Publicacions Matemàtiques, Vol 34 (1990), 307–309.

WEIGHTED ERGODIC THEOREMS FOR WEAKLY MIXING TRANSFORMATION GROUPS

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Abstract

In this paper we characterize weakly mixing transformation groups in terms of weighted ergodic theorems.

1. Introduction

Let $(\tau_t,\, t \in R)$ be a one parameter group of measure preserving transformations on a probability space (Ω, F, P) and let (T_t) denote the corresponding group of unitary operators on $L^2 = L^2(\Omega, F, P)$. We assume that the map t to $T_t f$ is continuous for each f in L^2.

The group (T_t) is ergodic if the only functions let fixed by T_t for all t are the constants. The group is called weakly mixing if for any f in L^2

$$\lim_{r\to\infty} \frac{1}{r} \int_0^r |(T_t f, f) - |E(f)|^2|^2 \, dt = 0,$$

where $E(f) = \int_\Omega f\, dP$.

Weak mixing implies ergodicity and then by the Birkhoff Ergodic Theorem we have

$$\lim_{r\to\infty} \frac{1}{r} \int_0^r T_t f(x)\, dt = E(f) \text{ a.e.}$$

From this it is easy to see that the convergence also holds in the L^2-norm. For a proof of these assertions see [1, Chapter 1].

We denote by W the class of all bounded complex valued functions p on R such that

$$m(p) = \lim_{r\to\infty} \frac{1}{r} \int_0^r p(t)\, dt \text{ exists.}$$

Let p be a function in W. For each $f \in L^2$ we consider the weighted averages

$$A_r(f,p)(x) = \frac{1}{r} \int_0^r p(t) T_t f(x)\, dt.$$

In what follows we prove that $A_r(f,p)$ converges in mean to $m(p)\, E(f)$ if, and only if, (T_t) is weakly mixing. We also show that this later condition is equivalent to the almost everywhere convergence of $A_r(f,p)$, when the weight p is a periodic function.

2. Statements and proofs

Theorem 1. *The following conditions are equivalent:*

(a) *(T_t) is weakly mixing*

(b) *For any p in W, $A_r(f,p)$ converges in mean to $m(p)\,E(f)$, for each $f \in L^2$.*

Proof: (a) $\Rightarrow$ (b). Let $f \in L^2$. We may assume without loss of generality that $E(f) = 0$. For a given $\varepsilon > 0$ let $B_\varepsilon = \{t \in R : |(T_t f, f)| \geq \varepsilon\}$. Since $|T_{-t}f, f)| = |(T_t f, f)|$, we see that B_ε is a symmetric set respect to zero. Also from

$$\frac{1}{r}\int_0^r |(T_t f, f)|^2\,dt \geq \varepsilon^2 \frac{|B_\varepsilon \cap [0,r]|}{r},$$

where vertical bars denote Lebesgue measure, we deduce that B_ε has zero density, i.e.

$$\lim_{r\to\infty} \frac{|B_\varepsilon \cap [-r,r]|}{r} = 0.$$

On the other hand, by Fubini's theorem we have

$$\begin{aligned}
\|A_r(f,p)\|_2^2 &= \frac{1}{r^2}\int_0^r\int_0^r p(t)\overline{p(s)}(T_{t-s}f, f)\,dt\,ds \\
&= \frac{1}{r}\int_0^r \overline{p(s)}(\frac{1}{r}\int_{-s}^{r-s} p(u+s)(T_u f, f)\,du)\,ds \\
&\leq \varepsilon\|p\|_\infty^2 + \|f\|_2^2\|p\|_\infty^2 \frac{|[-r,r] \cap B_\varepsilon|}{r}.
\end{aligned}$$

Letting $r \to \infty$, we obtain $\limsup_{r\to\infty}\|A_r(f,p)\|_2^2 \leq \varepsilon\|p\|_\infty^2$. Since ε is arbitrary, this implies $\|A_r(f,p)\|_2 \to 0$, as $r \to \infty$.

(b) $\Rightarrow$ (a). Suppose that (T_t) is not weakly mixing. Hence, there is a non-constant f such that $T_t f = e^{i\lambda t} f$, for some real number λ (see [1, p. 29]). If we take $p(t) = e^{-i\lambda t}$, then $A_r(f,p)(x) = f(x)$ does not converge to $m(p)\,E(f)$. ∎

Under the assumption that the map t to $T_t f$ is continuous for each $f \in L^2$, it is not hard to prove that if (T_t) is weakly mixing then T_α is itself weakly mixing for any $\alpha \neq 0$. Using this fact we can state the following result.

Theorem 2. *(T_t) is weakly mixing if, and only if, for any periodic function $p \in W$, $A_r(f,p)$ converges almost everywhere to $m(p)\,E(f)$, for each $f \in L^2$.*

Proof: Assume that (T_t) is weakly mixing. Let p be a bounded periodic function and let $\alpha > 0$ such that $p(\alpha + t) = p(t)$ for all real number t. Let f be a bounded function. To prove the almost everywhere convergence of $A_r(f,p)$ it suffices to show that $A_{\alpha n}(f,p)(x)$ tends to $m(p)\,E(f)$ for almost every x, as $n \to \infty$, n being integer.

We have

$$\begin{aligned} A_{\alpha n}(f,p)(x) &= \frac{1}{\alpha n}\int_0^{\alpha n} p(t)T_t f(x)\,dt = \\ &= \frac{1}{\alpha n}\sum_{j=0}^{n-1}\int_0^{\alpha} p(t)T_{t+j\alpha} f(x)\,dt \\ &= \frac{1}{n}\sum_{j=0}^{n-1} T_{j\alpha}[\frac{1}{\alpha}\int_0^{\alpha} p(t)T_t f(x)\,dt]. \end{aligned}$$

Since T_α is weakly mixing and therefore ergodic, it follows that

$$\lim_{n\to\infty} A_{\alpha n}(f,p)(x) = E(\frac{1}{\alpha}\int_0^{\alpha} p(t)T_t f(x)\,dt) = m(p)\,E(f)$$

for almost every x.

By virtue of the Maximal Ergodic Theorem (see [**2**, p. 690]) and the density of bounded functions in L^2, a standard argument implies the convergence of $A_r(f,p)$ for an arbitrary $f \in L^2$.

The converse follows from the proof of (b) $\Rightarrow$ (a) in Theorem 1. ■

Remark. Since all almost periodic function is the limit of a uniformly convergent sequence of trigonometric polynomials, Theorem 2 remains valid if we set almost periodic functions instead of periodic functions.

References

[1] I. CORNFELD, S. FOMIN AND YA. SINAI, "*Ergodic Theory*," Springer-Verlag, New York Inc., 1982.

[2] N. DUNFORD AND J. SCHWARTZ, "*Linear operators, Part I: General Theory*," Interscience, New York, 1958.

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Rebut el 7 de Novembre de 1989