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TRUNCATED POLYNOMIAL ALGEBRAS OVER THE STEENROD ALGEBRA

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Abstract

It is shown that the classification of polynomial algebras over the mod p Steenrod algebra is an essentially different problem from the classification of polynomial algebras truncated at height greater than p over the Steenrod algebra.

0. Introduction. Let $B = Z_p[y_{2n_1}, y_{2n_2}, \ldots, y_{2n_r}]$ be a polynomial algebra over $\mathcal{A}(p)$, the mod p Steenrod algebra, where y_{2n_i} has dimension $2n_i$ and p is an odd prime. If each n_i is prime to p, the results of [1] and [2] imply that the structure of B is well understood; in particular the set of dimensions $\{2n_1, 2n_2, \ldots, 2n_r\}$ is a union of sets given in the Clark-Ewing list of dimensions in the main Theorem of [3]. Earlier attempts in the 1960's and early 1970's to classify the set of dimensions occurring in B often depended only on the $\mathcal{A}(p)$-algebra structure of $B^{p+1} = Z_p[y_{2n_1}, y_{2n_2}, \ldots, y_{2n_r}]^{p+1}$, the polynomial algebra truncated at height $p+1$. The question of determining the dimensions of the generators of a truncated polynomial algebra $A = Z_p[x_{2n_1}, x_{2n_2}, \ldots, x_{2n_r}]^{p+1}$ over $\mathcal{A}(p)$, where each n_i is prime to p, is not well understood. It appears not to be known if the set of possible dimensions in the two cases coincide as is certainly the case when $r = 1$. The purpose of this note is to settle this question, for example, $Z_{11}[x_6, x_{10}]^{12}$ supports an $\mathcal{A}(11)$-structure, but $Z_{11}[y_6, y_{10}]$ does not.

The question has some topological significance. For example, if a product of p-local spheres, $\Pi S^{2n_i-1}_{(p)}$, $1 \le i \le r$, supports an $A(p)$ structure in the sense of [5], then each $n_i \in \{1, 2, \ldots, p\}$ and there exists a truncated polynomial algebra over $\mathcal{A}(p)$, $A = Z_p[x_{2n_1}, x_{2n_2}, \ldots, x_{2n_r}]^{p+1}$ [4]. If an addition, $1 < n_i < p$ for each i and there exists $B = Z_p[y_{2n_1}, y_{2n_2}, \ldots, y_{2n_r}]$, then the set of dimensions $\{2n_1, 2n_2, \ldots, 2n_r\}$ is given by the Clark-Ewing list. One would then expect that, up to homotopy, $\Pi S^{n_i-1}_{(p)}$ supports the structure of a topological group.

1. Clark's condition. First we apply a well known theorem of A. Clark [2]; it is clear that the proof holds for a polynomial algebra truncated at height greater than p.

Theorem 1.1. *Let A be a polynomial algebra truncated at height $p+1$ over the* mod p *Steenrod algebra. If $2m$ is the degree of a generator of A, then either $m \equiv 0$* mod p *or there exists a generator of dimension $2n$ with $n \equiv 1-p$* mod *m.*

If we restrict attention to $A = Z_p[s_{2n}, x_{2m}]^{p+1}$ as above, routine calculations imply that, for small primes, the pairs of integers $\{2n, 2m\}$ must lie in the table below.

$p=3$ $\{4,4\}$

$p=5$ $\{4,4\}$, $\{4,6\}$, $\{4,8\}$

$p=7$ $\{4,4\}$, $\{4,6\}$, $\{4,8\}$, $\{4,12\}$, $\{6,6\}$, $\{6,12\}$, $\{8,12\}$, $\{12,12\}$

$p=11$ $\{4,4\}$, $\{4,6\}$, $\{4,8\}$, $\{4,10\}$, $\{4,12\}$, $\{4,20\}$, $\{6,10\}$,
$\{8,20\}$, $\{10,10\}$, $\{10,20\}$, $\{12,16\}$, $\{20,20\}$

$p=13$ $\{4,4\}$, $\{4,6\}$, $\{4,8\}$, $\{4,12\}$, $\{4,14\}$, $\{4,24\}$, $\{6,6\}$, $\{6,10\}$,
$\{6,12\}$, $\{6,24\}$, $\{8,8\}$, $\{8,12\}$, $\{8,16\}$, $\{8,24\}$, $\{12,12\}$,
$\{12,18\}$, $\{12,24\}$, $\{16,24\}$, $\{24,24\}$

Comparing this list with the list of possible dimensions of the generators of a polynomial algebra $B = Z_p[y_{2n}, y_{2m}]$ over the Steenrod algebra [**3**], we see that the only pairs which do not appear in the latter are $\{6,10\}$ and $\{8,20\}$ when $p=11$ and $\{6,10\}$, $\{8,16\}$ and $\{12,18\}$ when $p=13$. We must therefore consider the possibility of defining the action of the Steenrod algebra on $A = Z_p[x_{2n}, x_{2m}]^{p+1}$ in these five cases.

2. Steenrod theorem. In his last published paper, Steenrod considered the question of defining the cyclic reduced powers on graded polynomial algebra. We believe that his results have never been applied. We summarize them in our context.

Let A be a graded algebra over the (unstable) mod p Steenrod algebra concentrated in even dimensions. Recall that the cyclic reduced powers are homomorphisms

$$p^q : A^{2n} \longrightarrow A^{2n+2q(p-1)}$$

satisfying:

(1) $p^0 =$ Identity,

(2) $p^q x = x^p$ if $2q = \dim x$,
(3) $p^q x = 0$ if $2q > \dim x$,
(4) $p^q(xy) = \sum_{i=0}^{q} p^i x p^{q-i} y$,
(5) If $a < pb$,

$$R(a,b) \equiv p^a p^b - \sum_{0}^{[a/p]} (-1)^{a+t} \binom{(p-1)(b-t)-1}{a-pt} p^{a+b-t} p^t = 0.$$

Now let $A = Z_p[x_{2n_1}, x_{2n_2}, \ldots, x_{2n_r}]^{p+1}$. Suppose that for each i, there are defined homomorphisms $\bar{p}^q$ by setting $\bar{p}^q x_{2n_i} = f_{q,i}$, $0 < q < 2n_i$, where $f_{q,i} \in A$ has dimension $2n_i + 2q(p-1)$, $\bar{p}^{n_i} x_{2n_i} = x_{2n_i}^p$ and $\bar{p}^q x_{2n_i} = 0$ for $q > n_i$ and extending $\bar{p}^q$ over $\bar{A}$ by linearity and the Cartan formula (4). Theorem 6.2 and Lemma 4.1 of [7] imply that these $\bar{p}^q$ define an action of the Steenrod algebra provided that $\bar{R}(a,b)x_{2n_i} = 0$ for all i where $(a,b) = (p^t, b)$, $t \geq 0$.

We now restrict attention to the five examples $A = Z_p[x_{2n}, x_{2m}]^{p+1}$ listed above where $n < m$. Suppose that $\bar{p}^1 x_{2n} = g_{1,n}, \bar{p}^1 x_{2m} = h_{1,m}$. We define $\bar{p}^i x_{2n}$ inductively for $1 < i < n$ by requiring $\bar{R}(1, i-1)x_{2n} = 0$, or equivalently, $\bar{p}^1 x_{2n} = (i)^{-1} \bar{p}^1 \bar{p}^{i-1} x_{2n}$, $\bar{p}^n x_{2n} = x_{2n}^p$ with similar definitions for x_{2m}. Dimensional reasons imply that $\bar{p}^p \equiv 0$ in all cases as $2mp - 2n < 2p(p-1)$ and therefore $R(p^t, b) \equiv 0$ for $t > 0$. Thus to show that the definition of the $\bar{p}^i$ defines an action of the Steenrod algebra it is sufficient to check that with the chosen $g_{1,n}$ and $h_{1,m}$, $\bar{R}(1, n-1)x_{2n} = 0$ and $\bar{R}(1, m-1)x_{2m} = 0$.

Theorem 2.1.

(a) $p = 11$, $Z_{11}[x_6, x_{10}]^{12}$ *and* $Z_{11}[x_8, x_{20}]^{12}$ *support an action of the Steenrod algebra which is unique up to algebra isomorphism over the Steenrod algebra.*

(b) $p = 13$, $Z_{13}[x_6, x_{10}]^{14}$, $Z_{13}[x_{12}, x_{18}]^{14}$ *and* $Z_{13}[x_8, x_{16}]^{14}$ *do not support an action of the Steenrod algebra.*

We consider the examples in turn.

(a) (1) $p = 11$, $\{6, 10\}$. We first prove uniqueness. Assume that $Z_{11}[x_6, x_{10}]^{12}$ supports an $\mathcal{A}(p)$-action. For dimensional reasons $p^1 x_6 = \alpha x_6 x_{10}^2$, $p^1 x_{10} = \beta x_{10}^3 + \gamma x_6^5$. Routine computations show that

$$p^3 x_6 = (3!)^{-1}[2\alpha\gamma^2 x_6^{11} + \alpha(\alpha + 2\beta)(\alpha + 4\beta)x_6^6 x_{10}^3 + 5\alpha\beta(\alpha + 2\beta)x_6^3 x_{10}^3] = x_6^{11}.$$

Therefore $\alpha\gamma^2 = 3$ and $\alpha + 2\beta = 0$. We can choose γ to have any non-zero value. So we set $\gamma = 5$. Then $\alpha = 1$ and $\beta = 5$. We deduce that if A supports an action of the Steenrod algebra, the action is unique up to algebra homomorphism over the Steenrod algebra and we can set $p^1 x_6 = x_6 x_{10}^2$, $p^1 x_{10} = 5(x_{10}^3 + x_6^5)$.

Therefore let $g_{1,3} = x_6 x_{10}^2$, $h_{1,5} = 5(x_0^3 + x_6^5)$ and define $\bar{p}^q x_6$, $\bar{p}^q x_{10}$ as described above. The calculation above implies that $\bar{R}(1,2)x_6 = 0$ and so we need only check that $\bar{R}(1,4)x_{10} = 0$. Routine calculation verifies that this is true.

(a) (2) $p = 11$, $\{8, 20\}$. Again assume that $Z_{11}[x_8, x_{12}]^{12}$ supports an $\mathcal{A}(p)$-action. For dimensional reasons, $p^1 x_8 = \alpha x_8 x_{20}$, $p^1 x_{20} = \beta x_{20}^3 + \gamma x_8^5$. Routine computations show that

$$p^4 x_8 = (4!)^{-1}[\alpha(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)x_8^6 x_{20} + \alpha\gamma(\alpha-\beta)(\alpha+2\beta)x_8^6 x_{20}^6 + \alpha\gamma^2(\alpha+3\beta)x_8^{11}] = x_8^{11}.$$

Therefore by suitable choice of x_{20}, we can assume that $\alpha = 1$. It follows that $\beta = 5$, $\gamma = \pm 4$. Replacing x_8 by $-x_8$ if necessary, we can assume that $\gamma = 4$, and therefore if A supports an action of the Steenrod algebra, the action is unique up to algebra homomorphism over the Steenrod algebra and we can set $p^1 x_8 = x_8 x_{10}$, $p^1 x_{20} = 5x_{20}^3 + 4x_8^5$.

Therefore let $g_{1,4} = x_8 x_{20}$, $h_{1,10} = 5x_{20}^3 + 4x_8^5$ and define $\bar{p}^q x_8$, $\bar{p}^q x_{20}$ as described above. Clearly $\bar{R}(1,3)x_8 = 0$ and so we need just to check that $\bar{R}(1,9)x_{20} = 0$. Routine calculation verifies that this is true.

(b) (3) $p = 13$, $\{6, 10\}$. Again suppose that $Z_{13}[x_6, x_{10}]^{14}$ supports an $\mathcal{A}(p)$ action. For dimensional reasons $p^1 x_6 = \alpha x_6^5 + \beta x_{10}^3$, $p^1 x_{10} = \gamma x_6^4 x_{10}$. Routine calculations show that

$$p^3 x_6 = (3!)^{-1}[6\alpha^3 x_6^{13} + \beta\{6\alpha^2 + (4\alpha+3\gamma)(5\alpha+3\gamma)\}x_6^8 x_{10}^3 + \beta^2(5\alpha+3\gamma)x_6^3 x_{10}^6] = x_6^{13}.$$

Therefore $\alpha^3 = 1$, $\beta\{6\alpha^2 + (4\alpha+3\gamma)(5\alpha+3\gamma)\}$ and $\beta^2(5\alpha+3\gamma) = 0$ in Z_{13}. This is possible only if $\beta = 0$, but this implies that $p^q x_{10} \in (x_6)$ which is not possible as $p^5 x_{10} = x_{10}^{13}$. Thus $Z_{13}[x_6, x_{10}]^{14}$ will not support an $\mathcal{A}(13)$ action.

(b) (4) $p = 13$, $\{8, 16\}$. Again suppose that $Z_{13}[x_8, x_{16}]^{14}$ supports an $\mathcal{A}(p)$ action. For dimensional reasons $p^1 x_8 = \alpha x_8^4 + \beta x_{16}^2 + \gamma x_8^2 x_{16}$, $p^1 x_{16} = \delta x_8 x_{16}^2 + \mu x_8^3 x_{16} + \theta x_8^5$. Routine calculations show that

$$\begin{aligned}
p^4 x_8 = (4!)\\
[\{&7\alpha^4 + 12\alpha\beta\theta^2 + 8\alpha\gamma\mu\theta + 6\alpha^2\theta\gamma + 11\gamma^2\theta^2 + 6\beta\theta^2\mu + \gamma\theta\mu^2 + 2\theta^2\gamma\delta\}x_8^{13} + \\
\{&9\alpha^2\gamma\mu + 3\alpha\beta\mu + 6\alpha\gamma\mu^2 + \beta\theta\mu^2 + \gamma\mu^3 + 8\theta\gamma\delta + 3\alpha^3\gamma + \\
&4\alpha\gamma^2\theta + 9\alpha^2\beta\theta + 8\alpha\theta\gamma\delta + \beta\gamma\theta^2 + 12\beta\theta^2\delta\}x_8^{11}x_{16} + \\
\{&5\alpha^3\beta + 11\beta^2\theta^2 + 6\beta\theta\gamma\mu + 11\alpha\beta\theta\gamma + 4\alpha^2\gamma^2 + 6\alpha\gamma^2\mu + 8\alpha^2\gamma\delta + 6\alpha\beta\delta\theta + \\
&\alpha^2\beta\mu + 8\alpha\gamma\delta\mu + 3\beta\theta\delta\mu + 7\gamma\delta\mu^2 + 8\beta\mu^3 + 9\theta\gamma^2\delta + 8\theta\gamma\delta^2 + 2\theta\gamma^3\}x_8^9 x_{16}^2 + \\
\{&\alpha^2\beta\gamma + 11\beta\theta\gamma^2 + 6\alpha\gamma^3 + 7\gamma^3\mu + 11\alpha\gamma^2\delta + 12\beta\theta\delta\gamma + 5\gamma^2\delta\mu + 12\beta\gamma\mu^2 + \\
&6\alpha^2\beta\delta + 10\theta\delta^2\beta + 12\alpha\beta\delta\mu + 12\gamma\delta^2\mu + 6\beta\mu^2\delta + 11\alpha\beta^2\theta + 3\beta^2\theta\mu\}x_8^7 x_{16}^3 + \\
\{&9\alpha^2\beta^2 + 6\beta^2\theta\gamma + 5\alpha\beta\gamma^2 + 3\beta\gamma^2\mu + \alpha\beta\gamma\delta + 11\beta^2\theta\delta + \alpha\beta^2\mu + 9\beta\gamma\delta\mu + \\
&9\beta^2\mu^2 + 11\gamma^4 + 7\gamma^3\delta + 3\gamma^2\delta^2 + 6\gamma\delta^3 + 3\alpha\beta\delta^2 + 9\beta\mu\delta^2\}x_8^5 x_{16}^4 + \\
\{&5\alpha\beta^2\gamma + \beta^3\theta + 4\gamma\beta^2\mu + \beta\gamma^3 + 4\beta\gamma^2\delta + 12\beta\gamma\delta^2 + 10\alpha\beta^2\delta + \beta^2\mu\delta + 11\beta\delta^3\}x_8^3 x_{16}^5 + \\
\{&11\alpha\beta^3 + 12\beta^3\mu + 3\beta^2\gamma^2 + 6\beta^2\gamma\delta + 9\beta^2\delta^2\}x_8 x_{16}^6] = x_8^{13}
\end{aligned}$$

One can show that this implies that $\beta = 0$, and so $p^q x_{16} \in (x_8)$ which is false as $p^8 x_{16} = x_{16}^{13}$. Thus $Z_{13}[x_8, x_{16}]^{14}$ will not support an $\mathcal{A}(13)$-action.

(b) (5) $p = 13$, $\{12, 18\}$. Similarly as in the last examples, we can see that, for dimensional reasons, $p^1 x_{12} = \alpha x_{12}^3 + \beta x_{18}^2$, $p^1 x_{18} = \gamma x_{12}^2 x_{18}$ and then

$$p^6 x_{12} = (6!)^{-1}[\{8\alpha^6 x_{12}^{13} + \beta(3\alpha^4\gamma + 9\alpha^3\gamma^2 + 4\alpha^2\gamma^3 + 4\alpha\gamma^4 + 6\gamma^5\}x_{12}^{10}x_{18}^2 + \beta^2\{5\alpha^4 + 7\alpha^2\gamma + 5\alpha^2\gamma^2 + 3\alpha\gamma^3\}x_{12}^7 x_{18}^4 + \beta^3\{7\alpha^3 + 9\alpha^2\gamma + 11\alpha\gamma^2 + \gamma^3\}x_{12}^4 x_{18}^6 + \beta^4\{7\alpha^2 + 12\alpha\gamma + 6\gamma^2\}x_{12}x_{18}^8] = x_{12}^{13}.$$

This is true only if $\beta = 0$, but this implies that $p^q x_{18} \in (x_{12})$ which is impossible as $p^9 x_{18} = x_{18}^{13}$ and so $Z_{13}[x_{12}, x_{18}]^{14}$ will not support an $\mathcal{A}(13)$-action.

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