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THE LIONS'S PROBLEM FOR GUSTAVSSON-PEETRE FUNCTOR

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Abstract

The problem of coincidence of the interpolation spaces obtained by use of the interpolation method of Gustavsson-Peetre generated by (parameters) quasi-concave functions is investigated. It is shown that a restriction of this method to the class of all non-trivial Banach couples gives different interpolation spaces whenever two different parameters satisfying some conditions are used.

1. Introduction

Let $\bar{X} = (X_0, X_1)$ be a compatible couple of Banach spaces (see [1] for fundamental definitions) and let $\mathcal{F}_\alpha$ be an *interpolation functor* depending on a parameter. In the theory of interpolation spaces is well-known the problem of Lions: whether an interpolation family $\{\mathcal{F}_\alpha\}$ depends effectively on its parameter. In the complex case, i.e. $\mathcal{F}_\alpha(\bar{X}) = [\bar{X}]_\alpha$, $0 < \alpha < 1$, Stafney [8] proved that (under certain auxiliary density assumptions on the couple $\bar{X}$) if $[\bar{X}]_{\alpha_0} = [\bar{X}]_{\alpha_1}$ for some $\alpha_0 \neq \alpha_1$, then $X_0 = X_1$.

The complete answer to the Lions's problem for the real interpolation method was given in [4].

In this paper we consider the Lions's problem for interpolation spaces generated by the functor of Gustavsson-Peetre.

2. Results

Throughout this section $\mathcal{P}$ denotes the set of all *quasi-concave* functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ (φ is quasi-concave if $0 < \varphi(s) \leq \max(1, s/t)\varphi(t)$ for all $s, t > 0$). By $\mathcal{P}_0$ we denote the subset of $\mathcal{P}$ consisting of all φ with $\varphi(t) \to 0$ as $t \to 0$ and $\varphi(t)/t \to 0$ as $t \to \infty$. For a Banach couple $\bar{X} = (X_0, X_1)$ and $\varphi \in \mathcal{P}_0$ the space $\langle \bar{X}, \varphi \rangle := G_\varphi(\bar{X})$ introduced in [2] consists of all $x \in X_0 + X_1$ such that

$$x = \sum_{\nu=-\infty}^{\infty} x_\nu \text{ (convergence in } X_0 + X_1), \; x_\nu \in X_0 \cap X_1$$

and for every finite subset $F \subset \mathbf{Z}$ and every $\xi = \{\xi_\nu\}_{\nu \in \mathbf{Z}}$ with $|\xi_\nu| \leq 1$, we have

$$(*) \qquad \|\sum_{\nu \in F} \xi_\nu \frac{2^{j\nu}}{\varphi(2^\nu)} x_\nu\|_{X_j} \leq C \qquad (j = 0, 1)$$

with C independent of F and ξ. It is well-known that $G_\varphi(\bar{X})$ is a Banach space with the norm $\|x\|_\varphi = \inf C$. Moreover G_φ is an *exact interpolation functor* (see [2], [3], [6]). For the other interesting descriptions of the functor G_φ and its properties (see [3], [5], [6]). If $\varphi(t) = t^\alpha$, $0 < \alpha < 1$, then we write $G_\alpha(\bar{X})$ instead of $G_\varphi(\bar{X})$.

In this section we will show that under some conditions on φ_0 and φ_1, $G_{\varphi_0}(\bar{X}) \neq G_{\varphi_1}(\bar{X})$ provided $\bar{X}$ is a *non-trivial* couple of Banach spaces, i.e. $X_0 \cap X_1$ is non-closed subspace of $X_0 + X_1$.

First we give auxiliary results. In what follows B_X denotes the closed unit ball of a Banach space X. The following lemma is a modification of Lemma 1 of [8] (for completeness sake we give a proof).

Lemma 1. *Let $\bar{X} = (X_0, X_1)$ be a couple of Banach spaces such that $X_0 \cap X_1 \neq \{0\}$. If there exist $0 < q < 1$ and $c > 0$ such that*

$$(1) \qquad \sup_{x \in B_{X_0}} \inf_{y \in X_0 \cap cB_{X_1}} \|x - y\|_{X_0} < q,$$

then $X_0 \subset X_1$ with continuous embedding.

Proof: Let $0 \neq x_0 \in B_{X_0}$, then there exists $y_1 \in X_0 \cap cB_{X_1}$ such that $r_1 = \|x_0 - y_1\|_{X_0} \leq q$ by (1). Now put $x_1 = r_1^{-1}(x_0 - y_1)$, provided $r_1 > 0$, then $\|x_1\|_{X_0} = 1$. Similarly, we get that $r_2 = \|x_1 - y_2\|_{X_0} \leq q$ for some $y_2 \in X_0 \cap cB_{X_1}$ and $\|x_2\|_{X_0} = 1$ for $x_2 = r_2^{-1}(x_1 - y_2)$, provided $r_2 > 0$.

Since $r_1 \leq q$, so $\|x_0 - y_1 - r_1 y_2\|_{X_0} \leq r_1 q \leq q^2$.

Proceeding by induction we see that there exists a sequence $\{y_k\} \subset X_0 \cap cB_{X_1}$ such that

$$\|x_0 - y_1 - \sum_{k=1}^{n} a_k y_{k+1}\|_{X_0} \leq q^{n+1}$$

holds for $r_n = \|x_{n-1} - y_n\|_{X_0} \leq q$, $x_n = r_n^{-1}(x_{n-1} - y_n)$, where $a_n = r_1 \ldots r_n$ for $n \in N$ (without loss of generality we assume that $r_n > 0$). This implies that $x_0 = y_1 + \sum_{n=1}^{\infty} a_n y_{n+1}$ (convergence in X_0). Further $y_1 + \sum_{n=1}^{\infty} a_n y_{n+1} \in X_1$ by $\|a_n y_{n+1}\|_{X_1} \leq cq^{n+1}$. Since $\bar{X}$ is a Banach couple, $x_0 \in X_1$ and thus $X_0 \subset X_1$ with continuous embedding. ■

Lemma 2. *Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let $\varphi \in \mathcal{P}_0$. Suppose that $x = \sum_{\nu=-\infty}^{\infty} x_\nu$ (convergence in $X_0 + X_1$) satisfies the condition $(*)$. Then for each positive integer N the following hold:*

(a) The series $\sum_{\nu=-\infty}^{N} x_\nu$ *is convergent in* X_0 *and* $\|\sum_{\nu=-\infty}^{N} x_\nu\|_{X_0} \leq C\varphi(2^N)$.

(b) The series $\sum_{\nu=N}^{\infty} x_\nu$ *is convergent in* X_1 *and* $\|\sum_{\nu=N}^{\infty} x_\nu\|_{X_1} \leq C\varphi(2^N)/2^N$.

Proof: (a) Fix positive integer N and put $S_k = \sum_{\nu=-k}^{N} x_\nu$ for $k \in \mathbb{N}$. Then for $m > k$, we have

$$\|S_m - S_k\|_{X_0} = \|\sum_{\nu=-m}^{-k-1} x_\nu\|_{X_0} = \|\sum_{\nu=-m}^{-k-1} \varphi(2^\nu)(\frac{1}{\varphi(2^\nu)})x_\nu\|_{X_0}$$
$$\leq C\varphi(2^{-k-1}) \to 0 \quad \text{as } k \to \infty$$

by $(*)$ and $\varphi \in \mathcal{P}_0$. This shows that $\{S_k\}$ converges in X_0. Since

$$\|S_k\|_{X_0} \leq C\varphi(2^N)$$

for every $k \in \mathbb{N}$, it follows that $\|\sum_{\nu=-\infty}^{N} x_\nu\|_{X_0} \leq C\varphi(2^N)$. In the similar way we get the proof of (b). ■

In the sequel for given two functions $\varphi_0, \varphi_1 \in \mathcal{P}$ we write $\varphi_{01}(t) = \varphi_0(t)/\varphi_1(t)$ for $t > 0$.

Theorem 1. *Let* $\bar{X} = (X_0, X_1)$ *be a couple of Banach spaces and let* $\varphi_0, \varphi_1 \in \mathcal{P}_0$. *Then* $G_{\varphi_0}(\bar{X}) \neq G_{\varphi_1}(\bar{X})$ *provided one of the following conditions holds*

(a) $X_0 \cap X_1$ *is non-closed subspace in* X_1 *and* $\varphi_{01}(2^\nu) \to 0$ *as* $\nu \to \infty$.

(b) $X_0 \cap X_1$ *is non-closed subspace in* X_0 *and* $\varphi_{01}(2^{-\nu}) \to 0$ *as* $\nu \to \infty$.

Proof: Let $\varphi_{01}(2^\nu) \to 0$ as $\nu \to \infty$. First we show that if $G_{\varphi_0}(\bar{X}) = G_{\varphi_1}(\bar{X})$, then

$$(**) \qquad G_{\varphi_0}(\bar{X}) \subset X_0.$$

To see this take $0 < \varepsilon < 1$ and $N \in \mathbb{N}$ such that

$$(2) \qquad \varphi_{01}(2^\nu) < \varepsilon/2C$$

for $\nu \geq N$, where C is a constant of embedding $G_{\varphi_1}(\bar{X})$ into $G_{\varphi_0}(\bar{X})$.

Now let $x \in G_{\varphi_0}(\bar{X})$ with $\|x\|_{\varphi_0} \leq 1$. Then $x = \sum_{\nu=-\infty}^{\infty} x_\nu$ (convergence in $X_0 + X_1$) for some $x_\nu \in X_0 \cap X_1$ and

$$(3) \qquad \|\sum_{\nu\in F} \xi_\nu \frac{2^{j\nu}}{\varphi_0(2^\nu)} x_\nu\|_{X_j} \leq 2\|\xi\|_{\ell_\infty}, \qquad j = 0, 1$$

for every finite subset $F \subset \mathbb{Z}$ and each $\xi = \{\xi_\nu\} \in \ell_\infty$.

Put $y = \sum_{\nu=-\infty}^{N-1} x_\nu$ and $c = 2\varphi_0(2^{N-1})$, then $y \in G_{\varphi_0}(\bar{X}) \cap cB_{X_0}$ by Lemma 2(a). Define $\{u_\nu\} \subset X_0 \cap X_1$ by $u_\nu = x_\nu$ for $\nu \in A = \{\nu \in \mathbf{Z} : \nu \geq N\}$ and $u_\nu = 0$ for $\nu \in \mathbf{Z} \setminus A$. Then $x - y = \sum_{\nu=-\infty}^{\infty} u_\nu$ (convergence in X_1) by Lemma 2. Moreover

$$\begin{aligned} &\|\sum_{\nu \in F} \xi_\nu \frac{2^{j\nu}}{\varphi_1(2^\nu)} u_\nu\|_{X_j} = \|\sum_{\nu \in F \cap A} \xi_\nu \varphi_{01}(2^\nu) \frac{2^{j\nu}}{\varphi_0(2^\nu)} x_\nu\|_{X_j} \\ &\leq \varepsilon C^{-1} \|\xi\|_{\ell_\infty}, \qquad j = 0, 1 \end{aligned}$$

holds for every finite subset $F \subset \mathbf{Z}$ and each $\xi = \{\xi_\nu\} \in \ell_\infty$, by (2) and (3). Hence $\|x - y\|_{\varphi_0} \leq C\|x - y\|_{\varphi_1} \leq \varepsilon$, so (**) holds by Lemma 1.

It is easy to see that for every $0 \neq x \in X_0 \cap X_1$ and $\varphi \in \mathcal{P}_0$ we have

$$\|x\|_\varphi \leq 2\|x\|_{X_0} / \varphi(\|x\|_{X_0} / \|x\|_{X_1}). \tag{4}$$

Now suppose that $X_0 \cap X_1$ is non-closed in X_1. Then there exists a sequence $\{x_n\} \subset X_0 \cap X_1$ such that $\|x_n\|_{X_0 \cap X_1} = 1$ and $\|x_n\|_{X_1} \to 0$ as $n \to \infty$. Further, assume by way of contradiction that $G_{\varphi_0}(\bar{X}) = G_{\varphi_1}(\bar{X})$. Thus by the above established inclusion (**), it follows that $\|x\|_{X_0} \leq K\|x\|_{\varphi_1}$ for some $K > 0$ and every $x \in G_{\varphi_1}(\bar{X})$. This implies that $\varphi_1(\|x_n\|_{X_1}^{-1}) \leq 2K$ for enough large $n \in \mathbf{N}$, by (4). A contradiction, since $\varphi_1(t) \to \infty$ as $t \to \infty$, by $\varphi_{01}(2^\nu) \to 0$ as $\nu \to \infty$. Thus the proof is finished if (a) holds. If the condition (b) holds, the proof is similar. ■

Remark 1. For each Banach couple $\bar{X}$ and each $\varphi \in \mathcal{P}_0$ the space $G_\varphi(\bar{X})$ is contained in the closure of $X_0 \cap X_1$ in $X_0 + X_1$. Thus $G_\varphi(\bar{X}) = X_0 \cap X_1$ provided $\bar{X}$ is a trivial couple, i.e. $X_0 \cap X_1$ is closed subspace in $X_0 + X_1$.

Corollary 1. *If $\bar{X} = (X_0, X_1)$ is a non-trivial Banach couple, then $G_\alpha(\bar{X}) \neq G_\beta(\bar{X})$ for each $\alpha, \beta \in (0,1)$, $\alpha \neq \beta$.*

Proof: It is easy to see that $\bar{X}$ is non-trivial couple if and only if $X_0 \cap X_1$ is a non-closed subspace in X_i, $i = 0$ or 1. Thus Theorem 1 applies. ■

Remark 2. Peetre [7] defined (for the case $\varphi(t) = t^\theta$, $0 < \theta < 1$) the interpolation functor $\langle\bar{X}\rangle_\varphi$ as the space of all sums $\sum_{\nu=-\infty}^{\infty} x_\nu$ (convergence in $X_0 + X_1$) such that $\{x_\nu/\varphi(2^\nu)\}$ and $\{2^\nu x_\nu/\varphi(2^\nu)\}$ are unconditionally convergent sequences in X_0 and X_1, respectively. If we consider the Lions's problem for the functor $\langle\cdot\rangle_\varphi$, then by the same way we obtain that Theorem 1 holds for this functor.

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