Lie algebra $A^{(s+k)}$;

part $(**)$ follows from the fact that the path object L^I of $L := A^{(s+k)}$ in $\underline{\mathcal{L}}_s$ is also a path object of L in the category of chain complexes. ■

Corollary 5.7. *The space $\mu(A^{(s+k)})$ is equivalent in Ho-$\underline{S}_r$ to the Eilenberg-MacLane space $K(A, r+k)$ and $H^{r+k}(X; A) \cong H^{s+k}(abL_X; A)$.*

Proof: In lemma 5.6, let X be a sphere S^{r+l} and let $L \xrightarrow{\sim} \lambda(S^{r+l})$ be a cofibrant model. By section 3, we have :

$$H_{s+i}(abL; R_k) = \begin{cases} R_k & \text{if} \quad i = l \\ 0 & \text{else} \end{cases}, \qquad \text{provided} \qquad 0 \le i \le k \ .$$

It follows that :

$$H^{s+k}(abL; A) = \begin{cases} A & \text{if} \quad l = k \\ 0 & \text{else} . \end{cases}$$

Remark now that an Eilenberg-MacLane space $K(A, r+k)$ is fibrant in $\underline{S}_r$; hence $K(A, r+k)$ is a fibrant model of $\mu(A^{(s+k)})$. Therefore, by remark 5.8 below :

$$H^{r+k}(X; A) \cong [X, K(A, r+k)]_{\underline{S}} \cong [X, K(A, r+k)]_{\underline{S}_r} \cong H^{s+k}(abL_X; A)$$

for $X \in \underline{S}_r$. ■

Remark 5.8. Let $X, Y \in \underline{S}_r$ and let Y be tame (i.e. fibrant). Then Y is also fibrant in $\underline{S}$ (3.2 of [3]) and we have $[X, Y]_{\underline{S}} = [X, Y]_{\underline{S}_r}$, for we may take the same cylinder object in both theories.

6. Samelson and Whitehead products

We refer to [10] for the definitions of Whitehead and Samelson products (to be recalled below) and a study of their interrelations. The objective of this section is to prove the following :

Proposition 6.1. *Let $X \in \underline{S}_r$ be a Kan complex and let $L_X \xrightarrow{\sim} \lambda(X)$ be a weak equivalence with L_X cofibrant. Let $r+k \geq l+m+1$ and let A be a cyclic R_k-module. Then, under the isomorphisms of theorem* 1.3, *the Whitehead product*

$$\pi_{l+1}(X;A) \times \pi_{m+1}(X;A) \to \pi_{l+m+1}(X;A)$$

corresponds to the map induced by the Lie bracket

$$\pi_l(L_X;A) \times \pi_m(L_X;A) \to \pi_{l+m}(L_X;A)\,.$$

Remark 6.2. The Whitehead product is identified with the Samelson product by way of the commutative diagram

$$\begin{tikzcd}
\pi_{l+1}(X;A) \times \pi_{m+1}(X;A) \arrow[r, "{[\,,\,]_W}"] \arrow[d, "(-1)^l\partial\times\partial"] & \pi_{l+m+1}(X;A) \arrow[d, "\partial"] \\
\pi_l(GX;A) \times \pi_m(GX;A) \arrow[r, "{[\,,\,]}"] & \pi_{l+m}(GX;A)
\end{tikzcd}$$

where $\partial : \pi_{n+1}(X;A) \to \pi_n(GX;A)$ is the connecting homomorphism in the path fibration over X.

To prove the proposition it suffices to show the corresponding statement about Samelson products, i.e. the Samelson product

$$\pi_l(GX;A) \times \pi_m(GX;A) \to \pi_{l+m}(GX;A)$$

corresponds to the Lie bracket as stated above.

To begin, we observe the following things :

(a) There is no need to take any precautions with respect to the p-primary parts for $p = 2, 3$, because by calculating in the "tame range" the usual difficulties are ruled out.

(b) We have to distinguish the cases where A is R_k and where A is finite. In addition, if A is finite, the situation where l or m is equal to s requires special attention. But we shall give the proof only for A finite with $l, m > s$, because the necessary adaptations of the arguments to the other cases are natural.

Next, we shortly recall the definition of Samelson products with coefficients using the notations of section 1 and we will immediatly consider only those in the "tame range".

Let $A = \mathbf{Z}/q\mathbf{Z}$ be a R_k-module. Let G be a s-reduced simplicial group. Let $\alpha \in \pi_l(G;A)$, $\beta \in \pi_m(G;A)$ with $l+m \leq s+k$. Let α, β be represented by maps $f : M(A,l-1) \to G$, resp. $g : M(A,m-1) \to G$. Consider the map $< f,g >: M(A,l-1) \times M(A,m-1) \to G$, $(x,y) \to f(x)g(y)f(x)^{-1}g(y)^{-1}$. Then, there is a factorization of $< f,g >$ as follows

$$\begin{tikzcd}
& M(A,l-1) \times M(A,m-1) \arrow[r, "{<f,g>}"] \arrow[d] & G \\
M(A,l+m-1) \arrow[r, "e"] & M(A,l-1) \wedge M(A,m-1) \arrow[ur, dashed, "c"'] &
\end{tikzcd}$$

and a canonical arrow e (see [10]) which exists only in Ho-$\underline{S}$ (by abuse of language we denote it as above). The Samelson product $[\alpha, \beta]$ is then represented by ce.

If $G \in \Delta$-$\underline{\mathrm{Laz}}_s$, then G will be considered as simplicial group or simplicial Lie algebra according to the context.

Lemma 6.3. *Let $G \in \Delta$-$\underline{Laz}_s$ with central series (G_i) defining the Lazard structure. Let $P := M(A, l-1) \times M(A, m-1)$. Assume that for some i the map of groups of homotopy classes*

$$[P, G] \to [P, G/G_i]$$

induced by $\Phi : G \to G/G_i$ is bijective.

Then $< f, g >$ is homotopic to $[f, g]$ where $[f, g]$ is defined by the Lie structure of G via the formula $[f, g](x, y) = [f(x), g(y)]$.

Proof: It suffices to prove that $\Phi(< f, g >) =< \Phi f, \Phi g >$ is homotopic to $[\Phi f, \Phi g]$. Set $\tilde{f} := \Phi f$, $\tilde{g} := \Phi g$. Then by the formula of Baker-Campbell-Haussdorff we have $< \tilde{f}, \tilde{g} >= [\tilde{f}, \tilde{g}] + v$ where v is a finite sum of iterated brackets of length ≥ 3 in $\tilde{f}, \tilde{g}$ with some coefficients. By the inverse of the Baker-Campbell-Haussdorff formula ([8]) v can be interpreted as an element in Γ_3 $(\mathrm{mor}_{\underline{S}}(P, G/G_i))$, the third term of the lower central series of the group $\mathrm{mor}_{\underline{S}}(P, G/G_i)$. But the group $[P, G/G_i]$ is nilpotent of class ≤ 2, because P is a product of two suspensions. Hence v is homotopic to zero. ■

Proof of proposition 6.1:

(a) By section 1 we have weak equivalences $GX \to \mathrm{Laz}GX \leftarrow N^* L_X$, hence there is an isomorphism $\pi_{\leq s+k}(N^* L_X; A) \cong \pi_{\leq s+k}(GX; A)$ compatible with the Samelson products.

(b) We verify the conditions of lemma 6.3 for $\mathrm{Laz}\, GX$. We will rely heavily on § 8 of [3] and will therefore adopt the corresponding notations. Let $Y := GX$ and $\Gamma_i Y$ the lower central series subgroups of Y ; let $EY := \mathrm{Laz}GX$ and $E_i Y$ the central series subgroups of EY defining the Lazard structure. Then, by [3, § 8], for i large and a given prime number p there are isomorphisms

$$\pi_{r+t}(Y/\Gamma_i Y) \otimes \mathbf{Z}_{(p)} \to \pi_{r+t}(EY/E_i Y) \otimes \mathbf{Z}_{(p)}$$

for $t \leq 2p - 4$, where $\mathbf{Z}_{(p)}$ denotes $\mathbf{Z}$ localized at p. Moreover for i large $\pi_{r+t}(Y/\Gamma_i Y) \cong \pi_{r+t}(Y)$. Hence we also have

$$\pi_{r+t}(Y) \otimes \mathbf{Z}_{(p)} \cong \pi_{r+t}(EY/E_i Y) \otimes \mathbf{Z}_{(p)} \ .$$

Let now p be a prime occuring as factor of q (recall $A = \mathbf{Z}/q\mathbf{Z}$). Then we have $p > \frac{k+3}{2}$, $l + m \leq s + k$ hence $r + 2p - 4 \geq r + k$. It follows that the map $EY \to EY/E_i Y$ induces isomorphisms of $\pi_{r+t}(-) \otimes \mathbf{Z}_{(p)}$ for $r + t \leq r + k$. The dimension of P is less than $r+k$. Hence $[P, EY] \to [P, EY/E_iY]$ is bijective (because these groups are torsion groups whose orders involve only the prime factors of q).

(c) Let f, g represent elements $\alpha \in \pi_l(N^*L_X; A)$, $\beta \in \pi_m(N^*L_X; A)$. Let $a \in \pi_l(L_X; A)$, $b \in \pi_m(L_X; A)$ correspond to α, β via the equivalence $L_X \to \to NN^*L_X$. Then it suffices to show that $[a, b] \in \pi_{m+l}(L_X; A)$ corresponds to $[\alpha, \beta]$ which is represented by $[f, g]$.

Hence we have to consider the following diagram :

$$\begin{tikzcd}
M(A,l-1) \times M(A,m-1) \arrow[r, "f\times g"] \arrow[d] & N^*L_X \times N^*L_X \arrow[r, "{[\,,\,]}"] & N^*L_X \\
M(A,l-1) \wedge M(A,m-1) \arrow[rru, dashed, "c"'] & & \\
M(A,l+m-1) \arrow[u, "e"'] & &
\end{tikzcd}$$

According to the procedure in section 1, we form the following diagram:

$$\begin{tikzcd}
N\mathbf{Z}(M(A,l-1) \times M(A,m-1)) \arrow[r] \arrow[d] & NN^*L_X \times NN^*L_X \arrow[r, "{[\,,\,]}"] & NN^*L_X \\
N\mathbf{Z}\,(M(A,l-1) \wedge M(A,m-1)) \arrow[rru, dashed] \arrow[rr, dashed, bend right, "\overline{c}"'] & L_X \times L_X \arrow[u] \arrow[r, "{[\,,\,]}"] & L_X \arrow[u] \\
N\mathbf{Z}M(A,l+m-1) \arrow[u, "\overline{e}"'] & &
\end{tikzcd}$$

To obtain $[a, b]$ we apply $\overline{c}\,\overline{e}$ to a canonical class in $H_{l+m}(N\mathbf{Z}M(A, l+m-1); A)$. ■

7. Application

Having identified enough algebraic invariants of spaces X in the model L_X of $\lambda(X)$ we can now do a little computation.

Proposition 7.1. *Let X be 3-connected, $\pi_4(X) \cong \mathbf{Z}[\frac{1}{2}]$ and let $y \in \pi_4(X)$ be a $\mathbf{Z}[\frac{1}{2}]$-module generator. Note that $H^4(X; R) \cong Hom(\pi_4(X), R)$ for any subring $R \subset \mathbf{Q}$, with $\frac{1}{2} \in R$. Let α_R be the corresponding generator for $H^4(X; R)$ with $\alpha_R(y) = 1$. Assume $\alpha_{\mathbf{Q}} \cup \alpha_{\mathbf{Q}} = 0$, but that for all R as above, $R \neq \mathbf{Q}$, the map $R \to H^8(X; R)$, $r \mapsto r(\alpha_R \cup \alpha_R)$, is injective. Then we have :*

(i) $[y,y]\otimes 1 \neq 0$ *in* $\pi_7(X)\otimes \mathbf{Q}$,

(ii) If $R\neq \mathbf{Q}$, $\frac{1}{2}$, $\frac{1}{3}\in R$, *then any homomorphism* $\pi_7(X)\to R$ *vanishes on* $[y,y]$. *In particular,* $[y,y]\otimes 1$ *is not in any direct summand* R *of* $\pi_7(X)\otimes R$.

Example 7.2. The proposition might be interesting because it applies to the classifying space $B\Gamma_3$ of C^p-foliations of codimension 3, $p\neq 4$. This is not the place to recall what is known about the structure of $\pi_* B\Gamma_3$. (E.g. it is proved by S. Hurder [**7**] that there is a surjective homomorphism $\pi_7 B\Gamma_3 \to \mathbf{R}^3$). But this result seems to be new.

We have to show that $B\Gamma_3$ satisfies the assumptions of the proposition. The condition on π_4 follows from theorem 2 of [**19**]. The condition on $\alpha_{\mathbf{Q}}\cup\alpha_{\mathbf{Q}}$ is Bott vanishing ; the condition on $\alpha_R\cup\alpha_R$ can be proved by following the method in the appendix of [**1**], see also [**20**]. Thus theorem D is established.

Remark 7.3. This application has been proposed to us by E. Vogt. He arrived at the result by different methods which are, however, similar in spirit. In [**15**], there will appear a proof within a different algebraization of tame homotopy theory.

Proof of proposition 7.1:

We work in the category $\underline{S}_3$ and may replace X by a taming (i.e. a fibrant model) of X.

We first construct a naive cofibrant model M of $\lambda(X)$ according to section 2. In degrees ≤ 7 it looks as follows :

7	w_j, $[\tilde{y}, ?]$
6	z_i, $[\tilde{y},\tilde{y}]$
5	?
4	?
3	$\tilde{y}$

We need one generator $\tilde{y}$ in degree 3 corresponding to y ; the differential is 0 in degree 3 and 4 and on decomposables of degree 7. The elements z_i and w_j denote generators. We set :

$$\partial w_j = r_j[\tilde{y},\tilde{y}] + \sum_i s^i_j z_i \ ,$$

with r_j, $s^i_j\in\mathbf{Z}$.

Assume now that $f:\pi_7(X)\to R$ is a homomorphism. We may look at f as a homomorphism $\pi_6(M\otimes R)\to R$ or a homomorphism $Z_6(M\otimes R)\to R$ which vanishes on boundaries, i.e. image $(\partial\otimes 1_R)$.

Clearly, $Z_6(M\otimes R)$ is a direct summand in $M\otimes R$, so we may finally interpret f as a homomorphism $M_6\otimes R\to R$ which vanishes on boundaries. Hence, setting $f([\tilde{y},\tilde{y}])=u$, $f(z_i)=u_i$, we have :

$$r_j u + \sum s^i_j u_i = 0 \quad \text{for all} \quad j\,. \tag{7.4}$$

We identified the cohomology of X as the cohomology of the module of indecomposables of M, the cup-product as given by the quadratic part of the differential. Thus we see :

$$(\alpha_R \cup \alpha_R)(w_j) = \pm 2r_j \ .$$

But, if (7.4) holds, $\frac{1}{2}u(\alpha_R \cup \alpha_R)$ is a coboundary, namely $\pm\delta(\psi)$ with $\psi(z_i) = u_i$. Hence equation (7.4) has no solution with $u \in R$, $u \neq 0$.

If $[\tilde{y}, \tilde{y}]$ were a boundary in $M \otimes \mathbf{Q}$, then we had $\alpha_{\mathbf{Q}} \cup \alpha_{\mathbf{Q}} \neq 0$. ■

References

1. BOTT, R. AND HEITSCH, J., A remark on the integral cohomology of $B\Gamma_q$, *Topology* **11** (1972), 141–146.
2. CENKL, B. AND PORTER, R., Modèles pour la théorie de l'homotopie modérée, *C.R.Acad.Sc.Paris, Série A* **290** (1980), 613–615.
3. DWYER, W.G., Tame homotopy theory, *Topology* **18** (1979), 321–338.
4. DWYER, W.G., "*The tame homotopy groups of a suspension*," Geometric Applications in Homotopy Theory II, Proceedings, Evanston 1977, Lecture Notes in Mathematics **658**, Springer-Verlag, 1978, pp. 165–168.
5. FÉLIX, Y. AND LEMAIRE, J.M., On the mapping theorem for Lusternik-Schnirelmann category, *Topology* **24** (1985), 41–43. Corrections: *Topology* **27**, (1988), p. 177.
6. FÉLIX, Y. AND LEMAIRE, J.M., On the mapping theorem for Lusternik-Schnirelmann category II, *Canad. J. Math.* **40** (1988), 1389–1398.
7. HURDER, S., The classifying space of smooth foliations, *Illinois J. Math* **29** (1985), 108–133.
8. LAZARD, M., Sur les groupes nilpotents et les anneaux de Lie, *Ann. Ec. Norm. Sup.* **71** (1954), 101–190.
9. MAY, J.P., "*Simplicial objects in algebraic topology*," Van Nostrand Mathematical Studies **11**, D. Van Nostrand Company, INC, 1967.
10. NEISENDORFER, J., Primary homotopy theory, *Memoirs Amer. Math. Soc.* **232** (1980).
11. NEISENDORFER, J., Lie algebras, coalgebras and rational homotopy theory for nilpotent spaces, *Pacific J. of Math.* **74** (1978), 429–460.
12. QUILLEN, D.G., "*Homotopical Algebra*," Lecture Notes in Mathematics **43**, Springer Verlag, 1967.
13. QUILLEN, D.G., Rational homotopy theory, *Ann. of Math.* **90** (1969), 205–295.
14. SCHEERER, H., "*Report on tame homotopy theory via differential forms*," Algebraic Topology, Rational Homotopy, Proceedings, Louvain-la-Neuve

1986, Lecture Notes in Mathematics **1318**, Springer Verlag, 1988, pp. 192–207.

15. SCHEERER, H., SCHUCH, K. AND VOGT, E., Tame homotopy theory via currents, Freie Universitaet Berlin, Fachbereich Mathematik. Preprint nº A 91–20.
16. SCHEERER, H. AND TANRÉ, D., Exploration de l'homotopie modérée de W.G.Dwyer, I, *C.R.Acad.Sci.Paris, Série I* **307** (1988), 783–785.
17. SCHEERER, H. AND TANRÉ, D., Exploration de l'homotopie modérée de W.G. Dwyer, II, *C.R.Acad.Sci.Paris, Série I* **307** (1988), 799–801.
18. SCHEERER, H. AND TANRÉ, D., Exploration de l'homotopie modérée de W.G.Dwyer, III, *C.R.Acad.Sci.Paris, Série I* **308** (1989), 181–183.
19. THURSTON, W., Foliations and groups of diffeomorphisms, *Bull. Amer. Math. Soc.* **80** (1974), 304–307.
20. VOGT, E., "*Bott vanishing and torsion in $H_*(B\Gamma)$*," Lecture Notes **9**, Centre de Recerca Matemàtica, Barcelona, 1984.
21. WARFIELD, R.B. JR., "*Nilpotent groups*," Lecture Notes in Mathematics **513**, Springer Verlag, 1976.
22. WHITEHEAD, G.W., "*Elements of homotopy theory*," Graduate Texts in Mathematics **61**, Springer Verlag, 1978.

Hans Scheerer: Mathematisches Institut
Arnimallee 2-6
D-1000 Berlin 33
GERMANY

Daniel Tanré: Université des Sciences et Techniques
de Lille Flandres Artois
U.F.R. de Mathématiques
59655 Villeneuve d'Ascq Cedex
FRANCE

Primera versió rebuda el 7 de Juliol de 1989,
darrera versió rebuda el 2 de Maig de 1990