ON THE MAXIMALITY OF THE SUM OF TWO MAXIMAL MONOTONE OPERATORS

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Abstract _

In this paper we deal with the maximal monotonicity of A + B when the two maximal monotone operators A and B defined in a Hilbert space X are satisfying the condition: $\bigcup_{\lambda \ge 0} \lambda$ (domB-domA) is a closed linear subspace of X.

In this note we study the maximal monotonicity of the sum of two maximal monotone operators by introducing a new weakened condition. The classical theorem of Rockafellar [5] and Brezis [3] tell us that A + B is a maximal monotone operator whenever A and B are so and dom $A \cap (int(dom B)) \neq \phi$. Attouch [1] dealt with the same problem with the condition: $0 \in int(dom A - dom B)$.

Our idea is to use Attouch and Brezis assumption kind, see [2]:

 $\bigcup_{\lambda \ge 0} \lambda(\operatorname{dom} B - \operatorname{dom} A) \text{ is a closed linear subspace.}$

Let X be a real Hilbert space with the norm $\|\cdot\|$, and scalar product $\langle\cdot,\cdot\rangle$.

Definition 4.1. A multivalued operator A in X is said to be monotone if for every $x_1, x_2 \in X$ and every $y_1 \in Ax_1$ and $y_2 \in Ax_2$ one has

$$\langle y_1-y_2, x_1-x_2\rangle \geq 0.$$

A is maximal monotone if it is maximal, relatively to the inclusion, in the set of all monotone operators.

Given A a maximal monotone operator in X, we shall denote by domA its domain (i.e. $x \in \text{dom}A$ if $Ax \neq \phi$), respectively by

$$A_{\lambda} = (1/\lambda)(I - J_{\lambda}^{A})$$
 and $J_{\lambda}^{A} = (I + \lambda A)^{-1}$ for $\lambda > 0$,

its Yosida approximation and resolvante and by A^0 its minimal section (i.e. A^0x is the projection of zero on Ax). See Brezis [3] for more details.

Theorem 4.2. Let X be a Hilbert space, A and B be two maximal monotone operators such that dom $A \cap \text{dom}B \neq \phi$ and \mathbb{R}_+ (domB-domA) is a closed linear subspace of X. Then A + B is maximal monotone.

Proof: Without loss of generality we can assume that $0 \in \text{dom}A \cap \text{dom}B$. Indeed, since there exists $x_0 \in \text{dom}A \cap \text{dom}B$, then the operators A_{x_0} and B_{x_0} defined by $A_{x_0}(x) = A(x_0 - x)$ and $B_{x_0}(x) = B(x_0 - x)$ are maximal monotone operators and satisfying: $0 \in \text{dom}A_{x_0} \cap \text{dom}B_{x_0}$ and

 $\mathbb{R}_+(\mathrm{dom}B_{x_0}-\mathrm{dom}A_{x_0})=\mathbb{R}_+(\mathrm{dom}B-\mathrm{dom}A).$

Let $x \in X$ and $\lambda > 0$, then, cf. [4], proposition 2.6 and lemma 2.6, $A + B_{\lambda}$ is a maximal monotone operator. Let u_{λ} be a solution of the inclusion $x \in u_{\lambda} + Au_{\lambda} + B_{\lambda}u_{\lambda}$.

From [4], lemma 2.5 and the fact that dom $A \cap \text{dom}B \neq \phi$, we deduce that the set $\{u_{\lambda}; \lambda > 0\}$ is bounded and included in \mathbb{R}_+ (dom*B*-dom*A*). Now from [4], theorem 2.4, we shall conclude that A + B is maximal monotone provided we show that $\sup_{\lambda > 0} ||B_{\lambda}u_{\lambda}|| < +\infty$.

Indeed, let us fix some $y \in X$, and prove that $\sup_{\lambda>0} \langle B_{\lambda} u_{\lambda}, y \rangle < +\infty$.

If $y \in \mathbb{R} + (\operatorname{dom} B - \operatorname{dom} A)$, there exists $\alpha > 0$, $a \in \operatorname{dom} A$ and $b \in \operatorname{dom} B$ such that $y = \alpha(b - a)$. Hence

$$\langle B_\lambda u_\lambda, y
angle = lpha (\langle B_\lambda u_\lambda, b
angle - \langle B_\lambda u_\lambda, a
angle).$$

Since B_{λ} is monotone then $\langle B_{\lambda}u_{\lambda} - B_{\lambda}b, u_{\lambda} - b \rangle \geq 0$ and consequently

$$\langle B_{\lambda}u_{\lambda}, y \rangle \leq \alpha(\langle B_{\lambda}u_{\lambda}, u_{\lambda} - a \rangle + ||B^{\circ}b|| \cdot ||u_{\lambda} - b||)$$

Here we use the fact $\sup_{\lambda>0} ||B_{\lambda}b|| = ||B^{\circ}b||$, see [4, prop. 2.6]. On the other hand, since $x \in u_{\lambda} + Au_{\lambda} + B_{\lambda}u_{\lambda}$, there exists $y_{\lambda} \in Au_{\lambda}$ such that $B_{\lambda}u_{\lambda} = x - u_{\lambda} - y_{\lambda}$. It follows that:

$$\langle B_{\lambda}u_{\lambda}, y \rangle \leq \alpha(\langle x - u_{\lambda} - y_{\lambda}, u_{\lambda} - a \rangle + ||B^{\circ}b|| \cdot ||u_{\lambda} - a||).$$

From the monotonicity of A we derive $\langle y_{\lambda} - A^{\circ}a, u_{\lambda} - a \rangle \geq 0$. Hence

$$\langle B_{\lambda}u_{\lambda}, y \rangle \leq \alpha(\|u_{\lambda} - a\|(\|x - u_{\lambda}\| + \|A^{\circ}a\|) + \|u_{\lambda} - b\| \cdot \|B^{\circ}b\|) = f(y)$$

Thus $\langle B_{\lambda}u_{\lambda}, y \rangle \leq f(y) < +\infty$ for every $y \in \mathbb{R}_{+}(\operatorname{dom} B - \operatorname{dom} A)$.

*If $y \notin \mathbb{R}_+(\operatorname{dom} B - \operatorname{dom} A)$, we shall have $\langle B_\lambda u_\lambda, y \rangle = 0$. Indeed, since $H = \mathbb{R}_+(\operatorname{dom} B - \operatorname{dom} A)$ is a closed linear subspace of X, then $X = H \oplus H^{\perp}$ (i.e. $H \cap H^{\perp} = \{0\}$ and $X = H + H^{\perp}$). On the other hand we have $B_\lambda u_\lambda \in B(J^B_\lambda u_\lambda)$, then $J^B_\lambda u_\lambda \in \operatorname{dom} D \subset H$, and since $u_\lambda \in H$ we get $B_\lambda u_\lambda = \frac{1}{\lambda}(u_\lambda - J^B_\lambda u_\lambda) \in H$. Hence $\langle B_\lambda u_\lambda, y \rangle = 0$, since $B_\lambda u_\lambda \in H$ and $y \in H^{\perp}$. We then have for every $y \in X$, $\sup_{\lambda>0} \langle B_\lambda u_\lambda, y \rangle < +\infty$, and from the Banach-Steinhaus theorem, we derive that $\{B_\lambda u_\lambda; \lambda > 0\}$ is bounded in X, which completes the proof of the theorem.

Remark 4.3. When dom A and dom B are convex, we can omit the assumption dom $A \cap \text{dom} B \neq \phi$, since $\mathbb{R}_+(\text{dom} B - \text{dom} A)$ is a closed linear subspace of X provided $0 \in (\text{dom} B - \text{dom} A)$.

Theorem 4.4. Under the assumptions of theorem 4.8, if we assume that $\mathbb{R}_+(co(dom B) - co(dom A))$ is a closed linear subspace, where co(dom A) is the convex hull of dom A, then A + B is still maximal monotone.

The proof of this theorem is similar to that of theorem 4.2.

Remark 4.5. It is clear that the assumption in theorem 4.4 is weaker than the condition of Rockafellar [5], Brezis [3], $int(donA) \cap donB \neq \phi$ and the condition of Attouch [1] that is: $0 \in int(domB - domA)$. More generally we can obtain the same result when $0 \in ri(co(domB - co(domA))$ (the relative interior) since this condition implies that of theorem 4.4.

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