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EMBEDDING TORSIONLESS MODULES IN PROJECTIVES

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Abstract

In this paper we study a condition right FGTF on a ring R, namely when all finitely generated torsionless right R-modules embed in a free module. We show that for a von Neuman regular (VNR) ring R the condition is equivalent to every matrix ring R_n is a Baer ring; and this is right-left symmetric. Furthermore, for any Utumi VNR, this can be strengthened: R is FGTF iff R is self-injective.

Introduction

When all injective modules over a ring R embed in a free R-module, the ring R must be quasi-frobenius, (= QF) and conversely. In this case every right or left R-module embeds in a projective R-module, and furthermore, the one-sided condition implies the two-sided condition ([**F–W**]). The two-sided condition that all cyclic R-modules embed in projectives implies R is QF, but here the one-sided condition is not sufficient.

Related to these rings are right IF rings initiated by Jain [**J**], Colby [**Co**], Damiano [**D**], and Würfel [**W**]. A ring R is **right IF** if every (injective) right module embeds in a flat module, i.e. if every injective right R-module is flat. These rings were characterized by Colby and Würfel in the one-sided case by the property that all finitely presented right R-modules embed in projective modules. Here, again, right does not imply the left condition ([**Co**]). However, two-sided IF are coherent (l.c.). A number of characterizations of IF rings are summarized in Section 4.

In this paper we study a condition that is considerably weaker, namely **right FGTF**: all finitely generated torsionless right R-modules embed in a projective equivalently in a free right R-module. We let **right FPTF** denote the same condition for finitely presented torsionless right R-modules.

We now introduce a concept that is germane to FGTF: A ring R is a **right ★-ring** (star ring) provided that every finitely generated right R-module has finitely generated dual.

Easy exercises establish that any left Noetherian ring is right ★, and that any right ★-ring is right FGTF. We establish here that over any von Neumann regular (= VNR) ring R that (right or left) FGTF and ★ rings are equivalent concepts. This relates to a theorem of Kobayashi that states that any right self-injective VNR is a right ★-ring. Furthermore, we establish the converse to Kobayashi's theorem for Utumi VNR's. This generalizes his own converse theorem for commutative VNR's to VNR's with (2-sided) self-injective maximal right quotient ring $Q^r_{max}(R)$, e.g. to Abelian VNR's. As a corollary we show that over an Abelian VNR ring R, that every matrix ring R_n is Baer iff R is self-injective. The proof requires two theorems of Utumi: (1) on the left self-injectivity of the maximal right quotient ring $Q^r_{max}(R)$ of a non-singular ring R; (2) on self-injectivity of continuous $n \times n$ matrix rings, $n \geq 2$.

We also show by example that the converse to Kobayashi's theorem fails for all non-Utumi VNR's.

In Section 5, we relate IF, FGTF and ★ rings.

1. Preliminaries

For a subset X of R, we let $X^\perp$ denote the right, ${}^\perp X$ the left, annihilator in R. Thus, a right ideal I is an annihilator iff $I = X^\perp$ for $X \subseteq R$. **Annulet** is a variant term for annihilator. Then, I is said to be **finitely annihilated (= FA)** provided that $I = X^\perp$ for a finite set X. If a ring satisfies the acc on annihilator left ideals (= R is a $\perp$acc ring), then every right annihilator of R is FA by a theorem of **[F1]**.

A right R-module is **torsionless** if M has the equivalent properties: (T1) M embeds in a direct product of copies of R; (T2) M canonically embeds in its bidual module M^{**}, (T3) For each nonzero $x \in M$ there exists $f \in M^* = \mathrm{Hom}_R(M, R)$ so that $f(x) \neq 0$.

Moreover, for any ring A, a cyclic right module A/I is:

(1) torsionless $\Leftrightarrow I = X^\perp$ for a subset X of A (= I is a right annihilator)

(2) embeddable in a free module $\Leftrightarrow I = X^\perp$ for $|X| < \infty$ (= I is FA).

We also record the fact mentioned in the abstract:

1.0. Proposition. *A right ★-ring is right FGTF.*

Proof: Trivial. (See, e.g. **[F2]**.) ■

1.1. Theorem. *A ring R is right FGTF iff each annihilator right ideal of the $n \times n$ matrix ring R_n is FA for every n.*

Proof: The proof of this Folkloric theorem is by "Morita theory", i.e. using the fact that under the Morita equivalence

$$\Theta \begin{cases} \mathrm{mod} - R \sim\!\!\rightarrow \mathrm{mod} - R_n \\ \qquad X \longrightarrow X^n \end{cases}$$

that n-generated right R-modules map onto cyclic right R_n-modules. Furthermore, a submodule X of free R-module, say $X \subseteq R^n$ maps onto a principal right ideal of R_n under Θ. Similarly, Θ preserves torsionless modules. ■

1.2. Corollary. *If a ring R embeds in left Noetherian ring A, then R is right FGTF.*

Proof: Then R_n embeds in a left Noetherian ring A_n, for every n, and thus satisfies the acc on annihilator left ideals. By [**F1**], then each annihilator right ideal is FA, so R is right FGTF. ■

2. Semi-Continuous and Baer Rings

A ring R is said to be **right semi-continuous** ([**F3**]) if R has the property:

right SC: every right ideal is an essential submodule of one generated by an idempotent.

In [**F3**] we indicated that this is equivalent to:

right CS: every complement right ideal is generated by an idempotent.

A ring R is **right continuous** provided that R is right SC and every right ideal I which is isomorphic to a right ideal generated by an idempotent is itself a right ideal generated by an idempotent.

A ring R is **VNR** (= von Neumann regular) iff each finitely generated right ideal is generated by an idempotent. (This is right left symmetric.) Obviously a VNR is right continuous iff R is right SC.

A VNR ring R is **Abelian** or **strongly regular** iff every idempotent in R is central. Moreover, an idempotent $e \in R$ is **Abelian** if eRe is an Abelian VNR.

2.1. Theorem. *If R is a right continuous VNR ring, then R is right self-injective under any of the following assumptions:*

(1) *R has no nonzero Abelian idempotents.*
(2) *$M_n(R)$ is right continuous for any $n > 1$ (Utumi* [U_1]*).*
(3) *R is directly indecomposable qua ring (e.g., R prime).*
(4) *Every primitive factor ring is Artinian.*

See [**G2**] for proofs.

A ring R is a **Baer ring** if every right annihilator is generated by an idempotent. (This is left-right symmetric.) By the Johnson-Utumi theory of maximal quotient rings (see [**F4**, chap. 19.]) every annihilator right ideal of a right non-singular ring R is a complement right ideal and, thus, is a direct summand when R is right self-injective, a fact that we record:

Theorem of R.E. Johnson and Y. Utumi. *The maximal right quotient ring $Q^r_{max}(R)$ of a right nonsingular ring R is a right self-injective VNR ring, hence a right continuous Baer ring.*

2.2. Theorem. *A VNR ring R is right FGTF iff every R_n is a Baer ring for every $n \geq 1$.*

Proof: This follows from Theorem 1.1, since every finitely generated one-sided ideal of a VNR ring is generated by an idempotent, and hence then so is its annihilator. ■

2.3. Theorem. *For a VNR ring, the f.a.e.:*

(1) *R is right FGTF.*
(2) *R is left FGTF.*
(3) *R is a right ★-ring.*
(4) *R is a left ★-ring.*
(5) *R_n is a Baer ring $\forall n$.*

Proof: Since "Baer ring" is left-right symmetric, (1) ⇔ (2) ⇔ (5) by Theorem 2.2. Moreover, (3) ⇒ (1) by Theorem 1.0, and (1) ⇒ (4) by Theorem 2.2 and Camillo's Theorem [C] cited below. (See Remark 2.5.A.) The remaining implications follow by symmetry. ■

An R-module is **coherent** if every finitely-generated submodule is finitely presented.

2.4 Theorem ([C]). *For any ring R the f.a.e.:*

(1) *R is right Π-coherent, i.e. any product of copies of R is a coherent right R-module.*
(2) *R is a left ★-ring.*
(3) *Annihilator right ideals of $M_n(R)$ are finitely generated, $\forall n \geq 1$.*

2.5A Remark. Any right coherent right FGTF ring R is right Π-coherent, hence a left ★-ring, whence left FGTF.
Thus by Camillo's theorem:

2.5B Corollary. *A left coherent right Π-coherent ring is left Π-coherent.*

Proof: R is left ★, hence left FGTF, so 2.5A applies. ■

2.5C Corollary. *A left coherent left ★-ring is left Π-coherent.*

2.5D Remark. We can easily deduce Theorem 1.0 from Theorem 2.4, using Theorem 1.1, since (3) of Theorem 2.4 implies the condition of Theorem 1.1.

Proof: This follows from Theorem 1.0, and the fact that any finitely generated submodule of a free module is coherent over a coherent ring. ■

3. Utumi Rings

As stated *supra* the Johnson-Utumi theorem in a right non-singular ring R, every right annulet is a complement right ideal. Utumi [U_2] considered the converse condition:

(right U) Every right complement is a right annulet.

We say R is **right Utumi** if R satisfies this condition, and R is **Utumi** if R is right & left Utumi. A **non-singular** ring R is a right and left non-singular ring.

Utumi's theorem. ([U_2]) . *In a non-singular ring R, the f.a.e.:*

(1) *The maximal right quotient ring $Q^r_{max}(R)$ is left self-injective.*
(2) $Q^r_{max}(R) = Q^\ell_{max}(R)$
(3) *R is Utumi.*

The next proposition is almost obvious.

3.1 Proposition. *A Baer VNR right Utumi ring R is right continuous.*

Proof: R is right SC (= CS), since every complement right ideal is generated by an idempotent. ■

3.2 Theorem. *For an Utumi VNR the f.a.e.:*

(1) *R is right FGTF.*
(2) *R is left FGTF.*
(3) *R is right self-injective.*
(4) *R is left self-injective.*

Proof: It follows from Proposition 3.1 and Theorem 2.2 that $M_n(R)$ is right continuous $\forall n \geq 1$ when R is left or right FGTF, hence from Theorem 2.1 that $M_n(R)$ is (right and left) self-injective. Thus (1) $\Rightarrow$ (3), and (2) $\Rightarrow$ (3).

If R is right self-injective then $M_n(R)$ is right continuous, since $M_n(R)$ is right self-injective. Thus, $M_n(R)$ is a Baer ring $\forall n \geq 1$. Then R is right and left FGTF by Theorem 1.1, so (3) $\Rightarrow$ (1). Then R is left self-injective by (2) $\Rightarrow$ (3), hence therefore (3) $\Rightarrow$ (4). Using Theorem 2.2, this completes the Proof. ■

3.3 Corollary. *An Abelian VNR ring R is right FGTF iff R is self-injective.*

3.4. Example. Any right but not left injective ring Q, e.g. the maximal right quotient ring $Q^r_{max}(R)$ of a VNR ring R that is not Utumi, is a left ★-ring that is not left self-injective.

It follows from Kobayashi's theorem that R is right ★-ring and from Theorem 2.3 that R is a left ★-ring. ■

4. If Rings

A number of these results are related to theorems of Gómez-Pardo and González, Goodearl, and Levy on when do finitely presented cyclic [**GP–G**], finitely generated nonsingular ([**G1**]), or finitely generated torsion-free modules respectively embed in free modules (see [**F2**]).

Moreover:

4.1. Theorem. (Colby [**Co**], Würfel[**W**])
All finitely presented right R-modules embed in a free module iff R is a right IF ring.

4.2. Theorem. ([**J**], [**Co**], [**W**], and [**GP–G**].)

The f.a.e.c.'s on a ring R.

(IF1) *R is right and left IF.*
(IF2) *R is (right & left) coherent and every finitely generated one-sided ideal is an annihilator.*
(IF3) *Annihilation defines a duality between finitely generated one-sided ideals.*
(IF4) *R is coherent and every flat module (either side) is FP-injective.*
(IF5) *The classes of flat left and $\aleph_0$-injective left R-modules are equal.*

A ring R is **right (F)GF** if every (finitely generated) right R-module embeds in a free right R-module. Every quasi-Frobenius (QF) ring is both right and left GF.

In this connection the following theorem is of interest.

4.3. Theorem. (Jain [**J**], Rutter [**R**]). *Any right FGF ring is right IF.*

Also of interest here are certain other theorems of Jain [**J**] which are summarized in [**F2**, p.35, Theorem 4.2 and Corollaries].

In [**F2**] we discussed various conditions that implied that right FGF rings are quasi-Frobenius (QF), including the assumption that R is left and right FGF. The general FGF problem, however, remains open. (However, see [**Me**].)

If F is a right cogenerator ring, then every right R-module is torsionless, hence R is then FGF iff FGTF. Whether R is necessarily QF is an open question, so the classification FGTF problem will remain open until the FGF problem for right cogenerator rings is settled. (Note, however, by a theorem of Onodera, any right and left cogenerator ring is self-injective and right self-injective right FGF rings are QF (see [**F2**] and [**F5**]).

5. If ★-Rings

An element $r \in R$ is **regular** if r is neither a right nor left zero divisor of R. Let R^* denote the set of zero divisors of R. A right R-module M is

torsion-free if:

$$xr = 0 \Rightarrow x = 0 \forall x \in M, r \in R^*.$$

A ring R is **right Ore** if and only if R has a classical right quotient ring

$$Q = Q^r_{Cl}(R) = \{a/b | a \in R, b \in R^*\}$$

where addition and multiplication in Q is canonical.

It is evident that over Q every right R-module is torsion-free.

Consider the property **right f.g.t.f.** = all finitely generated torsion-free right R-modules embed in a free module.

We now come to Levy's theorems.

Levy's first Theorem. *If R is a semi-prime right Ore ring then R is right f.g.t.f. iff R is a left Ore ring. In this case $Q^l_{Cl}(R) = Q^r_{Cl}(R)$ is semisimple.*

By a remark above, one sees that a ring $Q = Q^r_{Cl}(R)$ right FGF iff Q is right f.g.t.f. In this case, by [F−W] we have:

5.1. Theorem. ([F2]) *If R is left and right Ore, then R is right and left f.g.t.f. iff Q is QF.*

5.2. Corollary. ([F2]) *A commutative ring R is f.g.t.f. iff $Q_{Cl}(R)$ is QF.*

Since any torsionless module is torsion-free, one concludes that any commutative f.g.t.f. ring R, hence any R with $Q_{Cl}(R)$ QF, is right FGTF.

5.3. Remark. It follows from 5.2 that an integral domain R that is not coherent is an example of an FGTF ring that is not ★.

5.4. Theorem. *The f.a.e.c.'s on a ring R.*

(1) *R is a right and left FGTF and all finitely generated one-sided ideals are annulets.*

(2) *R is a right and left IF ★-ring.*

(3) *R is an IF and FGTF ring.*

Proof: (2) ⇒ (1) by Theorem 1.0 and Theorem 4.2, (IF2).

(1) ⇒ (2). All annihilator in any matrix ring R_n right or left ideals are FA by Theorem 1.1. Now let I be any right annulet. Now $L = {}^{\perp}I$ is FA, say $L = {}^{\perp}I_1$, for I_1 a finitely generated right ideal, which by hypothesis is an annulet. Thus, $L^{\perp} = I = I_1$ is finitely generated. By (IF3) of Theorem 4.2, R is an IF ring. By (IF2), R is coherent, hence R is a ★-ring by Theorem 2.4 and Remark 2.5A.

Obviously (2) ⇒ (3). Moreover (3) ⇒ (2) since IF implies R is coherent so Remark 2.5A applies. ■

6. Problems

1. Characterize the condition that all annihilator right ideals of $M_n(R)$ are finitely generated (for all $n \geq 1$) ideal-theoretically in R .

2. Same question for $acc\perp$ ($dcc\perp$) in $M_n(R)$.

In [**C**], Camillo raises the question:

3. (Question of [**C**]): If R is a (2-sided) Noetherian ring, is the polynomial ring $R[X]$ Π-coherent for any set X ?

Camillo verifies 3. for semiprime R and, in fact, for any R embeddable in an Artinian ring. (A semiprime Noetherian ring is Goldie and has Artinian classical ring of quotients.) However, as proved by Dean and Stafford [**D–S**], not every Noetherian ring can be embedded in an Artinian ring. Nevertheless, using a theorem of [**C–G**], Camillo [**C**] verifies 3. for any Noetherian ring R that is an algebra over a non-denumerable field.

4. Characterize right FGTF rings, equivalently, by Theorem 1.1, characterize when every annihilator right ideal of R_n, $n \geq 1$, is FA. (This characterization should be ideal - theoretically in R.)

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