We apply the Chebyshev coefficients $\lambda_f$ and $\lambda_b$, recently introduced by the authors, to obtain some results related to certain geometric properties of Banach spaces. We prove that a real normed space $E$ is an $L^1$-predual if and only if $\lambda_f(E) = 1/2$, and that if a (real or complex) normed space $E$ is a $P_1$ space, then $\lambda_b(E)$ equals $\lambda_b(K)$, where $K$ is the ground field of $E$.

In this note, $K$ will be the real or complex field, and $E$ a normed space over $K$; when we want to state a result only for the real case or the complex case, we will indicate it specifically. We will use the notations of [2], to which we refer for all concepts of the theory of normed spaces which may appear without defining them here.

If $S$ is a non empty subset of $E$, the number

$$r(S) = \inf_{y \in E} \sup_{x \in S} \|x - y\|$$

is called the Chebyshev radius of $S$, and $\delta(S)$ denotes the diameter of $S$.

**Definition.** We will call the *finite Chebyshev coefficient* of $E$ the real number

$$\lambda_f(E) = \sup \{r(S)/\delta(S) : S \subset E, S \text{ finite}, \delta(S) > 0\},$$

and the *bounded Chebyshev coefficient* of $E$ the real number

$$\lambda_b(E) = \sup \{r(S)/\delta(S) : S \subset E, 0 < \delta(S) < \infty\}.$$

It is easy to prove that, in general,

$$1/2 \leq \lambda_f(E) \leq \lambda_b(E) \leq 1.$$
Moreover, when $E$ is finite dimensional, we have $\lambda_f(E) = \lambda_b(E)$.

Specifically, the Chebyshev coefficients associated to the scalar fields are

$\lambda_f(\mathbb{R}) = \lambda_b(\mathbb{R}) = 1/2$

$\lambda_f(\mathbb{C}) = \lambda_b(\mathbb{C}) = 1/\sqrt{3}$.

Let us recall that a $\mathcal{P}_\alpha(K)$ space, where $\alpha$ is a real number greater than or equal to 1, is a Banach space $E$ for which any of following equivalent condition holds:

(i) Given two Banach spaces, $F$ and $G$, a linear isometry into, $\phi : F \to G$, and a bounded linear operator, $L : F \to E$, there exists a bounded linear operator $\hat{L} : G \to E$, which extends $L$, in the sense of $\hat{L} \circ \phi = L$, and such that $\|\hat{L}\| \leq \alpha \|L\|$ ($\alpha$-extension property).

(ii) Given a Banach space $F$, and a linear isometry, $\phi : E \to F$, there exists a projection, $P : F \to \phi(E)$, such that $\|P\| \leq \alpha$ ($\alpha$-projection property).

It is said that a Banach space $E$ is a $N_\alpha$ space, where $\alpha$ is a real number greater than or equal to 1, when there exists a collection $(E_\gamma)_{\gamma \in \Gamma}$ of finite dimensional subspaces of $E$, which is upwards directed, their union is dense in $E$ and every one of them is a $\mathcal{P}_\alpha(K)$ space. Note that a Banach space is an $L^1$-predual space if and only if it is a $N_\alpha$ space for every $\alpha > 1$ ([2, theorem 2, pg. 232]).

**Theorem 1.** If the Banach space $E$ is an $L^1$-predual, then

$\lambda_f(E) = \lambda_f(K)$.

**Proof:** We fix an $\alpha > 1$. Given that $E$ is an $L^1$-predual, it is a $N_\alpha$ space, and, so, there exists a collection $(E_\gamma)_{\gamma \in \Gamma}$ of subspaces of $E$ according to the definition above. We put $F = \bigcup_{\gamma \in \Gamma} E_\gamma$, which, because it is dense in $E$, satisfies

$\lambda_f(E) = \lambda_f(F)$.

Let $S$ be a finite subset of $F$ with more than one point. There exists $\gamma \in \Gamma$ such that $S \subseteq E_\gamma$, and, if we indicate with subindices the Chebyshev radii in subspaces of $E$, we have

$r(S) = r_{\gamma}(S) \leq r_{E_\gamma}(S) \leq \delta(S).\lambda_f(E_\gamma) \leq \delta(S).\alpha.\lambda_f(K)$,

where the last inequality is due to $E_\gamma \in \mathcal{P}_\alpha(K)$.

Therefore, $\lambda_f(E) = \lambda_f(F) \leq \alpha.\lambda_f(K)$, for every $\alpha > 1$, so $\lambda_f(E) \leq \lambda_f(K)$.

On the other hand, by the Hahn-Banach theorem, there exists a projection of norm 1 from $E$ to $K$, so $\lambda_f(K) \leq \lambda_f(E)$. $\blacksquare$

If $^*E$ is a non-standard enlargement of $E$, then, over the set $\text{fin}^*E = \{x \in ^*E : \exists y \in E, \|x - y\| \text{ is a finite hyperreal number}\}$ of the finite elements of $^*E$, consider the equivalence relation "$x$ is infinitely close to $y$", denoted by $x \equiv y$, and defined by $\|x - y\|$ is infinitesimal". In the quotient set, denoted $\bar{E}$, the norm $\|\bar{x}\| = st\|x\|$, $\bar{x} \in \bar{E}$, is defined, and the resulting normed space is called an infinitesimal hull of $E$. 
Lemma 2. $\lambda_f(\hat{E}) = \lambda_f(E)$.

Proof: Let $S$ be a finite subset of $E$ with more than one point. Then, $S$ is a finite subset of $\text{fin}^*E$ without infinitely close points.

It is obvious that $\delta(S) = \delta(S)$ and $r(S) \leq r(S)$. We suppose that $r(S) < r(S)$, and take a real number $t$ such that $r(S) < t < r(S)$. Then, there exists $c \in \text{fin}^*E$ such that $S \subseteq B[c, t]$, and so, $\|x - c\| < r(S)$ for every $x \in S$. Since $S$ is finite, $^*S = S$, and we have a $c \in ^*E$ such that $\|x - c\| < r(S)$ for every $x \in ^*S$. Applying the Transfer Principle, there exists a standard element $c \in E$ such that $\|x - c\| < r(S)$ for every $x \in S$, and again because $S$ is finite, this would imply $S \subseteq B[c, \rho]$, with $\rho = \max_{x \in S} \|x - c\| < r(S)$. Therefore, it is true $r(S) = r(S)$ and we conclude $\lambda_f(E) \leq \lambda_f(\hat{E})$.

Let $S$ be a finite subset of $\text{fin}^*E$ with some points not infinitely close. Then, $S$ is a $\cdot$-finite subset of $^*E$ with some points not infinitely close and $\hat{S}$ is a finite subset of $E$ such that $\delta(\hat{S}) \equiv \delta(S)$.

Since the relation $r(T) \leq \delta(T), \lambda_f(E)$ is true for every finite subset $T$ of $E$, by the Transfer Principle, we have $r(T) \leq \delta^*(T), \lambda_f(\hat{E})$ for every $\cdot$-finite $T$, and, in particular, $r(S) \leq \delta(\hat{S}), \lambda_f(\hat{E}) \equiv \delta(\hat{S}), \lambda_f(E)$.

Let $t$ be a hyperreal number such that $t > r(S)$, $t \equiv \delta(\hat{S}), \lambda_f(E)$. There exists a $c \in ^*E$ such that $S \subseteq B[c, t]$, and then,

$$||\hat{x} - \hat{c}|| = st \|x - c\| \equiv \|x - c\| \leq t \equiv \delta(\hat{S}). \lambda_f(E), \forall \hat{x} \in \hat{S}.$$ 

Since the first and last members are standard, we have $||\hat{x} - \hat{c}|| \leq \delta(\hat{S}). \lambda_f(E)$, for every $\hat{x} \in \hat{S}$, so that $r(\hat{S}) \leq \delta(\hat{S}), \lambda_f(E)$, and we conclude $\lambda_f(\hat{E}) \leq \lambda_f(E)$. ■

Theorem 3. If $E$ is a real Banach space such that $\lambda_f(E) = 1/2$, then $E$ is an $L^1$-predual.

Proof: We consider an infinitesimal hull $\hat{E}$ of $E$. Then, $\hat{E}$ has the radial intersection property $(2,4)$, that is, given four closed balls in $\hat{E}$ with the same radius, which intersect in pairs, the total intersection is non-empty.

Indeed, let $\rho$ be a positive real number, and let $\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4 \in \hat{E}$, such that $||\hat{x}_i - \hat{x}_j|| \leq 2\rho$, for $i, j = 1, 2, 3, 4$. We take $S = \{\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4\}$, a finite subset of $\hat{E}$ with $\delta(S) \leq 2\rho$, and, so, $r(S) \leq \rho$. For every natural number $p$, there exists $\hat{c}_p \in \hat{E}$ such that $||\hat{x}_i - \hat{c}_p|| < \rho + (1/2p)$, for $i = 1, 2, 3, 4$, and, consequently, there exists $c_p \in \text{fin}^*E$ such that $\|x_i - c_p\| < \rho + (1/p)$, for $i = 1, 2, 3, 4$. We consider now the sequence $(c_p)_{p\in N} \in E$, which can be enlarged to an internal sequence $(c_p)_{p\in \mathbb{N}}$ in $^*E$. The set of index $p \in \mathbb{N}$ such that $\|x_i - c_p\| < \rho + (1/p)$, for $i = 1, 2, 3, 4$ is an internal subset of $\mathbb{N}$ containing all standard natural numbers, and, so, if we work in a suitably saturated model (cf. [3]), it contains an infinite index, $\omega \in \mathbb{N}$. Then, the element $c_\omega \in ^*E$ is
finite, because \( \|x_i - c_i\| \leq \rho \), so that we can take \( \tilde{c}_i \in \tilde{E} \) thus verifying that
\[
\|\tilde{x}_i - \tilde{c}_i\| \leq \rho, \quad i = 1, 2, 3, 4.
\]

Therefore, \( \tilde{E} \) is an \( L^1 \)-predual ([2, theorem 6, pg. 212]), that is, \( \tilde{E}' \) is an \( L^1 \) space. Projecting \( E' \) over \( E \) by means of the function \( T \in \tilde{E}' \to T|_E \in E' \), we have that \( E' \) is also an \( L^1 \) space ([2, theorem 3, pg. 162]), and then \( E \) is an \( L^1 \)-predual. \( \blacksquare \)

**Lemma 4.** \( \lambda_b(E) \) is the infimum of the positive real numbers \( r \) such that for every \( \gamma > 0 \), whenever \( (x_\alpha)_{\alpha \in I} \subset E \) is a \( \gamma \)-Cauchy net (that is, given \( \varepsilon > 0 \), there exists \( \alpha_0 \in I \) such that \( \|x_\alpha - x_\beta\| \leq \gamma + \varepsilon \), for every pair of subindices \( \alpha, \beta \in I \) greater than or equal to \( \alpha_0 \)), \( (x_\alpha)_{\alpha \in I} \) has some \( r \)-Cauchy \( x \) in \( E \) (that is, given \( \varepsilon > 0 \), there exists \( \alpha_0 \in I \) such that \( \|x_\alpha - x\| \leq r \gamma + \varepsilon \), for every subindex \( \alpha \in I \) greater than or equal to \( \alpha_0 \)).

**Proof:** Let \( S \) be a bounded subset of \( E \), with more than one point. For every natural number \( n \), we consider \( S^{(n)} = S \times \{n\} \) and the bijection
\[
S \to S^{(n)}
\]
\[
x \mapsto x^{(n)} = (x, n).
\]
We take now \( I = \bigcup_{n \in \mathbb{N}} S^{(n)} \) with the order relation
\[
\alpha \leq \beta \iff \alpha = \beta \vee (\alpha = x^{(n)}, \beta = y^{(m)}, n < m),
\]
which makes \( I \) a directed set. Over it, we build the net \( (x_\alpha)_{\alpha \in I} \) defined by \( x_\alpha = x \) if \( \alpha \in I \) is such that \( \alpha = x^{(n)} \), for some \( n \in \mathbb{N} \). Thus, \( (x_\alpha)_{\alpha \in I} \) is a \( \delta(S) \)-Cauchy net in \( E \), with range \( S \), and such that for every \( y \in S \) and every \( \alpha \in I \), there exists a \( \beta \in I, \beta \geq \alpha \), which satisfies \( x_\beta = y \), that is, for every \( y \in S \) there exists and infinite index \( \beta \) which satisfies \( x_\beta = y \).

We call \( \lambda'(E) \) the infimum of the positive real numbers \( r \) such that for every \( \gamma > 0 \), every \( \gamma \)-Cauchy net has an \( \gamma \)-limit in \( E \), and let \( t \) be greater than \( \lambda'(E) \). The previously builted net \( (x_\alpha)_{\alpha \in I} \) has a \( t \delta(S) \)-limit \( x \in E \), and, so, \( \|x_\alpha - x\| \leq t \delta(S) \) for every infinite index \( \alpha \). Hence, we have \( \|y - x\| \leq t \delta(S) \) for every \( y \in S \), and, because both members are standard, \( \|x - y\| \leq t \delta(S) \) for every \( y \in S \), that is, \( S \subset B[s, t \delta(S)] \). Thus, \( r(S) \leq t \delta(S) \) for every \( t > \lambda'(E) \), and \( \lambda_b(E) \leq \lambda'(E) \).

Conversely, if \( (x_\alpha)_{\alpha \in I} \) is a \( \gamma \)-Cauchy net in \( E \) for some \( \gamma > 0 \), we put \( S_\alpha = \{x_\beta : \beta \in I, \beta \geq \alpha\} \), for every \( \alpha \in I \). Thus, every set \( S_\alpha \) is bounded and we can suppose that it has more than one point (otherwise the proof is trivial); therefore, given \( \varepsilon > 0 \), there exists \( \alpha \in I \) such that \( \delta(S_\alpha) < \gamma + \varepsilon / \lambda_b(E) \). Then,
\[
r(S_\alpha) \leq \delta(S_\alpha).\lambda_b(E) < (\gamma + \varepsilon / \lambda_b(E)).\lambda_b(E) = \gamma.\lambda_b(E) + \varepsilon,
\]
and, so, there exists, \( c_\alpha \in E \) such that \( \|x_\beta - c_\alpha\| \leq \gamma.\lambda_b(E) + \varepsilon \) for every \( \beta \geq \alpha \). Hence, \( c_\alpha \) is a \( (\gamma.\lambda_b(E) + \varepsilon) \)-limit of \( (x_\alpha)_{\alpha \in I} \) and, since this is true for every \( \varepsilon > 0 \) and for every \( \gamma \)-Cauchy net in \( E \), it follows that \( \lambda'(E) \leq \lambda_b(E) \). \( \blacksquare \)
Theorem 5. If $E$ is a $P_1(K)$ space, then $\lambda_0(E) = \lambda_0(K)$.

Proof: If we embed $K$ into $E$ by means of a linear isometry, identifying it with a one-dimensional subspace of $E$, the Hahn-Banach theorem assures the existence of a projection of norm 1, $P : E \to K$, whence we deduce $\lambda_0(K) \leq \lambda_0(E)$.

We will prove the reciprocal inequality in several stages:

(I) In the first place, we observe that if $E \in P_1(K)$ and $\tilde{E}$ is an infinitesimal hull of $E$, then $\lambda_0(\tilde{E}) \leq \lambda_0(E)$, because, considering the canonical linear isometry $E \to \tilde{E}$, there exists a contractive projection $\tilde{E} \to E$.

(II) Let $\Gamma$ be a non empty set. We denote by $l^\infty(\Gamma, K)$ the set of all bounded functions from $\Gamma$ to $K$, with the uniform norm. Giving to $\Gamma$ the discrete topology, we know that $l^\infty(\Gamma, K)$ is linearly isometric to the space $C(\beta \Gamma, K)$ of continuous functions with values in $K$ defined over the Stone-Čech compactification of $\Gamma$; so, $l^\infty(\Gamma, K) \in P_1(K)$ ([2]), and $\lambda_0(l^\infty(\Gamma, K)) \leq \lambda_0(l^\infty(\Gamma, K))$, by (I).

(III) We suppose that $\Gamma$ is a finite set. We will prove that, in this case, $\lambda_0(l^\infty(\Gamma, K)) \leq \lambda_0(K)$.

Because $l^\infty(\Gamma, K)$ is a finite dimensional, we know that its bounded and finite Chebyshev coefficients are equal, as are those of $K$.

Let $\rho$ be a real number, $\rho > \lambda_f(K)$, and let $S = \{x_1, \ldots, x_n\}$ be a finite subset of $l^\infty(\Gamma, K)$. Fixed $\gamma \in \Gamma$, we consider the finite subset of $K$ $S_\gamma = \{x_1(\gamma), \ldots, x_n(\gamma)\}$; then, $r(S_\gamma) / \delta(S_\gamma) \leq \lambda_f(K) < \rho$, and

$$r(S_\gamma) < \rho \cdot \delta(S_\gamma) = \rho \cdot \max_{1 \leq i, j \leq n} |x_i(\gamma) - x_j(\gamma)| \leq \rho \cdot \max_{1 \leq i, j \leq n} \|x_i - x_j\| = \rho \cdot \delta(S).$$

Then, there exists a centre $c_\gamma \in K$ such that $S_\gamma \subset B[c_\gamma, \rho \cdot \delta(S)] \subset K$. We define thus a function from $\Gamma$ in $K$ which associates $c_\gamma$ to every $\gamma$, and which satisfies

$$\|x_i - c\| = \sup_{\gamma \in \Gamma} |x_i(\gamma) - c| \leq \rho \cdot \delta(S), \quad i = 1, \ldots, n,$$

so that $S \subset B[c, \rho \cdot \delta(S)] \subset l^\infty(\Gamma, K)$. Therefore, $r(S) \leq \rho \cdot \delta(S)$, and we conclude $\lambda_f(l^\infty(\Gamma, K)) \leq \lambda_f(K)$.

(IV) The inequality $\lambda_0(l^\infty(\Gamma, K)) \leq \lambda_0(K)$ is valid also when $\Gamma$ is an infinite set.

Indeed, let $(x_\alpha)_{\alpha \in I}$ be a $\gamma$-Cauchy net in $l^\infty(\Gamma, K)$, for $\gamma > 0$. We take $X_0 = K \cup \Gamma \cup I$ in order to build a superstructure $X$ with base $X_0$ and over it a polysaturated nonstandard model satisfying the $R_0$-isomorphism property (cf. [3, sec. 0.4.]). In this case, we can identify $l^\infty(\Gamma, K)$ to $l^\infty(\omega, K)$, for every infinite natural number $\omega$, since $l^\infty(\omega, K)$ is isometrically isomorphic to $l^\infty(\omega, K)$, and this is so to $l^\infty(\Gamma, K)$ ([4, theorem 2.11]).

Let $p$ be a natural number. There exists an index $\alpha_p \in I$ such that $\|x_\alpha - x_\beta\| < \gamma + 1/(2p)$, when $\alpha, \beta \in *I$, $\alpha, \beta \geq \alpha_p$. We consider the set
\( S = \{ x_\alpha : \alpha \in I, \alpha \geq \alpha_p \} \), an internal \( * \)-bounded subset of \( \ell^\infty(\omega, K) \), where we can apply (III), and so

\[
\frac{r(S)}{\delta(S)} \leq \lambda_b(K) \\
r(S) \leq \lambda_b(K) \delta(S) \leq \lambda_b(K)(\gamma + 1/(2p)) < t,
\]

where \( t = \lambda_b(K)(\gamma + 1/p) \). Since \( *r(S) < t \), there exists \( c_p \in \ell^\infty(\omega, K) \) such that \( S \subset B[c_p, t] \) and \( \hat{c}_p \in \ell^\infty(\omega, K) = \ell^\infty(I, K) \) satisfies \( \| \hat{x}_\alpha - \hat{c}_p \| < t \), for \( \alpha \in I, \alpha \geq \alpha_p \).

Bearing in mind that \( \ell^\infty(I, K) \in \mathcal{P}_1(K) \), consider the natural embedding \( \ell^\infty(I, K) \to \ell^\infty(I', K) \); then, there exists a projection of norm 1, \( P : \ell^\infty(I, K) \to \ell^\infty(I', K) \), and \( x = P(\hat{c}_p) \) is an element of \( \ell^\infty(I', K) \) which satisfies

\[
\| x_\alpha - x \| = \| P(\hat{x}_\alpha) - P(\hat{c}_p) \| \leq \| \hat{x}_\alpha - \hat{c}_p \| < t, \alpha \in I, \alpha \geq \alpha_p.
\]

Thus, for every \( \rho > \lambda_b(K) \), taking a \( p \in \mathbb{N} \) greater than the real number \( \lambda_b(K)/\gamma(\rho - \lambda_b(K)) \), we have

\[
\| x_\alpha - x \| \leq \lambda_b(K)(\gamma + \frac{1}{p}) < \rho \gamma, \alpha \in I, \alpha \geq \alpha_p,
\]

that is, \( x \) is a \( \rho \gamma \)-limit of \( (x_\alpha)_n \) in \( \ell^\infty(I', K) \). Since this is valid for every \( \gamma \)-Cauchy net in \( \ell^\infty(I', K) \) and for any \( \gamma > 0 \) we conclude by Lemma 4 that \( \lambda_b(\ell^\infty(I', K)) \leq \lambda_b(K) \).

(V) Now, we can embed \( E \) linearly and isometrically into \( \ell^\infty(E, K) \) by means of the application

\[
\phi : E \to \ell^\infty(E, K) \\
x \mapsto \phi(x) : E \to K \\
y \mapsto \phi(x)(y) = f_y(x)
\]

where \( f_y \) is a continuous linear functional from \( E \) to \( K \) which satisfies \( \| f_y \| = 1 \) and \( f_y(y) = \| y \| \). The existence of which is guaranteed by the Hahn-Banach theorem. So, there exists a projection of norm 1, \( \ell^\infty(E, K) \to E \), which permits us to deduce the inequality \( \lambda_b(E) \leq \lambda_b(\ell^\infty(E, K)) \), and, from the result in the preceding paragraph, \( \lambda_b(E) \leq \lambda_b(K) \).

References


1980 Mathematics subject classifications: 46B20, 46B25

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Rebut el 18 de Desembre de 1989