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APPROXIMATION PROBLEMS IN MODULAR SPACES OF DOUBLE SEQUENCES

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Abstract

Let X denote the space of all real, bounded double sequences, and let Φ, φ, Γ be φ-functions. Moreover, let Ψ be an increasing, continuous function for $u \geq 0$ such that $\Psi(0) = 0$.

In this paper we consider some spaces of double sequences provided with two-modular structure given by generalized variations and the translation operator.

We apply the $\gamma(\bar{v}_\Phi, \bar{\rho}_\varphi)$-convergence in $\bar{X}(\Phi, \Psi)$ in order to obtain an approximation theorem by means of the (m, n)-translation, i.e. a result of the form $(\tau_{mn}x - x) \to 0$ in an Orlicz sequence space l^Γ.

1. Notation

1.1. A function φ defined in the interval $[0, \infty)$, continuous and nondecreasing for $u \geq 0$ and such that $\varphi(u) > 0$ for $u > 0, \varphi(u) \to \infty$ as $u \to \infty$ and $\varphi(0) = 0$, is called a φ-function. We will consider three φ-functions Φ, φ and Γ. Moreover, let Ψ be a nonnegative, nondecreasing function of $u \geq 0$ such that $\Psi(u) \to 0$ as $u \to 0+$, (see [**3**]).

1.2. Let X be the space of all real, bounded double sequences. Throughout this paper sequences belonging to X will be denoted by $x = (t_{\mu\nu}) = ((x)_{\mu\nu})$ or $(t_{\mu\nu})_{\mu,\nu=0}^{\infty} = ((x)_{\mu\nu})_{\mu,\nu=0}^{\infty}$ and $|x| = (|t_{\mu\nu}|)$, $y = (s_{\mu\nu})$, $x^p(t^p_{\mu\nu})$ for $p = 1, 2, \ldots$. By a convergent sequence we shall mean a double sequence converging in the sense of Pringsheim. The symbols X_d or X_i denote subspaces of the space X such that, for every fixed $\overline{\mu}$ and for every fixed $\overline{\nu}$ the sequences $(t_{\overline{\mu}\nu})$ and $(t_{\mu\overline{\nu}})$ are nonincreasing or nondecreasing, respectively.

1.3. Let $\rho_\varphi : X \to \langle 0, \infty)$ be a functional generated by the φ-function φ such that for arbitrary $x, y \in X$ and $\alpha, \beta \geq 0$.

1' $\rho_\varphi(0) = 0$,

1" $\rho_\varphi(x) = 0$ implies $x = 0$,

2' $\rho_\varphi(-x) = \rho_\varphi(x)$,

3' $\rho_\varphi(\alpha x + \beta y) \leq \rho_\varphi(x) + \rho_\varphi(y)$, for $\alpha + \beta = 1$,

3. Completeness of a two-modular space

3.1. We are now going to investigate the completeness of two-modular space $(\tilde{X}(\Phi,\Psi),\tilde{v}_\Phi,\tilde{\rho}_\varphi)$. The theorems on completeness of the spaces $\tilde{X}_{\rho_\varphi}$ and $\tilde{X}_\varphi(\Psi)$ with respect to the F-norm $\|\cdot\|_{\rho_\varphi}$ or the modular functional $\tilde{\rho}_\varphi$ have been obtained in [7] (compare also [5]). Let us remark that the space $\tilde{X}(\Phi,\Psi)$ is not complete with respect to $\|\cdot\|_{\rho_\varphi}$ and $\tilde{\rho}_\varphi$, respectively. Indeed, consider the following example.

Let $\Phi(u)=|u|$, $\varphi(u)=|u|$, $\Psi(u)=u^2$ and $x=(t_{\mu\nu})_{\mu,\nu=0}^\infty$, $x^p=(t^p_{\mu\nu})_{\mu,\nu=0}^\infty$, $p=1,2,\ldots$, where

$$t_{\mu\nu}=\begin{cases}\frac{1}{(\mu+1)(\nu+1)} & \text{for } \mu=\nu,\\ 0 & \text{elsewhere},\end{cases}\qquad t^p_{\mu\nu}=\begin{cases}t_{\mu\nu} & \text{for } \mu\le p \text{ and } \nu\le p,\\ 0 & \text{elsewhere}.\end{cases}$$

Since

$$\omega_\varphi(x^p;r,s)\le\sup_{m\ge r}\sup_{n\ge s}\sup_{p\ge\mu\ge m}\sup_{p\ge\nu\ge n}\frac{2}{(\mu+1)(\nu+1)}\le\frac{2}{(r+1)(s+1)},$$

$$rs\Psi\left(\omega_\varphi(x^p;r,s)\right)\le\frac{4}{(r+1)(s+1)}\to 0 \text{ as } r,s\to\infty$$

and

$$v_\Phi(x^p)=\sum_{1\le\mu,\nu\le p}(t_{\mu,\nu}+t_{\mu-1,\nu-1})=1+\frac{1}{(p+1)^2}+2\sum_{\mu=1}^{p-1}\frac{1}{(\mu+1)^2}<\infty,$$

then $x^p\in X(\Phi,\Psi)$. Further, if $r<p$ and $s<p$, we have

$$\omega_\varphi(x^p-x;r,s)\le\frac{2}{(p+1)^2},\; rs\Psi(\omega_\varphi(x^p-x;r,s))\le\frac{4}{(p+1)^2},$$

if $r\ge p$ and $s\ge p$, we have

$$\omega_\varphi(x^p-x;r,s)\le\frac{2}{(r+1)(s+1)},\; rs\Psi(\omega_\varphi(x^p-x;r,s))\le$$
$$\le\frac{4}{(r+1)(s+1)}\le\frac{4}{(p+1)^2}$$

and in consequence we obtain

$$\rho_\varphi(x^p-x)=\sup_{r,s}rs\Psi(\omega_\varphi(x^p-x;r,s))\le\frac{4}{(p+1)^2}\to 0 \text{ as } p\to\infty.$$

This shows that $x^p\to x$ in the F-norm of $X_\varphi(\Psi)$. Moreover, we have

$$rs\Psi(\omega_\varphi(x;r,s))\le\frac{4}{(r+1)(s+1)}\to 0 \text{ as } r,s\to\infty,$$

and so $x\in X_\varphi(\Psi)$. However

$$v_\Phi(x)=\sum_{\mu,\nu=1}^\infty|t_{\mu,\nu}+t_{\mu-1,\nu-1}|\ge 2\sum_{\mu,\nu=1}^\infty\frac{1}{(\mu+1)(\nu+1)}=\infty,$$

whence $x\notin X_\Phi$. Finally $x^p\in X(\Phi,\Psi)$, $\rho_\varphi(x^p-x)\to 0$ as $p\to\infty$, but $x\notin X(\Phi,\Psi)$.

3.2. In the sequel, for a given sequence $x \in X$ we define a new sequence $\overline{x} = (\overline{t}_{\mu\nu})_{\mu\nu=0}^{\infty}$ by the formulas

$$\overline{t}_{\mu\nu} = \begin{cases} t_{\mu 0} + a, & \text{for } \mu = 0,1,2,\dots \text{ and } \nu = 0, \\ t_{0\nu} + a, & \text{for } \mu = 0 \text{ and } \nu = 1,2,\dots, \\ t_{\mu\nu} + b, & \text{for } \mu \geq 1 \text{ and } \nu \geq 1, \end{cases}$$

where the constants a and b can be of the form $a = t_{\mu\nu} - t_{\mu 0}$, $b = t_{0\nu} - t_{00}$ ($\mu, \nu > 0$ are arbitrary indices). In the following we shall consider the sequence $\overline{x}$ defined by the constants $a = t_{11} - t_{10}$ and $b = t_{01} - t_{00}$.

Remark. The following identity holds $\tilde{v}_\Phi(\tilde{x}) = v_\Phi(\overline{x})$.

Proof: Since $\overline{x} \in \tilde{x}$, then by definition of $\tilde{v}_\Phi(\tilde{x})$ we have

$$(+) \qquad \tilde{v}_\Phi(\tilde{x}) \leq v_\Phi(\overline{x}).$$

Now, let $y = (s_{\mu\nu})_{\mu,\nu=0}^{\infty} \in \tilde{x}$, then $s_{\mu 0} = t_{\mu 0} + A$, $s_{0\nu} = t_{0\nu} + A$ for $\mu = 0,1,2,\dots$, $\nu = 1,2,\dots$ and $s_{\mu\nu} = t_{\mu\nu} + B$ for $\mu \geq 1$ and $\nu \geq 1$, where A and B are two arbitrary numbers. In the following we may define the sequence $\overline{y} = (\overline{s}_{\mu\nu})_{\mu,\nu=0}^{\infty}$, where $\overline{s}_{\mu 0} = t_{\mu 0} + A + a$, for $\mu = 0,1,2,\dots$, $\overline{s}_{0\nu} = t_{0\nu} + A + a$, for $\nu = 1,2,\dots$, and $\overline{s}_{\mu\nu} = t_{\mu\nu} + B + b$ for $\mu \geq 1$ and $\nu \geq 1$, with $a = t_{11} + B - t_{10} - A$ and $b = t_{01} = t_{00}$. Obviously, $v_\Phi(y) \geq v_\Phi(\overline{y})$ and $v_\Phi(\overline{y}) = v_\Phi(\overline{x})$. Hence, $v_\Phi(y) \geq v_\Phi(\overline{x})$ for every $y \in \tilde{x}$. In consequence

$$(++) \qquad \tilde{v}_\Phi(\tilde{x}) \geq v_\Phi(\overline{x}).$$

Finally, by (+) and (++) we obtain $\tilde{v}_\Phi(\tilde{x}) = v_\Phi(\overline{x})$. ■

3.3. Theorem. *Let Φ, φ be φ-functions and let Ψ be the function defined as in 1.1., which satisifies the condition:*

there exists a $u_0 > 0$ such that for every $\delta > 0$ there is an $\eta > 0$ satisfying the inequality $\Psi(\eta u) \leq \delta \Psi(u)$ for all $0 \leq u \leq u_0$.

Then, the two-modular space $\langle \tilde{X}(\Phi, \Psi), \tilde{v}_\Phi, \tilde{\rho}_\varphi \rangle$ is γ-complete.

Proof: Let us suppose that $\tilde{K}$ is a $\tilde{v}_\Phi$-ball in $\tilde{X}(\Phi, \Psi)$ and let $\tilde{x}^p \in \tilde{K}$ for $p = 1,2,\dots,(\tilde{x}^p)$ be a $\tilde{\rho}_\varphi$-Cauchy sequence. It is easily seen that the sequence $(\tilde{x}^p)$ is $\tilde{\rho}_\varphi$-convergent to an element $\tilde{x} \in \tilde{X}_\varphi(\Psi)$, (see [7] or compare [5]). In consequence $\tilde{x}^p \xrightarrow{\gamma} x$, where $\gamma = \gamma(\tilde{v}_\Phi, \tilde{\rho}_\varphi)$. Next, we show that $\tilde{x} \in \tilde{K}$. Taking the sequence (x^p), such that $x^p \in \tilde{x}^p$, $x^p \in X_\Phi$ we may define the sequence $(\overline{x}^p)$. Of course, we have

$$v_\Phi(k_0 \overline{x}^p) \leq M_0$$

for some positive numbers k_0 and M_0. If $\overline{x}^p = (\overline{t}^p_{\mu\nu})$, then

$$\sum_{\mu,\nu=1}^{\infty} \Phi\left(k_0 \left|\overline{t}^p_{m_{\mu-1},n_{\nu-1}} - \overline{t}^p_{m_{\mu-1},n_\nu} - \overline{t}^p_{m_\nu,n_{\nu-1}} + \overline{t}^p_{m_\mu,n_\nu}\right|\right) \leq M_0$$

for all increasing sequences (m_μ) and (n_ν) of positive integers and for $p = 1, 2, \dots$. Since $\bar{t}^p_{\mu\nu} \to \bar{t}_{\mu\nu}$ as $p \to \infty$ for every μ and ν, where $(\bar{t}_{\mu\nu}) = \bar{x}$, then we easily obtain

$$\sum_{\mu,\nu=1}^{\infty} \Phi\left(k_0 \left|\bar{t}_{m_{\mu-1},n_{\nu-1}} - \bar{t}_{m_{\mu-1},n_\nu} - \bar{t}_{m_\mu,n_{\nu-1}} + \bar{t}_{m_\mu,n_\nu}\right|\right) \le M_0$$

for $(m_\mu), (n_\nu)$, p as previously. Therefore $v_\Phi(k_0\bar{x}) \le M_0$. Applying the above remark, we obtain $\tilde{v}_\Phi(k_0\tilde{x}) \le M_0$, and consequently $\tilde{x} \in \tilde{K}$. ■

4. A theorem of approximation type

4.1. Let $\Phi, \varphi, \Psi, \Gamma$ be the functions defined as in part 1.1. We shall consider an Orlicz sequence space l^Γ and the space $\tilde{X}(\Phi, \Psi)$, and we shall apply the γ-convergence in $\tilde{X}(\Phi, \Psi)$ in order to formulate a theorem of the form $\tau_{mn}x - x \to 0$ in the space l^Γ.

Let us denote $T(x, m, n, \mu, \nu) = |(\tau_{mn}x)_{\mu\nu} - (x)_{\mu\nu}|$ and $M(x, m, n, \mu, \nu) = |t_{\mu+m,\nu+n} - t_{\mu+m,\nu} - t_{\mu,\nu+n} + t_{\mu,\nu}|$, for all m, n, μ, ν.

Lemma.

(a) *If $x \in X_d$, then $T(x, m, n, \mu, \nu) \le M(x, m, n, \mu, \nu)$ for all m, n, μ and ν.*
(b) *If $x \in X_i$, then $T(x, m, n, \mu, \nu) \le M(x, m, n, \mu, \nu)$ for all m, n, μ and ν.*

Proof (a): For $\mu < m$ and $\nu < n$ we have $T(x, m, n, \mu, \nu) = 0$.

If $\mu \ge m$ and $\nu < n$, then $T(x, m, n, \mu, \nu) = |t_{\mu+m,\nu} - t_{\mu,\nu}| \le |(t_{\mu,\nu+n} - t_{\mu+m,\nu+n}) + (t_{\mu+m,\nu} - t_{\mu,\nu})| = M(x, m, n, \mu, \nu)$.

If $\mu < m$ and $\nu \ge n$, then $T(x, m, n, \mu, \nu) = |t_{\mu,\nu+n} - t_{\mu,\nu}| \le |(t_{\mu+m,\nu} - t_{\mu+m,\nu+n}) + (t_{\mu,\nu+n} - t_{\mu,\nu})| = M(x, m, n, \mu, \nu)$.

For $\mu \ge m$ and $\nu \ge n$ we have $T(x, m, n, \mu, \nu) = |t_{\mu+m,\nu+n} - t_{\mu,\nu}| \le |(t_{\mu+m,\nu+n} - t_{\mu,\nu}) + (t_{\mu,\nu} - t_{\mu+m,\nu}) + (t_{\mu,\nu} - t_{\mu,\nu+n}) = M(x, m, n, \mu, \nu)$.

Finally $T(x, m, n, \mu, \nu) \le M(x, m, n, \mu, \nu)$ for all m, n, μ and ν. ■

Proof (b): For $\mu < m$ and $\nu < n$, $(\tau_{mn}x)_{\mu\nu} = t_{\mu\nu}$, then $T(x, m, n, \mu, \nu) = 0$.

If $\mu \ge m$ and $\nu < n$, then $T(x, m, n, \mu, \nu) = |t_{\mu+m,\nu} - t_{\mu,\nu}| \le |(t_{\mu,\nu} - t_{\mu+m,\nu}) + (t_{\mu+m,\nu+n} - t_{\mu,\nu+n})| = M(x, m, n, \mu, \nu)$.

If $\mu < m$ and $\nu \ge n$, then $T(x, m, n, \mu, \nu) = |t_{\mu,\nu+n} - t_{\mu,\nu}| \le |(t_{\mu,\nu} - t_{\mu,\nu+n}) + (t_{\mu+m,\nu+n} - t_{\mu+m,\nu})| = M(x, m, n, \mu, \nu)$.

For $\mu \ge m$ and $\nu \ge n$ we have $T(x, m, n, \mu, \nu) = |t_{\mu+m,\nu+n} - t_{\mu,\nu}| \le |(t_{\mu,\nu} - t_{\mu+m,\nu+n}) + (t_{\mu,\nu+n} - t_{\mu,\nu}) + (t_{\mu+m,\nu} - t_{\mu,\nu})| = M(x, m, n, \mu, \nu)$.

Thus $T(x, m, n, \mu, \nu) \le M(x, m, n, \mu, \nu)$ for all m, n, μ and ν. ■

4.2. Let us suppose that the functions Φ, φ, Γ and Ψ satisfy the following condition:

(i) There exist positive constants a, b, u_0 such that

$$\Gamma(au) \leq b\Phi(u)\Psi(\varphi(u)) \text{ for } 0 \leq u \leq u_0.$$

First let us remark that the condition (i) is equivalent to the following one:

(ii) For every $u_1 \geq 0$ there exists a constant $c > 0$ such that

$$\Gamma(cu) \leq b\Phi(u)\Psi(\varphi(u)) \text{ for } 0 \leq u \leq u_1, \text{ (for a proof see [5]).}$$

4.3. Let the functions $\Phi, \varphi, \Psi, \Gamma$ satisfy the assumptions 1.1. and 4.2., and let $v_\Phi(\lambda x) < \infty$ for a $\lambda > 0$.

Theorem 1. *If $x \in X_d$ or $x \in X_i$, then*

$$(*) \qquad \sum_{\mu,\nu}^{\infty} \Gamma\left(c\lambda \left|(\tau_{rs}x)_{\mu\nu} - (x)_{\mu\nu}\right|\right) \leq brs\Psi(\omega_\varphi(\lambda x; r, s))v_\Phi(\lambda x)$$

for all nonnegative integers r and s, where c and b are some positive constants.

Proof: We limit ourselves to the case when $x \in X_d$. By Lemma we have $|(\tau_{mn}x)_{\mu\nu} - (x)_{\mu\nu}| \leq |t_{\mu,\nu} - t_{\mu+m,\nu} - t_{\mu,\nu+n} + t_{\mu+m,\nu+n}|$ for arbitrary m, n, μ and ν. Let a positive constant λ and integers r and s be given. Since x is a bounded sequence, taking $u_1 = 4\lambda \sup_{\mu,\nu} |t_{\mu,\nu}|$, and choosing $m \geq r$, $n \geq s$ arbitrary, by (i) we obtain

$$\Gamma(c\lambda M(x,m,n,\mu,\nu)) \leq b\Phi(\lambda M(x,m,n,\mu,\nu))\Psi(\varphi(\lambda M(x,m,n,\mu,\nu)))$$

for all m, n, μ, ν such that $\lambda M(x,m,n,\mu,\nu) \leq u_1$. We have

$$\sum_{\mu,\nu=0}^{\infty} \Gamma(c\lambda|(\tau_{mn}x)_{\mu\nu} - (x)_{\mu\nu}|) \leq$$

$$\leq b\Psi(\sup_{m\geq r}\sup_{n\geq s}\sup_{\mu\geq m}\sup_{\nu\geq n}\varphi(\lambda M(x,m,n,\mu,\nu))) \sum_{\mu\geq m,\nu\geq n} \Phi(\lambda M(x,m,n,\mu,\nu)) =$$

$$= b\Psi(\omega_\varphi(\lambda x; r, s)) \sum_{k,l=1}^{\infty} \sum_{\mu=km}^{(k+1)m-1} \sum_{\nu=ln}^{(l+1)n-1} \Phi(\lambda M(x,m,n,\mu,\nu)) =$$

$$= b\Psi(\omega_\varphi(\lambda x; r, s)) \sum_{k,l=1}^{\infty} \sum_{u=m}^{2m-1} \sum_{v=n}^{2n-1} \Phi(\lambda|t_{km+u,ln+v} - t_{km+u,(l-1)n+v} -$$

$$- t_{(k-1)m+u,ln+v} + t_{(k-1)m+u,(l-1)n+v}|) =$$

$$= b\Psi(\omega_\varphi(\lambda x; r, s)) \sum_{u=m}^{2m-1} \sum_{v=n}^{2n-1} \sum_{k,l=1}^{\infty} \Phi(\lambda|t_{km+u,ln+v} - t_{km+u,(l-1)n+v} -$$

$$- t_{(k-1)m+u,ln+v} + t_{(k-1)m+u,(l-1)n+v}|) \leq$$

$$\leq b\Psi(\omega_\varphi(\lambda x; r, s)) \sum_{u=m}^{2m-1} \sum_{v=n}^{2n-1} v_\Phi(\lambda x) = bmn\Psi(\omega_\varphi(\lambda x; r, s))v_\Phi(\lambda x).$$

Finally we obtain

$$\sum_{\mu,\nu=0}^{\infty} \Gamma(c\lambda|(\tau_{mn}x)_{\mu\nu} - (x)_{\mu\nu}|) \leq bmn\Psi(\omega_{\varphi}(\lambda x; r, s))v_{\Phi}(\lambda x)$$

for some positive constants c, b, λ and for all $m \geq r$, $n \geq s$, where r, s are nonnegative integers. Hence, taking $m = r$ and $n = s$, we get the inequality $(*)$. ∎

Theorem 2. *Let Φ, φ, Γ be φ-functions (Φ convex) and let Ψ have the same properties as in the previous theorem. Let $x \in \tilde{x} \in \tilde{X}(\Phi, \Psi)$ and $x \in X_d$ (or $x \in X_i$). Then $\tau_{rs}x - x \in l^{\Gamma}$ for all $r, s \geq 0$, and $\tau_{rs}x - x \to 0$ in the sense of modular convergence in l^{Γ}.*

Proof: First, let us remark that the condition $x \in X(\Phi, \Psi)$ implies that $v_{\Phi}(\lambda x) < \infty$ and $rs\Psi(\omega_{\varphi}(\lambda x; r, s)) < \varepsilon$ for sufficiently small $\lambda > 0$ and for sufficiently large r and s, where ε is an arbitrary positive number. But, an easy computation shows that if the φ-function Φ is convex then the conditions $x \in X_{\Phi}$ and $v_{\Phi}(kx) < \infty$ for some positive constant k are equivalent. Applying this observation and Theorem 1, we conclude that $\tau_{rs}x - x \in l^{\Gamma}$ for all nonnegative integers r and s. In order to get the condition $\tau_{rs}x - x \to 0$ in the sense of modular convergence in l^{Γ}, it will be necessary to take $r, s \to \infty$, in the inequality $(*)$. ∎

Theorem 3. *Let $x^p = (t^p_{\mu\nu})_{\mu,\nu=0}^{\infty} \in X_{\Phi}$, $t^p_{\mu 0} = t^p_{0\nu} = 0$ for $p = 1, 2, \ldots$ where $\mu, \nu = 0, 1, 2, \ldots$, and let x^p, $p = 1, 2, \ldots$ belong to the v_{Φ}-ball in X_{Φ}, where Φ is an increasing φ-function. Then the set of sequences (x^p) is uniformly bounded.*

Proof: By assumption $v_{\Phi}(k_0 x^p) \leq M_0$ for $p = 1, 2, \ldots$, where k_0, M_0 are some positive constants. In consequence, we have

$$\Phi(k_0|t^p_{\mu\nu}|) = \Phi(k_0|t^p_{00} - t^p_{0\nu} - t^p_{\mu 0} + t^p_{\mu\nu}|) \leq v_{\Phi}(k_0 x^p) \leq M_0.$$

Now, applying the properties of φ-function Φ we obtain that there exists a positive constant M such that $|t^p_{\mu\nu}| \leq M$ for $\mu, \nu = 0, 1, 2, \ldots$. ∎

Theorem 4. *Let Γ, Φ, φ be φ-functions (Φ and φ are convex) and let Ψ be a nonnegative, nondecreasing function of $u \geq 0$ such that $\Psi(u) \to 0$ as $u \to 0+$. Let us suppose that the functions Φ, φ, Ψ and Γ satisfy the condition 4.2.(i). Moreover, let (x^p) be a sequence such that $t^p_{\mu 0} = t^p_{0\nu} = 0$ for $\mu, \nu = 0, 1, 2, \ldots$, $p = 1, 2, \ldots$, $x^p \in \tilde{x}^p$, $\tilde{x}^p \in \tilde{X}(\Phi, \Psi)$, $\tilde{x}^p \xrightarrow{\gamma} 0$ as $p \to \infty$ in $(\tilde{X}(\Phi, \Psi), \tilde{v}_{\Phi}, \tilde{\rho}_{\varphi})$. Then $\tau_{rs}x^p - x^p \to 0$ with respect to modular convergence in l^{Γ}, as $p \to \infty$, uniformly for $r \geq 0$ and $s \geq 0$.*

Proof: The condition $\tilde{x}^p \xrightarrow{\gamma} 0$ implies that $\tilde{x}^p \in \tilde{K}$, where $\tilde{K}$ is a $\tilde{v}_{\Phi}$-ball, with parameters k_0, M_0, and by Theorem 3 we have $|t^p_{\mu\nu}| \leq M$ for all μ, ν, p

with an $M > 0$. Choosing $u_1 = 4\lambda M$, $c = a\frac{u_0}{u_1}$, where $0 < \lambda < k_0$, and applying the inequality $(*)$, we obtain

$$(+) \qquad \sum_{\mu,\nu=0}^{\infty} \Gamma(c\lambda|(\tau_{rs}x^p)_{\mu\nu} - (x^p)_{\mu\nu}|) \le b\rho_\varphi(\lambda x^p)v_\Phi(\lambda x^p) \le bM_0\rho_\varphi(\lambda x^p).$$

By assumption there exists a $\lambda > 0$ such that for every $\varepsilon > 0$ there is an integer P for which

$$\tilde{\rho}_\varphi(2\lambda\tilde{x}^p) = \inf\{\rho_\varphi(y) : y \in 2\lambda\tilde{x}^p\} < \varepsilon$$

for all $p > P$. In consequence there exist $y^p \in 2\lambda\tilde{x}^p$, such that

$$(++) \qquad \rho_\varphi(y^p) < \varepsilon \text{ for } p > P.$$

Since

$$\rho_\varphi(\lambda x^p) = \rho_\varphi\left(\frac{y^p + (2\lambda x^p - y^p)}{2}\right) \le \rho_\varphi(y^p) + \rho_\varphi(2(\lambda x^p - \frac{1}{2}y^p))$$

and

$$\frac{1}{2}y^p - \lambda x^p \in \overline{c},$$

then we have

$$(+++) \qquad \rho_\varphi(\lambda x^p) \le \rho_\varphi(y^p), \text{ for } p > P.$$

By the inequalities $(++)$ and $(+++)$ we obtain

$$\rho_\varphi(\lambda x^p) < \varepsilon$$

for sufficiently large p. Finally, the condition $(+)$ implies that $\tau_{rs}x^p - x^p \to 0$ with respect to modular convergence in l^Γ as $p \to \infty$, uniformly for $r, s \ge 0$. ■

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