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A NOTE ON CHARACTERIZATION OF MOISHEZON SPACES

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Abstract

In this note a necessary and sufficient condition for a compact complex space X to be Moishezon is obtained; it can be seen as the existence of a line bundle L on X such that, for some point $x \in X$, the first cohomology groups of X with values respectively in $L \otimes m_x$ and $L \otimes m_x^2$, vanish. (Here m_x denote the ideal sheaf at x).

1. Grauert and Riemenschneider [1], [7] conjectured a characterization of a Moishezon space in terms of "almost (quasi) positive" coherent sheafs on it. The problem in proving this was to obtain Moishezonness of a compact complex space X if it carries an almost positive coherent sheaf. From then a number of characterizations of Moishezonness with additional assumptions to the hypothesis of the conjecture have been obtained [7], [12], [9], [5], [10]. For example, in [8] it was assumed that X is Kähler. Some of these are in such a way that the proofs can be obtained by using Kodaira's techniques: namely blowups, Kodaira's Vanishing and embedding theorems. Siu [11] has succeeded in proving a stronger version than the conjecture. His proof is by using the powerful theorem of Hirzebruch-Riemann-Roch and giving estimates on the dimensions of the cohomology groups of X with coefficients in a power of a line bundle which has a non strictly positive curvature form. It deals, more generally with one of the fundamental questions of obtaining holomorphic sections for non strictly positive line bundles.

The present note gives a characterization of Moishezon spaces using Kodaira's techniques.

2. Since compact complex analytic space can be desingularized and coherent analytic sheaves can be made free (modulo torsion) by proper modifications, a characterization of Moishezon space X can be stated in terms of compact complex manifold X and line bundles (locally free sheaves) over X. We prove the following:

Theorem. *Let X be an irreducible, n-dimensional compact complex manifold. Then X is Moishezon (i.e. the transcendence degree of the field of meromorphic functions on X is equal to the complex dimension of X) if and only if there exists a line bundle L over X such that for $r = 1, 2$,*

$$H^1(X, L \otimes m_x^r) = 0 \text{ for some point } x \in X$$

(m_x denotes the ideal sheaf of x).

Proof: To prove the "if" part, consider the exact sequences

$$0 \longrightarrow m_x^r \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/_{m_x^r} \longrightarrow 0, \; r = 1, 2$$

Then one can deduce the exactness of

$$\text{(a)} \quad H^0(X, L) \xrightarrow{\alpha_r} H^0(X, L \otimes \mathcal{O}_X/_{m_x^r}) \longrightarrow H^1(X, L \otimes m_x^r) \text{ for } r = 1, 2$$

using the given vanishing for $r = 1$, we get from (a) that the map α_r is onto. This means that the global holomorphic sections of L generate the stalk $L_x = H^0(X, L \otimes \mathcal{O}_X/_{m_x})$. Hence the meromorphic mapping $s : X \to \mathbb{P}^m$ induced by a basis $s_1, \ldots, s_m$ of $H^0(X, L)$ is holomorphic at $x \in X$ and separates points in a neighbourhood of x.

For $r = 2$, the sequence (a) gives the surjection of $\alpha_2 : H^0(X, L) \to H^0(X, L \otimes \mathcal{O}_X/_{m_x^2})$. This implies that s has maximal rank at x. That is, s defines a closed embedding near x. Then for a suitable neighbourhood U of x in X, there exist meromorphic functions $f_1, \ldots, f_n$ on $\mathbb{P}^m$ such that $d(f_{1|_{s(U)}}) \wedge \cdots \wedge d(f_{n|_{s(U)}})(s(x)) \neq 0$ (since $s(U)$ becomes on n-dimensional complex submanifold of an open subset of $\mathbb{P}^m$). Since s is meromorphic, the functions $f_1, \ldots, f_n$ can be lifted to meromorphic functions $g_1, \ldots, g_n$ on X which are algebraically and analytically independent with $dg_1 \wedge \cdots \wedge dg_n(x) \neq 0$ (by a theorem in [6]). Thus X is Moishezon. Conversely assume that X is Moishezon. Then by Theorem 4 of [7] there exists a line bundle H on X which is positive on a dense open set U. Let $x \in U$. Consider the blow up $(\tilde{X}, \Pi)$ of X at x. Let E_x be the line bundle on $\tilde{X}$ associated to the divisor $\Pi^{-1}(x)$. Then $E^*_{x|_{\Pi^{-1}(x)}}$ is positive by standard arguments [2], [3], [7]. Since $\Pi^* H$ is semipositive everywhere and positive on $\Pi^{-1}(U - x)$ there exists a positive integer μ such that for $r = 0, 1$, $T = \Pi^* H^\nu \otimes E_x^{*^{n+r}}$ is semipositive everywhere and positive in $\Pi^{-1}(U)$ for all $\nu \geq \mu$ (note that $E^{*^{n+r}}$ is positive in a neighbourhood of $\Pi^{-1}(x)$ and trivial outside $\Pi^{-1}(U)$ ([3]). Observe that $\tilde{X}$ is Moishezon, being a modification of a Moishezon space [4]. Hence by applying Theorem 3 of [6] for a trivial bundle and T, it follows that

$$H^l(\tilde{X}, \Pi^* H^\nu \otimes E_x^{*^{n+r}} \otimes K_{\tilde{X}}) = 0, \; l \geq 1, \; \nu \geq \mu, \; r = 0, 1.$$

By Kodaira's formula [3], $K_{\tilde{X}} = \Pi^* K_X \otimes E_x^{n-1}$, we get, in particular,

$$H^1(\tilde{X}, \Pi^*(H^\mu \otimes K_X) \otimes E_x^{*^r}) = 0, \; r = 1, 2$$

i.e. $H^1(X, L \otimes m_x^r) = 0$, $r = 1, 2$, by taking $L = H^\mu \otimes K_X$. ∎

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