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MONOID RINGS THAT ARE FIRS

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Abstract

It is well-known that the monoid ring of the free product of a free group and a free monoid over a skew field is a fir. We give a proof of this fact that is more direct than the proof in the literature.

History. In essence, the result is due to P. M. Cohn [3],[4] who showed that over a division ring the monoid ring of a free monoid is a fir, and that the monoid ring described in the abstract is a semifir, from which it follows fairly easily that it is a fir since it is hereditary. There are four other proofs in the literature: one due to J. Lewin [7] using Schreier rewriting techniques; one due to G. M. Bergman [1] using free products; one due to P. M. Cohn and W. Dicks [6] using localization in firs; and one due to R. W. Wong [8] using only the normal form in a free group.

Definitions and notation. Let X, Y be two disjoint sets. Let M be the free product of the free monoid on X and the free group on Y. Let G be the free group on $X \cup Y$. We view $M \subseteq G$.

Each $c \in G$ has a unique *normal form* $c = c_1 c_2 \dots c_n$, such that $n \geq 0$, $c_i \in X \cup X^{-1} \cup Y \cup Y^{-1}$ and $c_i c_{i+1} \neq 1$; in this case we write $l(c) = n$. We remark that if $c \in M$ then each $c_i \in X \cup Y \cup Y^{-1}$.

Let $<$ be any well-order of $X \cup X^{-1} \cup Y \cup Y^{-1}$. We extend this to the length-lexicographic well-order $<$ of G, that is, if $c = c_1 c_2 \dots c_n$, $d = d_1 d_2 \dots d_m$ are elements of G in normal form then we write $c < d$ to mean that either $l(c) < l(d)$, or $l(c) = l(d)$ and for some j such that $1 \leq j \leq n$ we have $c_1 = d_1, \dots, c_{j-1} = d_{j-1}$ and $c_j < d_j$.

Let K be a skew field. We write $K[M]$ for the monoid ring, and view it as a subring of the group ring $K[G]$. An element x of $K[G]$ has a unique expression in the form $x = \sum_{c \in G} x(c)c$ where $x: G \to K$ is a function which takes the value 0 except for a finite number of elements of G. We shall treat interchangeably the elements of $K[G]$ and their corresponding functions. We define $\operatorname{Supp} x = \{c \in G \mid x(c) \neq 0\}$.

We have functions

$$l: K[G] \to \{-\infty\} \cup \mathbf{N}, \text{ such that } l(x) = \max\{l(c) \mid c \in \operatorname{Supp} x\},$$

$$\deg\colon K[G] \to \{-\infty\} \cup G, \text{ such that } \deg(x) = \max\{c \mid c \in \operatorname{Supp} x\},$$

where we understand $\max \emptyset = -\infty$.

It is easy to see that $l(x) = l(\deg(x))$ for all $x \neq 0$, and $\deg(K[M]) = \{-\infty\} \cup M$. We call $l(x)$ the *length* of x.

Let $x \in K[G]$ and $a \in G$. If $l(ac) = l(a) + l(c)$ for all $c \in \operatorname{Supp} x$, we shall write $a \cdot x$ to mean ax; otherwise $a \cdot x$ is undefined.

Let I be a left ideal of $K[M]$ and $x \in I$. We shall say that x is *isolated in* I if, whenever $x = \sum_{i=1}^{n} r_i x_i$, $r_i \in K[M]$, $x_i \in I$, then $\deg(x_j) \geq \deg(x)$ for some j with $1 \leq j \leq n$.

Let $a \in G$. We set

$$a \cdot G = \{b \in G \mid b = a \cdot c \quad \text{for some} \quad c \in G\}.$$

We define the *right transduction* with respect to a to be the function

$$[\ ]^a\colon G \longrightarrow \{0\} \cup G$$

such that $[a \cdot b]^a = b$ for all $a \cdot b \in a \cdot G$ and $[d]^a = 0$ for all $d \in G \setminus a \cdot G$. This extends by linearity to $K[G]$, i.e. $[\sum_{c \in G} x(c)c]^a = \sum_{c \in G} x(c)[c]^a$.

It is clear that $[K[M]]^a \subseteq K[M]$ for all $a \in G$.

Observe that if $a \in G$, $b \in G \setminus \{1\}$ such that $ab = a \cdot b$ and $x \in K[G]$ then $[ax]^{a \cdot b} = [x]^b$.

The result. First we state a lemma.

Lemma. *Let x, y be elements of $K[G]$ with $x \neq 0$, and a be a nontrivial element of G, so $a = b \cdot c$ for some $c \in X \cup Y \cup X^{-1} \cup Y^{-1}$. Then*

(i) $[yx]^a = [y]^a x - c^{-1} \cdot \sum\limits_{d \in a \cdot G} y(d)[x]^{(c \cdot [d]^a)^{-1}} + \sum\limits_{d \in G \setminus a \cdot G} y(d)[dx]^a.$

(ii) $l([yx]^a - [y]^a x + c^{-1} \cdot y(a)[x]^{c^{-1}}) < l(x).$

(iii) $l(c[yx]^a - c[y]^a x - c \cdot y(b)[x]^c) < l(x).$

Proof: Since all the expressions involved are K-linear in x and y, it suffices to consider this case where $x, y \in G$. There are three cases.

CASE 1. $y \in a \cdot G$

Here $y = a \cdot y'$ where $y' = [y]^a$ and (i) reduces to

$$[yx]^a = [y]^a x - c^{-1} \cdot [x]^{(c \cdot y')^{-1}}$$

which can be rewritten as

(i') $$[cy'x]^c = y'x - c^{-1} \cdot [x]^{(c \cdot y')^{-1}}.$$

CASE 1A. $x \in y'^{-1} \cdot c^{-1} \cdot G$. Then $cy'x \in G \setminus c \cdot G$ and both sides of (i') reduces to 0.

Since $cy'x = [x]^{(c \cdot y')^{-1}}$, (iii) holds in this case.

To see (ii) be consider the cases $y' \neq 1$ and $y' = 1$. If $y' \neq 1$, then $y \neq a$ and so $l(c^{-1} \cdot [x]^{(c \cdot y')^{-1}}) < l(x)$. If $y' = 1$ then $y = a$ and so $[y]^a x = x = c^{-1}[x]^{c^{-1}}$, (ii) holds in this case.

CASE 1B. $x \in G \setminus y'^{-1} \cdot c^{-1} \cdot G$. Then $cy'x \in c \cdot G$ and both sides of (i') reduces to $y'x$.

Moreover $y \neq b$, thus (iii) reduces to $l(0) < l(x)$.

To see (ii) consider the cases $y' \neq 1$ and $y' = 1$. If $y' \neq 1$, then $y \neq a$, and (ii) reduces to $l(0) < l(x)$. If $y' = 1$, then $[x]^{c^{-1}} = 0$ and (ii) reduces to $l(0) < l(x)$.

CASE 2. $y = b$

Here $[yx]^a = [bx]^a = [x]^c$, which gives (i) in this case, (ii) reduces to $l([x]^c) < l(x)$ and (iii) reduces to $l(0) < l(x)$.

CASE 3. $y \in G \setminus a \cdot G$ and $y \neq b$

Here (i) reduces to the triviality $[yx]^a = [yx]^a$.

CASE 3A. $[yx]^a = 0$. In this case (ii) and (iii) reduces to $l(0) < l(x)$.

CASE 3B. $[yx]^a \neq 0$. Since $y(a) = y(b) = 0$ and $[y]^a = 0$, (ii) and (iii) reduce to $l([yx]^a) < l(x)$ and $l(c[yx]^a) < l(x)$ respectively. So in this case it suffices to show that $l([yx]^a) < l(x) - 1$. Here $yx = a \cdot d$ where $d = [yx]^a$. It is easy to see that there exist $e, y', x' \in G$ such that $y = y' \cdot e$, $x = e^{-1} \cdot x'$ and $y' \cdot x' = a \cdot d$. Since $y \in G \setminus a \cdot G$ it follows that $y' \in G \setminus a \cdot G$. Hence there exists $f \in G \setminus \{1\}$ such that $a = y' \cdot f$, so $y' \cdot x' = a \cdot d = y' \cdot f \cdot d$ and $x' = f \cdot d$. Thus $l(x') - l(f) = l(d) = l([yx]^a)$. Since $l(x) = l(x') + l(e)$ we see $l([yx]^a) = l(x) - l(e) - l(f)$. Thus it suffices to show that $l(e) + l(f) \geq 2$. We know $l(f) \geq 1$. If $e = 1$ then $y = y'$ and $a = y \cdot f$, but $y \notin a \cdot G$ and $y \neq b$, thus $l(f) \geq 2$. ∎

Theorem (Lewin [7], Cohn [3]). *$K[M]$ is a fir.*

Proof: Let I be a left ideal of $K[M]$. We set

$$I^* = \{x \in I \mid x \quad \text{is isolated in} \quad I\}$$

and introduce an equivalence relation $\sim$ in I^* by defining $x \sim y$ if $\deg(x) = \deg(y)$, for all $x, y \in I^*$.

Let B be a complete set of representatives of the $\sim$-classes in I^*. We shall show that B is a left $K[M]$-basis of I.

To see that I is generated by B, let us suppose that it is not true, and choose $z \in I \setminus K[M]B$ of minimum possible degree.

If z is not isolated in I then exists an expression $z = \sum r_i z_i$ with $r_i \in K[M]$, $z_i \in I$ and $\deg(z_i) < \deg(z)$. By the minimality of the degree of z, $z_i \in K[M]B$; hence $z \in K[M]B$, a contradiction.

If z is isolated in I then exists $x \in B$ with $\deg(z) = \deg(x)$, so there exists a unique $r \in K$ such that $\deg(z - rx) < \deg(z)$. Now $z - rx \in I$ and by the minimality of the degree of z, $z - rx \in K[M]B$; hence $z \in K[M]B$, a contradiction.

These contradictions show that I is generated by B, and it remains to show that B is left $K[M]$-independent. Suppose then that it is dependent, so there exist distinct $x_1, x_2 \dots x_n$ in B and nonzero $y_1, y_2, \dots y_n$ in $K[M]$ such that $\sum_{i=1}^{n} y_i x_i = 0$.

Since the x_i are distinct elements of B, we may assume that

$$\deg(x_n) > \deg(x_{n-1}) > \dots > \deg(x_1).$$

We shall use right transduction with respect to the element $a = \deg(y_n)$. Since x_n is isolated in I, it follows that $a \neq 1$, so $a = b \cdot c$ for some $c \in X \cup Y \cup Y^{-1}$.

Consider the element

$$W = \sum_{i=1}^{n} [y_i x_i]^a - \sum_{i=1}^{n} [y_i]^a x_i = -\sum_{i=1}^{n} [y_i]^a x_i = -y_n(a) x_n - \sum_{i=1}^{n-1} [y_i]^a x_i,$$

it is clear that $W \in I$. Since x_n is isolated in I, we see $\deg(W) \geq \deg(x_n)$. By part (i) of the Lemma,

$$W = \sum_{i=1}^{n} (-c^{-1} \cdot \sum_{d \in a \cdot G} y_i(d) [x_i]^{(c \cdot [d]^a)^{-1}} + \sum_{d \in G \setminus a \cdot G} y_i(d) [d x_i]^a),$$

and by part (ii) of the Lemma,

$$l(W + c^{-1} \cdot \sum_{i=1}^{n} y_i(a) [x_i]^{c^{-1}}) < l(x_n).$$

Since $\operatorname{Supp}(y_i(a) c^{-1} \cdot [x_i]^{c^{-1}}) \subseteq \operatorname{Supp} x_i$ for all $i = 1, \dots, n$, and $\deg(x_i) < \deg(x_n)$ for all $i = 1, \dots, n-1$, thus $\deg(W) = \deg(y_n(a) c^{-1} \cdot [x_n]^{c^{-1}}) = \deg(x_n) \in c^{-1} \cdot G$. In particular $c^{-1} \in M$, and there exists a unique $r \in K$ such that $\deg(x_n - rW) < \deg(x_n)$. Now from the equation $x_n = (x_n - rW) + (rc^{-1}) cW$ and the fact x_n is isolated in I we see that $\deg(cW) \geq \deg(x_n)$. By part (i) of the Lemma,

$$cW = \sum_{i=1}^{n} (-\sum_{d \in a \cdot G} y_i(d) [x_i]^{(c \cdot [d]^a)^{-1}} + c \cdot \sum_{d \in G \setminus a \cdot G} y_i(d) [d x_i]^a),$$

and by part (iii) of the Lemma,

$$l(cW - c \cdot \sum_{i=1}^{n} y_i(b) [x_i]^c) < l(x_n).$$

Since $\operatorname{Supp}(y_i(b)c \cdot [x_i]^c) \subseteq \operatorname{Supp} x_i$ for all $i = 1, \dots, n$, and $\deg(x_i) < \deg(x_n)$ for all $i = 1, \dots, n-1$, thus $\deg(x_n) = \deg(y_n(b)c \cdot [x_n]^c) \in c \cdot G$, which contradicts the fact that $c \cdot G \cap c^{-1} \cdot G = \emptyset$.

Thus B is a basis for I, and I is free as left $K[M]$-module.

By the symmetry of the hypotheses, every right ideal is free as right module, so $K[M]$ is a fir. ∎

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