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LEAVES OF MARKOV LOCAL MINIMAL SETS IN FOLIATIONS OF CODIMENSION ONE

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Abstract

The authors continue their study of exceptional local minimal sets with holonomy modeled on symbolic dynamics (called Markov LMS [C- C 1]). Here, an unpublished theorem of G. Duminy, on the topology of semiproper exceptional leaves, is extended to every leaf, semiproper or not, of a Markov LMS. Other topological results on these leaves are also obtained.

Introduction

Let M be a compact, orientable manifold, $\mathcal{F}$ a transversely orientable C^2 foliation of M of codimension one. Each component of ∂M, if there are any, is to be a leaf.

Let X be a local minimal set (LMS) of $\mathcal{F}$. That is, there is an open, connected, $\mathcal{F}$-saturated subset $U \subseteq M$ and X is a minimal set of $\mathcal{F}|U$. A special class of such sets, called Markov LMS, was introduced in [C-C 1]. These are exceptional LMS such that the holonomy pseudogroup of the foliated set $(X, \mathcal{F}|X)$ is, in a certain sense, generated by a *subshift of finite type* (see §1). We will see, in §6, that every such subshift can occur for suitable Markov minimal sets in suitable C^∞-foliated 3-manifolds.

Let $\mathcal{E}(L)$ denote the set of *ends* of a leaf $L \subset X$, a compact, totally disconnected metrizable space of ideal points of L at infinity. Let $\mathcal{E}^*(L)$ denote the closed subspace of ends that are asymptotic to L. In case X is a minimal set of $\mathcal{F}$ (*i.e.*, we take $U = M$), it is clear that $\mathcal{E}(L) = \mathcal{E}^*(L)$.

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Theorem 1. *Let $X \subset M$ be a Markov LMS of $\mathcal{F}$ and let L be any leaf of $\mathcal{F}|X$. Then $\mathcal{E}^*(L)$ is homeomorphic to the Cantor set.*

G. Duminy has proven this for *semiproper* leaves in an arbitrary exceptional LMS (unpublished). It is unknown whether his result extends to all leaves of the LMS.

Definition. A leaf L is resilient if it has a holonomy contraction and if L itself is captured by this contraction.

Definition. A handle in L is a compact, connected, nonseparating submanifold H of codimension one, $\partial H = \emptyset$. The genus of L is the maximal number of pairwise disjoint handles in L that are linearly independent in $H_*(L; \mathbb{R})$.

Theorem 2. *Let $X \subset M$ be a Markov LMS of $\mathcal{F}$. Then X contains exactly a countable infinity of resilient leaves. Furthermore, either genus$(L) = 1$, for each resilient leaf $L \subset X$, the remaining leaves of $\mathcal{F}|X$ having genus 0, or every leaf $L \subset X$ has infinite genus and every end $\epsilon \in \mathcal{E}^*(L)$ is a cluster point of handles.*

It was shown in [C-C 1] that each *semiproper* leaf in X is resilient, but that there are only finitely many such leaves.

1. Markov LMS

Let $m \geq 2$ be an integer and let $\mathcal{K}$ denote the Cantor set $\{1, 2, \dots, m\}^{\mathbf{N}}$ with typical element $\iota = (i_1, i_2, \dots)$. Let $P = (p_{ij})$ be an $m \times m$ matrix with entries from the set $\{0, 1\}$. For many choices of P, the set

$$\mathcal{K}_P = \{\iota \in \mathcal{K} \mid p_{i_j i_{j+1}} = 1, 1 \leq j \leq \infty\}$$

is also a Cantor set. In any case, the map $\sigma : \mathcal{K}_P \longrightarrow \mathcal{K}_P$, defined by

$$\sigma(i_1, i_2, i_3, \dots) = (i_2, i_3, \dots)$$

is called a *subshift of finite type*, or a *topological Markov chain* [Wa]. A subshift of finite type is locally a homeomorphism. The pseudogroup Γ_σ on $\mathcal{K}_P$, generated by the local one-one restrictions of σ, is said to be generated by the subshift σ.

If $X \subset M$ is an exceptional LMS for $\mathcal{F}$, fix $T \subset \mathbb{R}$, a finite union of open, bounded intervals, let $\overline{T}$ denote the closure of T, a compact one-manifold, and fix an imbedding $\rho : \overline{T} \hookrightarrow U$, transverse to $\mathcal{F}$ and such that $C = X \cap \rho(T)$ is a Cantor set. Here, of course, U is an open, connected, $\mathcal{F}$-saturated subset of M such that X is a minimal set for $\mathcal{F}|U$. Let Γ denote the pseudogroup on T induced by the holonomy of the foliation $\mathcal{F}$ and let $\Gamma|C$ be the induced pseudogroup on C. By an abuse, we suppress all mention of ρ hereafter.

Definition. Suppose that the above choices can be made so that there is a subshift of finite type $\sigma : \mathcal{K}_P \longrightarrow \mathcal{K}_P$ and maps $\tau : C \longrightarrow C$ and $\eta : C \longrightarrow \mathcal{K}_P$ having the following properties:

(1) τ is locally a homeomorphism and $\Gamma|C$ is generated by the local one-one restrictions of τ;
(2) η is a continuous surjection;
(3) $\eta^{-1}(\eta(x))$ is either a singleton or the pair of endpoints of a gap of C, for every $x \in C$;
(4) $\eta \circ \tau = \sigma \circ \eta$.

Then τ is said to be essentially a subshift of finite type generating $\Gamma|C$, C is called a Markov Γ-minimal set, and X is called a Markov LMS of $\mathcal{F}$.

The constructions of Sacksteder [**Sa 1**], Raymond [**Ra**], Hector [**He**], Ghys and Sergiescu [**G-S**], Inaba [**In**], and others provide examples of Markov minimal sets of a foliation $\mathcal{F}$. It is not hard to modify these to produce Markov LMS that are not minimal sets of $\mathcal{F}$.

Various elementary properties of Markov LMS were treated in [**C-C 1**,§1]. For the sake of completeness, we review these briefly.

Definition. Let $S = (\{I_1, \ldots, I_m\}, \{h_1, \ldots, h_m\}, P)$, where P is an $m \times m$ matrix of 0's and 1's, $I_j \subset \mathbb{R}$ is a compact, nondegenerate interval, $h_j : D(h_j) \longrightarrow R(h_j)$ is an orientation preserving homeomorphism between open, bounded intervals, and $I_j \subset R(h_j)$, $1 \leq j \leq m$. Set $X_j = h_j^{-1}(I_j)$. Let Γ_S denote the pseudogroup generated by $\{h_j\}_{j=1}^m$. If the following properties hold, then S is a Markov system and Γ_S is a Markov pseudogroup:

(1) $R(h_i) \cap R(h_j) = \emptyset$, $i \neq j$;
(2) $p_{ij} = 1$ implies that $I_j \subseteq X_i$;
(3) $p_{ij} = 0$ implies that $I_j \cap D(h_i) = \emptyset$.

Assume that S as above is a Markov system. Let $\{i_1, i_2, \ldots, i_n\} \subseteq \{1, 2, \ldots, m\}$ and let $p_{i_k i_{k+1}} = 1$, for $1 \leq k \leq n-1$. Then $w = h_{i_1} \circ \cdots \circ h_{i_n}$ is defined in Γ_S and $X_{i_n} \subset D(w)$. For all such elements w, set $|w| = n$ and set $I_w = w(X_{i_n})$. Finally, set

$$Z = \bigcap_{n=1}^{\infty} \left(\bigcup_{|w|=n} I_w \right)$$

$$Z_0 = Z \setminus \operatorname{int}(Z).$$

For $x \in R(h_i)$, set $t(x) = h_i^{-1}(x) \in D(h_i)$. This well defines

$$t : \bigcup_{i=1}^{m} R(h_i) \longrightarrow \bigcup_{i=1}^{m} D(h_i),$$

locally a homeomorphism, such that $t(Z_0) \subseteq Z_0$. Set

$$\tau_S = t|Z_0 : Z_0 \longrightarrow Z_0,$$

a continuous map that is locally a homeomorphism, and remark that $\Gamma_{\mathcal{S}}|Z_0$ is generated by $\tau_{\mathcal{S}}$.

Let $\iota = (i_n)_{n=1}^{\infty} \in \mathcal{K}_P$, and let w_n be defined inductively as $w_{n-1} \circ h_{i_n}$. Then $I_{w_{n+1}} \subseteq I_{w_n}$ and the set $I_\iota = \bigcap_{n=1}^{\infty} I_{w_n}$ is either a singleton or a nondegenerate, compact interval. Let $\dot{I}_\iota$ denote the set-theoretic boundary of I_ι , a set with one or two elements. It is clear that $Z = \bigcup_{\iota \in \mathcal{K}_P} I_\iota$ and $Z_0 = \bigcup_{\iota \in \mathcal{K}_P} \dot{I}_\iota$. Define $\eta_{\mathcal{S}} : Z_0 \longrightarrow K_P$ by $\eta_{\mathcal{S}}(\dot{I}_\iota) = \iota$ and remark that $\eta_{\mathcal{S}} \circ \tau_{\mathcal{S}} = \sigma \circ \eta_{\mathcal{S}}$. Thus, we can say that $\tau_{\mathcal{S}}$ is essentially a subshift of finite type.

1.1. Lemma. *The set K_P is a Cantor set and Γ_σ-minimal if and only if there exists a $\Gamma_{\mathcal{S}}$-minimal Cantor set $C \subseteq Z_0$ such that $\eta_{\mathcal{S}}(C) = K_P$. In this case $Z_0 \setminus C$ is a union of at most countably many $\Gamma_{\mathcal{S}}$-orbits, each of which accumulates exactly on C.*

For the elementary proof, see [**C-C 1**, (1.1)]. If, in fact, $\Gamma_{\mathcal{S}}$ is of class at least C^2 , it is true that $Z_0 \setminus C$ is a union of at most *finitely* many $\Gamma_{\mathcal{S}}$-orbits, but this follows from a much more difficult theorem [**C-C 1**, (6.1)].

1.2. Lemma. *Let X be a Markov LMS of $\mathcal{F}$and let C, Γ, P, K_P, τ, and η be as in the definition of such a set. Then this data can be chosen so that there exists a Markov sub-pseudogroup $\Gamma_{\mathcal{S}} \subseteq \Gamma$, defined relative to the matrix P, such that $C \subseteq Z_0$, $\eta_{\mathcal{S}}|C = \eta : C \longrightarrow \mathcal{K}_P$, $\tau_{\mathcal{S}}|C = \tau$, and $\Gamma_{\mathcal{S}}|C = \Gamma|C$.*

This was essentially proven in [**C-C 1**, (1.2)]. The relatively minor adjustments that are needed for the above formulation are left as an exercise.

Let $X \subset U$ be a Markov LMS. Thus $C = X \cap T$ is a Markov Γ-minimal set. We fix a Markov sub-pseudogroup $\Gamma_{\mathcal{S}} \subseteq \Gamma$ as in (1.2).

Let $x \in C$ and let $|\Gamma_{\mathcal{S}}(x)|$ denote the digraph of the orbit $\Gamma_{\mathcal{S}}(x)$ relative to the set $\{h_1, \ldots, h_m\}$ of generators. Here, each positively oriented edge denotes an application of some h_i, $1 \leq i \leq m$, hence moving backward along an edge denotes an application of τ. By elementary symbolic dynamics, it is easy to see that this graph contains at most one cycle and that, if $v \in |\Gamma(x)|$ is a vertex not on a cycle, every positively directed edgepath out of v of infinite length meets no cycle. Moreover, infinitely many vertices of such an edgepath are"branch points", in the sense that at least two distinct, positively directed, infinitely long edgepaths of $|\Gamma_{\mathcal{S}}(x)|$ emerge from each of them. These observations have certain easy consequences.

1.3. Lemma. *The space $\mathcal{E}(|\Gamma_{\mathcal{S}}(x)|)$ of ends of the graph $|\Gamma_{\mathcal{S}}(x)|$ is a Cantor set.*

We will prove Theorem 1 by showing that, under the hypotheses of that theorem, if L_x is the leaf of $\mathcal{F}|X$ passing through $x \in T \cap X = C$, then there is a homeomorphism $\mathcal{E}(|\Gamma_{\mathcal{S}}(x)|) \cong \mathcal{E}^*(L_x)$.

Definition. For $x \in C$, let $\Gamma_x \subset \Gamma$ denote the isotropy pseudogroup of x. The group

$$\mathcal{H}_x(L_x, X) = \mathcal{H}_x(\Gamma(x), C) = \mathcal{H}_x(\Gamma_{\mathcal{S}}(x), C)$$

of germs at x of elements $\gamma \in \Gamma_x|C$ is called the holonomy group (at x) either of the leaf L_x relative to X or of the orbit $\Gamma(x)$ relative to C.

1.4. **Lemma.** *The holonomy group $\mathcal{H}_x(\Gamma_{\mathcal{S}}(x), C)$ of $\Gamma(x)$ relative to C is either trivial or infinite cyclic and generated by a contraction that is unique in a suitable neighborhood of x in C. The nontrivial cases are precisely the ones in which $|\Gamma_{\mathcal{S}}(x)|$ contains a cycle, the contraction being the holonomy produced by a positively directed loop around this cycle.*

The proof is an exercise in elementary symbolic dynamics and, at any rate, will be found in [C-C 1, (3.2)].

1.5. **Corollary.** *A leaf L of $\mathcal{F}|X$ is resilient if and only if the holonomy group of L relative to X is infinite cyclic.*

Our proof of Theorem 2 will show that the above holonomy contractions are compactly supported. Such a contraction determines a compactly supported cocycle on the leaf L that is nontrivial in $H^*(L; \mathbb{Z})$. That is, if the graph $|\Gamma_{\mathcal{S}}(x)|$ has a cycle, that cycle is Poincaré dual to a handle on the leaf through x.

Remark that so far in this section we have not needed smoothness hypotheses. For the following, smoothness of class at least C^2 seems to be necessary.

1.6. **Lemma.** *If $\mathcal{F}$ is of class C^2 and $X \subset M$ is a Markov LMS, then the Markov system $\mathcal{S}$ of (1.2) can be chosen so that I_ι is a singleton, for each $\iota \in K_P$. In particular, $\eta : C \longrightarrow K_P$ is a homeomorphism.*

Proof: We modify the Markov system $\mathcal{S}$ of (1.2). If I_ι is a nondegenerate interval, then [C-C 1,(3.4)] implies that $\iota = (J_0, J_1, J_1, \ldots, J_1, \ldots)$, where J_0 and J_1 are finite sequences of elements of $\{1, 2, \ldots, m\}$. By applying a suitable nonnegative power of τ to I_ι, we may assume that $J_0 = \emptyset$. We set $J_1 = (j_1, j_2, \ldots, j_q)$. If I_{j_k} and $\tau^{k-1}(I_\iota)$ share an endpoint x, delete $\{x\} \cup \tau^{k-1}(\mathrm{int}(I_\iota))$ from I_{j_k}. Otherwise, delete the interval $\tau^{k-1}(\mathrm{int}(I_\iota))$ from I_{j_k}. Similarly, delete $\{\tau(x)\} \cup \tau^k(\mathrm{int}(I_\iota))$ or $\tau^k(\mathrm{int}(I_\iota))$ from X_{j_k}. These operations should be performed for $1 \leq k \leq q$. This breaks each I_{j_k} and X_{j_k} into a finite number of subintervals $I_{j_k 1}, \ldots, I_{j_k m_k}$ and $X_{j_k 1}, \ldots, X_{j_k m_k}$ (remark that possibly $j_k = j_l$, $k \neq l$). The generator h_{j_k} of $\Gamma_{\mathcal{S}}$ is then broken into $h_{j_k i} : X_{j_k i} \longrightarrow I_{j_k i}$, $1 \leq i \leq m_k$, $1 \leq k \leq q$. It is elementary to check that this modification of $\mathcal{S}$ is again a Markov system for the Markov LMS X. Finally, again because of the C^2 hypothesis, there are only finitely many $\Gamma_{\mathcal{S}}$-orbits in Z_0 that are semiproper [C-C 1, (6.1)], so the above procedure only needs to be repeated finitely often to produce the desired Markov system. ∎

1.7. Corollary. *If $\mathcal{F}$ is of class C^2, if $X \subset M$ is a Markov LMS, and if $\epsilon > 0$ is given, then the Markov system S of (1.2) can be modified to a Markov system S^*, for the same LMS X, such that the intervals X_i^* and I_i^* are all of length less than ϵ and each generator h_i^* is the restriction of a suitable generator $h_{j(i)}$ from S.*

Proof: Apply [C-C 1, (4.1)] in conjunction with (1.6). ■

2. The combinatorial arguments.

The dynamical description of $\mathcal{F}$ is given via a suitable choice of open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of M by Frobenius coordinate charts. Each $\overline{U}_\alpha$ is to lie in the interior of some Frobenius chart and an $\mathcal{F}$-plaque in U_α is to meet at most one $\mathcal{F}$-plaque in U_β, for all $\alpha, \beta \in \mathcal{A}$.

The set of (closed) $\mathcal{F}$-plaques in $\overline{U}_\alpha$ can be identified with a compact, imbedded, $\mathcal{F}$-transverse arc $R_\alpha \hookrightarrow M$, the preimage R_α being, itself, a compact subinterval of $\mathbf{R}$. We arrange that these subintervals be disjoint in $\mathbf{R}$ with disjoint images in M. If $P_\alpha \in R_\alpha$ and $P_\beta \in R_\beta$ are interpreted as closed plaques, we will write

$$\gamma_{\beta\alpha}(P_\alpha) = P_\beta$$

if and only if P_α and P_β overlap in the sense that

$$\operatorname{int}(P_\alpha) \cap \operatorname{int}(P_\beta) \neq \emptyset.$$

This defines the set of generators $G_\mathcal{F} = \{\gamma_{\beta\alpha}\}_{\alpha,\beta \in \mathcal{A}}$ of the holonomy pseudogroup $\Gamma_\mathcal{F}$ of $\mathcal{F}$, a pseudogroup on the compact one-manifold $R_\mathcal{F} = \bigcup_{\alpha \in \mathcal{A}} R_\alpha$. Thus, a word of length n in these generators, when applied to a plaque $P \in R_\mathcal{F}$, amounts to a chain $(P = P_0, P_1, \ldots, P_n)$ of plaques such that P_{i-1} and P_i overlap, $1 \leq i \leq n$.

Let X be a Markov LMS of $\mathcal{F}$. We can assume that the imbedding $\overline{T} \hookrightarrow U$, relative to which X is of Markov type, is the imbedding of a compact submanifold $\overline{T} \subseteq R_\mathcal{F}$ induced by $R_\mathcal{F} \hookrightarrow M$.

By (1.2), there is a set $G_S = \{h_i\}_{1 \leq i \leq m}$, generating a Markov sub-pseudo group $\Gamma_S \subseteq \Gamma_\mathcal{F}$, such that $\Gamma_\mathcal{F}|C = \Gamma_S|C$. Each h_i is locally a composition of $\gamma_{\alpha\beta}$'s. By (1.7), we can break each h_i up into finitely many of its restrictions to disjoint, open subintervals of $D(h_i)$, on each of which h_i is a pure composition of $\gamma_{\alpha\beta}$'s. Thus, *wlog*, we assume that each h_i is a reduced word in the generators $G_\mathcal{F}$.

We must take into account the structure of the foliated manifold $(\widehat{U}, \widehat{\mathcal{F}})$ obtained by completing $(U, \mathcal{F}|U)$ relative to any Riemannian metric inherited from M [**Di**]. Let

$$\widehat{U} = K \cup V_1 \cup \cdots \cup V_r$$

be a decomposition into a compact *nucleus* K (a foliated manifold with convex corners) and *arms* V_i (foliated interval bundles), as described in [**Di**] (also, see [**C-C 2**], [**H-H**], [**Go**]). Let $\mathcal{B} \subseteq \mathcal{A}$ be the subset such that $\overline{U}_\alpha \subset \text{int}(K)$ if and only if $\alpha \in \mathcal{B}$. Let $R = \bigcup_{\alpha\in\mathcal{B}} R_\alpha$.

The nucleus can be chosen large enough to engulf any specified compact subset of U. Thus, without loss of generality, we can assume that the Frobenius cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ has been chosen so that $\overline{T} \subseteq R$ and the expressions for the generators h_i of $\Gamma_\mathcal{S}$ as words in the generators $\gamma_{\alpha\beta}$ of $\Gamma_\mathcal{F}$ involve only indices $\alpha, \beta \in \mathcal{B}$. Let $G = \{\gamma_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{B}}$ and let Γ be the pseudogroup on R generated by the set G. Thus, $\Gamma_\mathcal{S}|C = \Gamma|C$.

Definition. The foliated manifold $(M', \mathcal{F}')$ is defined by setting $M' = \bigcup_{\alpha\in\mathcal{B}} \overline{U}_\alpha$ and $\mathcal{F}' = \mathcal{F}|M'$. The subset $X' \subset M'$ is the $\mathcal{F}'$-saturation of C.

Remark. Even if the foliated manifold $(M, \mathcal{F})$ has some degree of smoothness, $\partial M'$ is only piecewise smooth, being divided by corners of various descriptions into smooth pieces that are each either transverse or tangent to $\mathcal{F}'$.

In light of our discussion so far, the following is an exercise.

2.1. Lemma. *The set X' is an exceptional minimal set of $\mathcal{F}'$. Each leaf L' of $\mathcal{F}'|X'$ is contained in a unique leaf L of $\mathcal{F}|X$ and the correspondence $L \mapsto L'$ is one-one between the leaves of $\mathcal{F}|X$ and of $\mathcal{F}'|X'$.*

For $\delta \in \mathcal{B}$ and $x \in R_\delta \cap X'$, we can choose $\gamma_x \in \Gamma$ to be a word of shortest length in the elements of G such that $\gamma_x(x) \in C$. By compactness of R_δ and of $R_\delta \cap X'$, there are finitely many compact subintervals with interiors covering $\text{int}(R_\delta)$, each meeting X' in a set that lies entirely in the domain of some such γ_x and is carried by γ_x into C. That is, as δ ranges over $\mathcal{B}$ and x ranges over $R_\delta \cap X'$, we can arrange that only finitely many γ_x are distinct. Remark that, if $x \in C$, then γ_x is an identity.

Finally, whenever $\alpha, \beta \in \mathcal{B}$ and $x \in D(\gamma_{\alpha\beta}) \cap X'$, we can write

$$\gamma_{\alpha\beta}(x) = y = \gamma_y^{-1} \circ w^x_{\alpha\beta} \circ \gamma_x(x),$$

where γ_y and γ_x are as above and $w^x_{\alpha\beta}$ is a reduced word in the generators $G_\mathcal{S}$. Again, by compactness, there will only be finitely many distinct words $w^x_{\alpha\beta}$ that occur as α and β range over $\mathcal{B}$ and x ranges over $D(\gamma_{\alpha\beta})$.

Definition. For $w \in \Gamma_\mathcal{S}$ a pure reduced word in the generators $G_\mathcal{S}$, the length of that word is $\|w\|_\mathcal{S}$. Similarly, for a pure reduced word $\gamma \in \Gamma$ in the generators G, the length is denoted by $\|\gamma\|$.

It is clear from the above discussion that we can fix a choice of $N \in \mathbf{Z}^+$ that is simultaneously an upper bound to each of the following :

(a) $\|\gamma_x\|$, for each $x \in X \cap R$;
(b) $\|h_j\|$, $1 \leq j \leq m$;
(c) $\|w^x_{\alpha\beta}\|_\mathcal{S}$, for each $x \in C$ and for all $\alpha, \beta \in \mathcal{B}$.

Remark also that this implies

(d) $\|w^z_{\alpha\beta}\| \leq N^2$.

Let L be a leaf of $\mathcal{F}|X$, $x \in L \cap T$, and let L' be the corresponding leaf of $\mathcal{F}'|X'$ as in (2.1). Then the graphs $|\Gamma(x)|$ and $|\Gamma_S(x)|$ will be defined relative to the generators G and G_S, respectively, and so will be quite different. Also, since $\gamma_{\beta\alpha} = \gamma^{-1}_{\alpha\beta}$, we will agree that both $\gamma_{\beta\alpha}$ and $\gamma_{\alpha\beta}$ are to label the same edge with opposite directions. By contrast, each edge of $|\Gamma_S(x)|$ has been given a unique label h_j and a corresponding preferred direction (*cf.* §1).

The graph $|\Gamma(x)|$ is the 1-skeleton of the nerve of the plaque-cover of L', hence the first of the following lemmas is evident.

2.2. Lemma. *The spaces $\mathcal{E}(L')$ and $\mathcal{E}(|\Gamma(x)|)$ are homeomorphic.*

2.3. Lemma. *The spaces $\mathcal{E}(|\Gamma(x)|)$ and $\mathcal{E}(|\Gamma_S(x)|)$ are homeomorphic.*

Proof: For $\alpha, \beta \in \mathcal{B}$, let $z \in L \cap D(\gamma_{\alpha\beta})$ and let $y = \gamma_{\alpha\beta}(z)$. Recall that

$$\gamma_{\alpha\beta}(z) = \gamma_y^{-1} \circ w^z_{\alpha\beta} \circ \gamma_z(z),$$

where $\|\gamma_z\|$, $\|\gamma_y^{-1}\|$, and $\|w^z_{\alpha\beta}\|_S$ are all bounded above by the integer N and $\|w^z_{\alpha\beta}\|$ is bounded above by N^2.

The vertices of $|\Gamma_S(x)|$ are contained in the set of vertices of $|\Gamma(x)|$ and this inclusion extends to a mapping $|\Gamma_S(x)| \longrightarrow |\Gamma(x)|$, not of graphs but of topological spaces. Indeed, an edge of $|\Gamma_S(x)|$ labeled by h_j is carried onto an edgepath in $|\Gamma(x)|$ corresponding to the expression for h_j in terms of the generating set G.

The above mapping induces $\lambda : \mathcal{E}(|\Gamma_S(x)|) \longrightarrow \mathcal{E}(|\Gamma(x)|)$, a continuous map. This is a surjection since every vertex of $|\Gamma(x)|$ can be joined to a vertex of $|\Gamma_S(x)|$ by an edgepath in $|\Gamma(x)|$ of length at most N.

We must show that λ is one-one. Let ϵ and ϵ' be distinct elements of $\mathcal{E}(|\Gamma_S(x)|)$. Let $\{x_k\}_{k=1}^{\infty}$ and $\{x'_k\}_{k=1}^{\infty}$ be sequences of vertices of $|\Gamma_S(x)|$ converging to ϵ and ϵ' respectively. Then, there is a vertex x_* of this graph such that, for k sufficiently large, any edgepath in $|\Gamma_S(x)|$ joining x_k and x'_k must pass through x_*. Indeed, x_* will be one of the points at which the graph "branches".

Let $D(x_*, \nu) \subset |\Gamma(x)|$ be the set of vertices that can be joined to x_* by an edgepath of this graph of length at most ν. In order to show that $\lambda(\epsilon) \neq \lambda(\epsilon')$, it will be sufficient to find an integer $\nu > 0$ such that, for k sufficiently large, every edgepath in $|\Gamma(x)|$ joining x_k and x'_k must meet $D(x_*, \nu)$. For simplicity of notation, let $z = x_k$ and $z' = x'_k$. An edgepath joining these points corresponds to a word

$$\gamma = \gamma_{\alpha_q\alpha_{q-1}} \circ \cdots \circ \gamma_{\alpha_1\alpha_0}$$

such that $\gamma(z) = z'$. Let $z(0) = z$ and

$$z(p) = \gamma_{\alpha_p\alpha_{p-1}} \circ \cdots \circ \gamma_{\alpha_1\alpha_0}(z),$$

$1 \leq p \leq q$. Since $z \in C$ and $z' = z(q) \in C$, we see that γ_z and $\gamma_{z'}$ are identities, hence that

$$\gamma(z) = w(z) = w^{z(q-1)}_{\alpha_q\alpha_{q-1}} \circ \cdots \circ w^{z(0)}_{\alpha_1\alpha_0}(z).$$

Here, each $w^{z(i-1)}_{\alpha_i\alpha_{i-1}}$ is a word in the generators G_S, hence the above expression represents an edgepath in $|\Gamma_S(x)|$ joining z and z' and, *as such*, passes through x_*. Let us say that $w^{z(p-1)}_{\alpha_p\alpha_{p-1}}$ represents a segment meeting x_*. It follows that $z(p) \in D(x_*, N^2 + N)$.

Since endsets are compact Hausdorff spaces, we conclude that λ is a homeomorphism. ■

2.4. Corollary. *The space $\mathcal{E}(L')$ is homeomorphic to the Cantor set.*

Proof: By (2.2) and (2.3), there is a homeomorphism

$$\mathcal{E}(L') \cong \mathcal{E}(|\Gamma_S(x)|)$$

and, by (1.3), the space $\mathcal{E}(|\Gamma_S(x)|)$ is a Cantor set. ■

2.5. Corollary. *If X is a Markov minimal set of $\mathcal{F}$, then the conclusion of Theorem 1 is true.*

Indeed, we can take $U = M = M'$, $\mathcal{F} = \mathcal{F}'$, $X = X'$, $L = L'$ and apply (2.4).

2.6. Lemma. *If $X \subset M$ is a Markov LMS of $\mathcal{F}$, then X contains exactly a countable infinity of resilient leaves.*

Proof: By (1.1), K_P is a Γ_σ-minimal Cantor set. An elementary consequence is that exactly a countable infinity of $(i_1, i_2, \ldots, i_k, \ldots)$ are of the periodic form $(J, J, \ldots, J, \ldots)$ J being a finite string of integers from $\{1, \ldots, m\}$. By (1.4) and (1.5), the Markov LMS X contains exactly a countable infinity of resilient leaves. ■

2.7. Lemma. *Let $X \subset M$ be a Markov LMS of $\mathcal{F}$, let $L \subset X$ be a resilient leaf, and let $x \in L \cap T \subset C$. Then, the holonomy contraction that represents the generator of $\mathcal{H}_x(L', X') = \mathcal{H}_x(L, X) \cong \mathbf{Z}$ is compactly supported on L'.*

Proof: We can assume that $x \in |\Gamma_S(x)|$ lies on the unique cycle w of that graph and we view w as an edgeloop based at x. let $D(w, \nu) \subset |\Gamma(x)|$ denote the set of vertices that can be joined to a vertex of w by an edgepath in this

It is a consequence of Theorem 3 that, for a suitable (open) normal neighborhood W of F in $\widehat{U}$, each component $\widetilde{F}$ of $X \cap \overline{W}$ is diffeomorphic to F_∞ in such a way that the projection $p : \widetilde{F} \longrightarrow F$, defined by projection along the normal fibers in $\overline{W}$, is identified with $\pi : F_\infty \longrightarrow F$. Here $\overline{W}$ will be diffeomorphic to the manifold Q which is obtained from $F_\natural \times [x, b]$ by the identification map

$$\varphi : H_+ \times [x, b] \longrightarrow H_- \times [x, b],$$

$$\varphi(x_+, t) = (x_-, f(t)).$$

The manifold Q is naturally foliated by leaves diffeomorphic to F_∞ and the natural projection $\rho : Q \longrightarrow F$ restricts to π on each of these leaves. The diffeomorphism $\overline{W} \cong Q$ carries each component of $X \cap \overline{W}$ onto one of these leaves and identifies ρ with projection along the normal fibers of $\overline{W}$. This picture and the proof are, by now, standard in geometric foliation theory. For example, see [C-C 2, §6] for the completely analogous situation of totally proper leaves winding in on leaves at lower levels. The situation there is precisely the one described here, but with X replaced by a single leaf.

Let $F_1, \ldots, F_q$ be the components of $\partial\widehat{U}$ on which X accumulates, let $H_1, \ldots, H_q$ be the respective handles, and let $W_1, \ldots, W_q$ be the corresponding normal neighborhoods just described. Let $F_{q+1}, \ldots, F_p$ be the remaining components of $\partial\widehat{U}$ and $W_{q+1}, \ldots, W_p$ respective normal neighborhoods of these in $\widehat{U}$ that do not meet X. We can assume that, for $q + 1 \leq j \leq p$, $\overline{W}_j \cap K$ is a finite union of closed Frobenius charts for $\widehat{\mathcal{F}}$. Finally, let $M' = K \setminus \bigcup_{1 \leq i \leq p} W_i$. By Theorem 3, $\mathcal{F}|X$ has no holonomy outside of M'. At this point, the following is clear.

3.1. Claim. *The Frobenius cover $\{U_\alpha\}_{\alpha \in A}$ for the foliated manifold $(M, \mathcal{F})$ can be chosen so that the manifold M', as described above, coincides with the manifold M' constructed in §2.*

As in §2, we set $\mathcal{F}' = \mathcal{F}|M'$, a foliation with exceptional minimal set X'. Let L be a leaf of $\mathcal{F}|X$ and let L' be the unique leaf of $\mathcal{F}'|X'$ contained in L (2.1). Because the foliated manifold $(M', \mathcal{F}')$ has been fashioned with greater care than in §2, the leaf L' is a manifold with boundary, but with no corners. Indeed, (M', F') is a foliated manifold with corners of various descriptons, and these corners divide $\partial M'$, as usual, into a part tangent to $\mathcal{F}$, denoted $\partial_\tau M'$, and a part transverse to $\mathcal{F}$, denoted $\partial_\pitchfork M'$. The components of $\partial L'$ are exactly the components of $L \cap \partial_\pitchfork M'$, each being a copy either of H_i, $1 \leq i \leq q$, or of ∂B_j, $1 \leq j \leq r$. There are infinitely many such components. Finally, each component of $L \setminus \mathrm{int}(L')$ has as boundary exactly one component of $\partial L'$ and either lies entirely in W_i, $1 \leq i \leq q$, or in some V_j, $1 \leq j \leq r$.

3.2. Proposition. *The inclusion $L' \hookrightarrow L$ induces an imbedding $\lambda : \mathcal{E}(L') \hookrightarrow \mathcal{E}(L)$ with image $\mathcal{E}^*(L)$.*

Proof: Since $\mathcal{E}(L') = \mathcal{E}^*(L')$, the image of λ is contained in $\mathcal{E}^*(L)$. We must show that λ maps $\mathcal{E}(L')$ one-one onto $\mathcal{E}^*(L)$.

Let ϵ and ϵ' be distinct ends of L'. Let $\Delta \subset \operatorname{int}(L')$ be a compact subset separating ϵ and ϵ'. Since the components of $L \setminus \operatorname{int}(L')$ correspond one-one to the components of $\partial L'$, it is clear that Δ, as a compact subset of L, also separates $\lambda(\epsilon)$ and $\lambda(\epsilon')$, hence these ends are distinct in $\mathcal{E}^*(L)$.

Let $\epsilon \in \mathcal{E}^*(L)$. Let $\cdots \subset U_k \subset U_{k-1} \subset \cdots \subset U_1 \subset L$ be a fundamental system of neighborhoods of ϵ. Since $\emptyset \neq U_k \cap M' = U_k \cap L'$ (the second equality is a consequence of (2.1)) we can choose $x_k \in U_k \cap L'$. By passing to a subsequence, if necessary, we can assume that the sequence $\{x_k\}_{k=1}^{\infty}$ converges to an end $\eta \in \mathcal{E}(L')$. It follows that $\lambda(\eta) = \epsilon$. ■

3.3. Corollary. *Theorem 1 is true.*

Proof: Apply (2.4) and (3.2). ■

3.4. Proposition. *Let L be a resilient leaf of $\mathcal{F}|X$. Then the holonomy contraction that represents the generator of $\mathcal{H}_x(L, X) \cong \mathbb{Z}$ is compactly supported on the leaf L.*

Proof: As usual, the contraction on L defines a nontrivial cohomology class $\theta \in H^1(L; \mathbb{Z})$. If we view this class as a homomorphism

$$\theta : \pi_1(L, x) \longrightarrow \mathbb{Z},$$

we see, by the geometric consequences of Theorem 3, that it vanishes on any loop that is freely homotopic to a loop on $L \setminus \operatorname{int}(L')$. Thus, θ is supported in $\operatorname{int}(L')$ where, by (2.7), it is compactly supported. ■

At this point, the proof of (2.8) applies without change.

3.5. Corollary. *Theorem 2 is true.*

4. The fundamental contraction.

Let X be an exceptional LMS, not necessarily of Markov type, and let F be any proper or semiproper leaf of the foliation $\mathcal{F}$. By transverse orientability, F has two sides, denoted F^+ and F^-. Assume that X is asymptotic to F on at least one side, say on F^+. Let $0 \in F$ and let $[0, \epsilon]$ be a parametrized, $\mathcal{F}$-transverse arc, issuing from 0 on the side F^+. Let $]a, b[$ be a component of $]0, \epsilon[\setminus X$ such that $[a, b] \subset]0, \epsilon[$.

As in §3, h_σ denotes holonomy in $[0, \epsilon]$ defined by a loop σ on F, based at 0, and $\sigma * H$ denotes the homological intersection number of σ with a handle H.

Definition. Let $\sigma : [0, 1] \longrightarrow L$ be a loop based at $\sigma(0) = \sigma(1) \in H$. If $\sigma|\,]0, 1[$ is in general position with respect to H, if $\sigma|\,]0, \delta]$ and $\sigma|\,[1 - \delta, 1[$ both lie on the same side of H for small δ, and if σ always crosses H in the same direction, then we will say that σ is in normal position with respect to H.

4.1. Theorem.(Duminy). *If $\epsilon > 0$ is sufficiently small, then there is a handle $H \subset F$ and a loop σ on F, based at 0, such that $\sigma * H = 1$ and $h_\sigma = f$ is a contraction of $[0, \epsilon]$ to 0. Furthermore, for some integer $\kappa \geq 0$ and every loop τ on F, based at 0 and in normal position with respect to H,*

$$h_\tau(f^k(b)) = f^{k+r*H}(b),$$

for each integer $k \geq \kappa$, provided that $h_\tau(f^k(b))$ is defined and $\leq b$.

Remark. If F happens to be a semiproper leaf that lies in X itself and if F^+ is the *nonproper* side of F, this theorem leads rather easily to the result that $\mathcal{E}^*(F)$ contains no isolated points, hence is a Cantor set.

Let U be an open, connected, $\mathcal{F}$-saturated set such that X is a minimal set of $\mathcal{F}|U$. Let F be a component of $\partial\widehat{U}$ on which X accumulates in $\widehat{U}$. Our main application of (4.1) will be to this situation, the side F^+ being the side bordered by U, so F^+ will be a *proper* side of F. The conclusion of (4.1) will be our starting point for the proof of Theorem 3.

Unfortunately, Duminy has never made his proof of (4.1) available to the mathematical public. The result is crucial for current research into the structure of exceptional minimal sets, so we have prepared an account [**C-C 3**] for informal circulation.

Definition. The handle $H \subset F$, given by (4.1), will be called a holonomy handle.

5. The proof of Theorem 3

As in §3, the decomposition $\widehat{U} = K \cup V_1 \cup \cdots \cup V_r$ induces a decomposition $F = A \cup B_{j_1} \cup \cdots \cup B_{j_t}$, where F is a component of $\partial\widehat{U}$. By suitably renumbering, we take $B_{j_i} = B_i$, $1 \leq i \leq t$.

Let $x_i \in B_i$ and let J_i be the fiber over x_i of the interval bundle $V_i \longrightarrow B_i$, $1 \leq i \leq t$. Let P_0 denote an $\mathcal{F}$-plaque containing the point $0 \in A$. Edgeloops on $|\Gamma_{\mathcal{F}}(0)|$, based at 0, induce a holonomy sub-pseudogroup $\Gamma_F \subset \Gamma_{\mathcal{F}}$, defined on open neighborhoods of 0 in $[0, \epsilon[$ and fixing 0. The usual holonomy group $\mathcal{H}_0(F)$ is the group of germs at 0 of the elements of Γ_F.

We say that a chain $\rho = (P_0, P_1, \ldots, P_q)$ of plaques, without repetitions, is a *simple chain* at P_0, as is the holonomy element $h_\rho \in \Gamma_{\mathcal{F}}$ that it induces. A chain $\sigma = (P_0, P_1, \ldots, P_q, P_0)$ and the associated $h_\sigma \in \Gamma_F$ will be called a *simple loop* at P_0 if $(P_0, P_1, \ldots, P_q)$ is a simple chain. Finally, if $\rho = (P_0, \ldots, P_q)$ is a simple chain and $\sigma = (P_q, P_{q+1}, \ldots, P_{q+s}, P_q)$ is a simple loop at P_q, then

$$\tau = \rho\sigma\rho^{-1} = (P_0, \ldots, P_q, P_{q+1}, \ldots, P_{q+s}, P_q, \ldots, P_0)$$

and

$$h_\tau = h_\rho^{-1} \circ h_\sigma \circ h_\rho \in \Gamma_F$$

are called *basic loops* at P_0. Every element of Γ_F, restricted to a suitable neighborhood of 0 in $[0,\epsilon[$, can be written as a composition of basic loops at P_0. Thus, the (germs of) basic loops generate $\mathcal{H}_0(F)$.

Let $A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_k \subseteq \cdots \subset F$ be an exhaustion of F by compact, connected submanifolds with boundary. These should be chosen so that the components of $F \setminus A_k$ are not relatively compact in F and so that these components correspond one-one to the components of ∂A_k. Let G_0 denote the set of basic loops at P_0 which, as plaque chains, consist entirely of plaques meeting A_0. Inductively, suppose that G_k has been defined, some $k \geq 0$. Then $G_{k+1} \setminus G_k$ is to be the set of basic loops of the form $h_\rho^{-1} \circ h_\sigma \circ h_\rho$, where ρ is a simple chain at P_0 that involves only plaques that meet A_k and contains just one plaque that meets ∂A_j, $1 \leq j \leq k$, and σ is a basic loop that involves only plaques that meet $A_{k+1} \setminus A_k$. View each G_k as a subset of Γ_F and let $G_\sharp = \bigcup_{k=0}^{\infty} G_k \subseteq \Gamma_F$.

5.1. Lemma. *For $\epsilon > 0$ sufficiently small, and for each $g \in G_\sharp$, either g or g^{-1} is defined on $[0,\epsilon]$ and maps that interval into itself. Furthermore, $G_\sharp$ generates $\mathcal{H}_0(F)$.*

Proof: Since G_0 is finite, choose $\epsilon > 0$ so small that both g and g^{-1} are defined on $[0,\epsilon]$ with images in $[0,\epsilon+\delta[$, for some $\delta > 0$ and each $g \in G_0$. Either $g(\epsilon)$ or $g^{-1}(\epsilon) \in [0,\epsilon]$. Making ϵ smaller, if necessary, we make sure that h_ρ is defined on $[0,\epsilon]$, for each simple chain ρ at P_0 in A. Since V_i is a foliated interval bundle over B_i, $1 \leq i \leq t$, it is true that either g or g^{-1} sends $[0,\epsilon]$ to itself, for each $g \in G_\sharp \setminus G_0$.

The fundamental group $\pi_1(F,0)$ can be defined via the nerve of the plaque cover of F (provided the Frobenius cover of M has been suitably chosen). Clearly the set of edgeloops in $|\Gamma_F(0)|$, corresponding to G_0, generates $\pi_1(A_0,0)$. By the Van Kampen theorem and induction on k, the set of edgeloops, corresponding to G_k, generates $\pi_1(A_k,0)$, hence those corresponding to $G_\sharp$ generate $\pi_1(F,0)$. The natural surjection $\pi_1(F,0) \longrightarrow \mathcal{H}_0(F)$ is then used to prove that $G_\sharp$ generates $\mathcal{H}_0(F)$. ■

Definition. Let $g \in G_\sharp \setminus G_k$. The k-representation of g is $h_\rho^{-1} \circ h_\sigma \circ h_\rho$, where ρ is a simple chain in A_k and σ is a basic loop involving only plaques that meet $F \setminus A_k$.

Such a k-representation always exists and is unique. Its usefulness lies in the fact that, while $G_\sharp \setminus G_k$ is generally an infinite set, the k-representations involve only finitely many of the simple chains h_ρ.

5.2. Lemma. *Let $0 < r < 1 < s$. Then there is an integer $k \geq 0$ such that, whenever $g \in G_\sharp \setminus G_k$ and $g = h_\rho^{-1} \circ h_\sigma \circ h_\rho$ is the k-representation, then $r < h'_\sigma(x) < s$, for each $x \in h_\rho(D(g))$.*

This lemma is just a slight variation on [C-C 2, (2.9)], so we omit the proof.

5.3. Lemma. *If $\epsilon > 0$ and $\delta > 0$ are suitably chosen, each as small as desired, then every $g \in G_\sharp$ defines $g : [0, \epsilon] \longrightarrow [0, \epsilon + \delta[$. Furthermore, given $\eta > 0$, k can be chosen so large that $|g(u) - u| < \eta$, for each $g \in G_\sharp \setminus G_k$, $0 \leq u \leq \epsilon$.*

Proof: Let ϵ' be as small as desired with the property ascribed to ϵ in (5.1). Let $\eta > 0$. The generalized Kopell lemma [**C-C 2**, (2.8)] is a consequence of (5.2) (*cf.* [**C-C 2**, p. 190]) and implies that, if k is sufficiently large and if $g \in G_\sharp \setminus G_k$ maps $[0, \epsilon']$ into itself, then $|g(u) - u| < \eta$, $0 \leq u \leq \epsilon'$. By taking $\eta \leq \epsilon'/2$, we see that g^{-1} maps $[0, \epsilon'/2]$ into $[0, \epsilon']$ and that

$$|g^{-1}(u) - u| = |g^{-1}(u) - g(g^{-1}(u))| < \eta,$$

$0 \leq u \leq \epsilon'/2$. That is, both g and g^{-1} send $[0, \epsilon'/2]$ into $[0, \epsilon']$ *and* satisfy

$$|g(u) - u| < \eta > |g^{-1}(u) - u|.$$

Since G_k is finite, we can choose $\epsilon \in]0, \epsilon'/2[$ so small that g sends $[0, \epsilon]$ into $[0, \epsilon']$, for each $g \in G_k$, hence for each $g \in G_\sharp$. Take $\delta \in]\epsilon' - \epsilon, \epsilon'[$ and obtain all assertions. ■

5.4. Corollary. *Let $0 < r < 1 < s$. Then ϵ and δ can be chosen as in (5.3) and $k \geq 0$ can be chosen so that, for all $g \in G_\sharp \setminus G_k$ and for $0 \leq u \leq \epsilon$, the inequalities $r < g'(u) < s$ hold. If, furthermore, $\{g_1, \ldots, g_n\} \subseteq G_k$ is a subset such that no g_i is germinally equivalent to a contraction to 0, then the choice of ϵ can be made so small that $r < g_i'(u) < s$, $1 \leq i \leq n$, $0 \leq u \leq \epsilon$.*

Proof: By (5.2), we choose $k \geq 0$ such that, for $g \in G_\sharp \setminus G_k$ with k-representation $g = h_\rho^{-1} \circ h_\sigma \circ h_\rho$, we have

$$\sqrt{r} < h_\sigma'(h_\rho(u)) < \sqrt{s},$$

$0 \leq u \leq \epsilon$. By elementary calculus,

$$g'(u) = \frac{h_\rho'(u) h_\sigma'(h_\rho(u))}{h_\rho'(g(u))}$$

$0 \leq u \leq \epsilon$, and each of the finitely many functions h_ρ' is continuous. By making ϵ (and, if desired, δ) small enough, we guarantee that

$$\sqrt{r} < \frac{h_\rho'(u)}{h_\rho'(g(u))} < \sqrt{s},$$

$0 \leq u \leq \epsilon$, hence

$$r < g'(u) < s,$$

$0 \leq u \leq \epsilon$. There remain the elements $g_i \in G_k$, $1 \leq i \leq n$. Since none of these restricts to a contraction to 0 on any neighborhood of 0 in $[0, \epsilon]$, we see that $g_i'(0) = 1$, $1 \leq i \leq n$. Thus, by making $\epsilon > 0$ possibly smaller, we complete the proof ■

At this point, we consider a Markov LMS, $X \subset M$, assumed to be a minimal set of $\mathcal{F}|U$ and to accumulate on $F \subseteq \partial\widehat{U}$ in $\widehat{U}$. We let $]a, b[$ be a component of $]0, \epsilon[\setminus X$, as in §4, and consider the holonomy handle $H \subset F$ and the holonomy contraction $f = h_\sigma$ given by (4.1). For each integer $n \geq 0$ and for $u \in D(f)$, we set $u_n = f^n(u)$.

5.5. Claim. *Without loss of generality, we can assume that the $\mathcal{F}$-transverse 1-manifold T, as in §1, is $]a_1, b_0[$. This implies that $C \subset [b_1, a_0]$.*

Proof: (1) We construct a holonomy imbedding $\psi : C \longrightarrow]a_1, b_0[$. If this has image $X \cap]a_1, b_0[$, the first assertion follows. We will consider the contrary possibility in step (2).

If there is an integer $k \in \{1, \ldots, m\}$ such that $p_{ik} = 0$, $1 \leq i \leq m$, assume *wlog* that $k = m$. The Markov system S can be replaced by a Markov system S', indexed by $\{1, \ldots, m-1\}$, without loss of information. This replaces the subshift with one on a Cantor subset of $\{1, \ldots, m-1\}^{\mathbb{N}}$ and replaces C with a Cantor subset C'. By finite repetition of this process, we assume that, for each index $k = 1, \ldots, m$, there is an index $i \in \{1, \ldots, m\}$ with $p_{ik} \neq 0$.

From the above paragraph and elementary symbolic dynamics, it follows that the points $y \in C$ such that $\eta(y)$ is a periodic point in $\{1, \ldots, m\}^{\mathbb{N}}$ are dense in $C \cap I_k$, $1 \leq k \leq m$. Since only finitely many periodic points can pertain to a semiproper Γ-orbit [C-C 1, (6.2)], we choose $y_k \in C \cap I_k$ such that C clusters on y_k from *both* sides and $\eta(y_k)$ is periodic. Then there is a holonomy contraction θ_k of I_k to y_k. Let φ_k be an element of holonomy of $\mathcal{F}|U$ that carries y_k to $x_k \in]a_1, b_0[$. It is easy to arrange that $x_1, \ldots, x_k$ all be distinct. For a suitably large integer n, $\psi_k = \varphi_k \circ \theta_k^n$ is defined on all of I_k, $1 \leq k \leq m$, and $\{\psi_k(I_k)\}_{k=1}^m$ is a set of disjoint subintervals of $]a_1, b_0[$. Set

$$\psi = \bigcup_{k=1}^{m} \psi_k | (C \cap I_k) : C \longrightarrow]a_1, b_0[,$$

the desired holonomy imbedding. Clearly, $\psi(C)$ is an open-closed subset of $X \cap]a_1, b_0[$.

(2) We now assume that $C \subset]a_1, b_0[$. If $K = X \cap]a_1, b_0[\setminus C \neq \emptyset$, then K is a Cantor set. For each $x \in K$, choose γ_x as in §2, a holonomy element with $\gamma_x(x) \in C$. As remarked in §2, we can arrange that only finitely many γ_x be distinct. Thus, $\{R(\gamma_x)\}_{x \in K}$ is actually a finite set of open intervals.

By (1.7), we can assume that the Markov system S is such that, for each $x \in K$ and $1 \leq i \leq m$, either $X_i \cap R(\gamma_x) = \emptyset$ or $X_i \subset R(\gamma_x)$. Select an open

interval U_i about each X_i with these same relations to the intervals $R(\gamma_x)$. One then enlarges $\mathcal{S}$ by adding the generators $\gamma_x^{-1}|U_i$ and the intervals $\gamma_x^{-1}(X_i)$. This Markov system generates the Cantor set $X \cap]a_1, b_0[$, so we can take $T =]a_1, b_0[$.

(3) Since $]a_1, b_1[\cap X = \emptyset =]a_0, b_0[\cap X$, it follows that $C \subset [b_1, a_0]$. ■

As in §1, Γ denotes the pseudogroup on $T =]a_1, b_0[$ that is induced by $\Gamma_{\mathcal{F}}$. We have a Markov system

$$\mathcal{S} = (\{I_1, \ldots, I_m\}, \{h_1, \ldots, h_m\}, P)$$

and associated pseudogroup $\Gamma_{\mathcal{S}} \subseteq \Gamma$, such that $\Gamma|C = \Gamma_{\mathcal{S}}|C$. Since $C \subset [b_1, b_0[$, we can take $I_j \subset [b_1, b_0[$, $1 \leq j \leq m$. Let $\Gamma_{\mathcal{S}}^+$ denote the set of words in nonnegative powers of the generators $\{h_1, \ldots, h_m\}$.

We can assume that $H \subset \text{int}(A)$, hence by cutting A along H we produce a compact, connected manifold A'. It is fairly obvious how to select a subset of basic loops in A that should be thought of as the basic loops on A'. Let $G_0' = \{g_1, \ldots, g_m\} \subset G_0$ be the elements of holonomy corresponding to the basic loops in A'. Then, if $k \geq 0$ is sufficiently large, the set $\{f\} \cup G_0'$ generates the same holonomy on $[0, b_k]$ as does G_0. By renumbering, replace b_0 with b_k. Thus, if $G_i' = G_0' \cup G_1 \cup \cdots \cup G_i \cup \cdots$, we can assume that $G_1 = G_1' \cup \{f\}$.

In order to prove Theorem 3, it should be clear that proving the following two propositions will suffice.

5.6. Proposition. *For $k \geq 0$ sufficiently large, G_1' fixes every point of $X \cap [0, b_k]$.*

5.7. Proposition. *It is possible to choose the decomposition of $\widehat{U}$ so that the total holonomy of the foliated interval bundle $J_i \hookrightarrow V_i \longrightarrow B_i$ fixes every point of $X \cap J_i$, $1 \leq i \leq t$.*

Proof: (Using (5.6)) By [**C-C 1**, Theorem 1], each of the semiproper leaves $L \subset X$ has a holonomy contraction on its nonproper side, say on L^+, and this contraction is unique relative to X. Apply (4.1) to the side L^+ so as to conclude that this contraction is compactly supported. By [**C-C 1**, Theorem 2], there are only finitely many semiproper leaves in X, so we choose the customary decomposition $\widehat{U} = K \cup V_1 \cup \cdots \cup V_r$ in such a way that K engulfs the holonomy handle on each of them.

Parametrize J_i as $[0, 1]$, $1 \leq i \leq t$. By (5.6), the loops on B_i, based at x_i, fix every point of $X \cap [0, \delta]$, for some $\delta \in]0, 1]$. The subset of $X \cap]0, 1[$, fixed by these loops, is closed in $]0, 1[$, hence, if it is not all of $X \cap]0, 1[$, we can choose the above $\delta \in X \cap]0, 1[$ in such a way that some loop on B_i at x_i induces a germinally nontrivial element of holonomy on $[\delta, \delta + \epsilon[\cap X$ that fixes δ. Again appealing to [**C-C 1**, Theorem 1], we see that, for $\epsilon > 0$ sufficiently small, the holonomy around this loop gives a contraction of $[\delta, \delta + \epsilon[$ to δ.

But this loop also fixes each point of $[0,\delta] \cap X$ and, in a Markov LMS, each holonomy contraction to δ contracts a whole neighborhood of δ in $]0,1[\cap X$ to δ (elementary symbolic dynamics). Thus, δ must lie on a semiproper leaf and this contradicts the fact that the holonomy handle on that leaf does not meet V_i. ■

In order to prove (5.6), we first review certain points from [**C-C 1**, §6], where the finiteness of the number of semiproper leaves in X was established.

By the parametrization of T as the subinterval $]a_1, b_0[\subset \mathbf{R}$, we view Γ_S as a pseudogroup on $\mathbf{R}$ and place the following definition.

Definition. A point $x \in \mathbf{R}$ is Γ_S^+-uniform if there is an open neighborhood V of x in $\mathbf{R}$ and a number $\nu > 0$ such that $g'(u)/g'(v) \leq \nu$, for each $g \in \Gamma_S^+$ and for all $u, v \in V \cap D(g)$. The set of Γ_S^+-uniform points is denoted by $\mathcal{U}^+$ and its complement in $\mathbf{R}$ by $\mathcal{B}^+$.

Remark. Since $\mathcal{U}^+$ is open in $\mathbf{R}$ and, by default, contains $\mathbf{R} \setminus]a_1, b_0[$, the set $\mathcal{B}^+$ is compact.

In the following, τ again denotes the essential subshift that generates $\Gamma|C$. A τ-cycle is a subset $\{x, \tau(x), \ldots, \tau^k(x)\} \subset C$ such that $k \geq 1$ and $\tau^k(x) = x$. The minimal such k is the *length* of the τ-cycle.

5.8. Lemma. *The set $\mathcal{B}^+ \cap C$ is the union of at most finitely many τ-cycles. Furthermore, $\mathcal{B}^+$ meets at most finitely many components of $T \setminus C$ and the points of C bordering these components fall into finitely many τ-cycles.*

Proof: This was essentially proven in [**C-C 1**]. Indeed, by [**C-C 1**, (6.8), (6.18), and (6.19)] the set $\mathcal{B}^+ \cap C$ and the set of components of $T \setminus C$ that meet $\mathcal{B}^+$ are both finite. Let $x \in \mathcal{B}^+$ and let V be a neighborhood of x in $\mathbf{R}$. For each integer $N \geq 0$, let W_N be the set of words of length N in positive powers of $\{h_1, \ldots, h_m\}$. By [**C-C 1**, (6.5)], we see that $\bigcup\{g(V \cap D(g)) \mid g \in W_N\}$ meets $\mathcal{B}^+$, for all choices of V and N. From these facts the assertions follow easily. ■

Definition. The set $\mathcal{G}$ consists of all $g \in \Gamma$ such that

(a) g maps $[b_1, b_0[$ diffeomorphically onto itself;
(b) $g|C$ has fixed points $p < q$ with $g'(p) = 1 = g'(q)$ and such that $C \cap]p, q[\neq \emptyset$;
(c) if $x \in C \cap]p, q[$ and if $g(x) = x$, then $g'(x) \neq 1$.

5.9. Lemma. *There are constants $B > 0$ and $0 < r_0 < 1$ such that, if $g \in \mathcal{G}$ and p, q are as in the above definition, there is a choice of $h \in \{g, g^{-1}\}$ and of $x \in]p, q[$ for which one of the following holds.*

(1) $h(x) = x$ *and* $h'(x) < r_0$.
(2) $p < h(x) < x$ *and* $(x - p)/(x - h(x)) \leq B$.
(3) $x < h(x) < q$ *and* $(q - x)/(h(x) - x) \leq B$.

Proof: (1) Let $m_i = \min(C \cap I_i)$ and $M_i = \max(C \cap I_i)$, $1 \leq i \leq m$. These points border gaps of C, so we fix $n_i < m_i$ and $N_i > M_i$ in these gaps such that n_i and $N_i \in R(h_i)$. Let

$$\widetilde{B} = \max_{1 \leq i \leq m} \left\{ \frac{N_i - m_i}{N_i - M_i}, \frac{M_i - n_i}{m_i - n_i} \right\}.$$

The points m_i and M_i lie on semiproper $\Gamma_{\mathcal{S}}$-orbits, hence there are contractions γ_i and $\eta_i \in \Gamma_{\mathcal{S}}$ to m_i and M_i, respectively, defined on the nonproper sides of these orbits and generating the infinite cyclic, relative holonomy groups $\mathcal{H}(\Gamma_{\mathcal{S}}(m_i), C)$ and $\mathcal{H}(\Gamma_{\mathcal{S}}(M_i), C)$ [C-C 1, Theorem 1].

Let $0 < r_0 < 1$ be such that, for each value of $i = 1, \ldots, m$, if $\gamma_i'(m_i)$ and/or $\eta_i'(M_i)$ are not equal to 1, then they are less than r_0. If all of these derivatives are 1, then $0 < r_0 < 1$ can be fixed arbitrarily.

(2) If $\mathcal{B}^+ \neq \emptyset$, let λ be the length of the longest of the τ-cycles given by (5.8) and let $\mu = \lambda^2$. If $\mathcal{B}^+ = \emptyset$, fix μ arbitrarily.

For $w = h_{i_n} \circ \cdots \circ h_{i_1} \in \Gamma_{\mathcal{S}}^+$, let $w_k = h_{i_k} \circ \cdots \circ h_{i_1}$, $0 \leq k \leq n$. Here we let $w_0 = id$.

Let $\mathcal{W}_i$, $1 \leq i \leq m$, denote the set of all $w \in \Gamma_{\mathcal{S}}^+$ such that

(a) $I_i \subset D(w)$;

(b) either $|w| < \mu$ or $w_\mu(I_i) \cap \mathcal{B}^+ = \emptyset$.

By the definition of $\mathcal{B}^+$, we can choose the points n_i and N_i of step (1) and a constant $\bar{B} > 0$ such that $w'(u)/w'(v) \leq \bar{B}$, for each $w \in \mathcal{W}_i$ and for all $u, v \in [n_i, N_i] \cap D(w)$, $1 \leq i \leq m$.

Finally, set $B = \widetilde{B} \cdot \bar{B}$.

(3) Let $g \in \mathcal{G}$ and p, q be as above. Since $C \cap]p, q[\neq \emptyset$, there is a unique shortest $w \in \Gamma_{\mathcal{S}}^+$, possibly the identity, such that $[p, q] \subset D(w^{-1})$, $w^{-1}(p) = p_* \in I_i$, $w^{-1}(q) = q_* \in I_j$, and $i \neq j$. Let $g_* = w^{-1} \circ g \circ w$ with $D(g_*) = [p_*, q_*]$.

Clearly m_j and M_i both lie in $[p_*, q_*] \subset D(w)$. Let $\bar{m} = w(m_j) \in [p, q]$ and $\bar{M} = w(M_i) \in [p, q]$. Since $p_* \in I_i$ and $q_* \in I_j$ and $I_i \cap I_j = \emptyset$, it is also clear that

$$M_i < N_i < q_*,$$
$$p_* < n_j < m_j,$$

and we set $\bar{n} = w(n_j) \in [p, q]$, $\bar{N} = w(N_i) \in [p, q]$.

(4) Suppose that either $\bar{m}$ or $\bar{M}$ is a fixed point of g. We will show that property (1) in the statement of the lemma follows.

For definiteness, suppose that $g(\bar{M}) = \bar{M}$, the argument in the alternative case being completely parallel. There are two cases.

Case 1. Suppose that $p_* = M_i$, hence $p = \bar{M}$. Since $C \cap]p, q[\neq \emptyset$ and since M_i is the left endpoint of a gap of C, so is $\bar{M} = p$. We take $x \in]p, q[$ to be

the other end of the gap. Then the fact that $w^{-1}(p) = p_* = M_i$ implies that $w^{-1}(x) = m_k$, some $k \in \{1, \dots, m\}$. Also, $g(\bar{M}) = \bar{M}$ implies that $g(x) = x$, while $x \in C \cap]p, q[$ then implies that $g'(x) \neq 1$. Either g or g^{-1} will be a contraction to x and we choose $h = g^{\pm 1}$ to be that contraction. Finally, let $h_* = w^{-1} \circ h \circ w = g_*^{\pm 1}$. Then $1 > h'(x) = h'_*(m_k)$, so h_* is a positive power of γ_k and step (1) implies that $h'(x) < r_0$, as desired.

Case 2. Suppose that $p_* < M_i$. Then, if we take $x = w(M_i) = \bar{M} \in]p, q[$, we can argue exactly as in Case 1.

(5) By step (4), we can assume that neither $\bar{m}$ nor $\bar{M}$ is fixed by g. We will show that either property (2) or (3) in the statement of the lemma follows. We consider three possible cases.

Case 1. Suppose that $w \in \mathcal{W}_i$. Again choosing $h = g^{\pm 1}$ appropriately, we see that $h_*(N_i) < M_i$ and $h(\bar{N}) < \bar{M} < \bar{N}$. From step (3), $\bar{M}$ and $\bar{N} \in [p, q]$, $w(M_i) = \bar{M}$, and $w(N_i) = \bar{N}$. By these remarks and the mean value theorem,

$$\frac{\bar{N} - p}{\bar{N} - h(\bar{N})} \leq \frac{\bar{N} - p}{\bar{N} - \bar{M}} = \frac{w'(\xi)}{w'(\eta)} \cdot \left(\frac{N_i - p_*}{N_i - M_i} \right)$$

with ξ and η both $\in]p_*, N_i[$. Since $w \in \mathcal{W}_i$, we see that $w'(\xi)/w'(\eta) \leq \tilde{B}$. Also, $N_i - p_* \leq N_i - m_i$, so

$$\frac{\bar{N} - p}{\bar{N} - h(\bar{N})} \leq \tilde{B} \cdot \left(\frac{N_i - m_i}{N_i - M_i} \right) \leq \tilde{B} \cdot \tilde{B} = B.$$

Thus, $x = \bar{N}$ satisfies property (2) in the statement of the lemma.

Case 2. Suppose that $w \in \mathcal{W}_j$. An argument completely parallel to that in Case 1 shows that $\bar{n} < h(\bar{n}) < q$ and that $(q - \bar{n})/(h(\bar{n}) - \bar{n}) \leq B$, so we take $x = \bar{n}$, obtaining property (3) in the statement of the lemma.

Case 3. Suppose that $w \notin \mathcal{W}_i$ and $w \notin \mathcal{W}_j$. This cannot actually happen, as we now show.

We have $|w| \geq \mu$ and $w_\mu(I_i) \cap \mathcal{B}^+ \neq \emptyset \neq w_\mu(I_j) \cap \mathcal{B}^+$. It follows from (5.8) that there are points $y_i \in I_i$ and $y_j \in I_j$, belonging to τ-cycles of respective lengths $\lambda(i) \leq \lambda$ and $\lambda(j) \leq \lambda$, where λ is as in step (2), and $w_\mu(y_i)$ and $w_\mu(y_j)$ belong to these same τ-cycles. Let $k = \lambda(i)\lambda(j) \leq \lambda^2 = \mu$. Let g_i and g_j be the words of lengths $\lambda(i)$ and $\lambda(j)$ fixing y_i and y_j respectively. It follows that

$$w_k = g_i^{\lambda(j)} = g_j^{\lambda(i)} = h_{i_k} \circ \cdots \circ h_{i_1}.$$

But then $y_i = w_k(y_i) \in I_{i_k}$, so $i_k = i$, while, similarly, $y_j = w_k(y_j) \in I_{i_k}$ and $i_k = j$. This contradicts the fact that $i \neq j$. ■

Proof of (5.6): Choose θ so that $f''(\xi)/f'(\eta) \leq \theta$, for all ξ and $\eta \in [0, \epsilon]$. Choose $r \in]r_0, 1[$, sufficiently close to 1 that $(e^{\theta\epsilon} B)^{-1} > 1 - r$. Here, of course, r_0 and B are as in (5.9).

By (5.4), choose $k \geq 0$ so large that $g'(x) > r$, for each $g \in G'_{\natural}$, $0 \leq x \leq b_k$. We want to show that every $g \in G'_{\natural}$ fixes every $x \in X \cap [0, b_k]$. If not, there exists $g \in G'_{\natural}$ and $n \geq k$ so that g moves some points in $X \cap [b_{n+1}, b_n] \subset [0, b_k]$. Fix a choice of such g and deduce a contradiction as follows.

Let $g_0 = f^{-n} \circ g \circ f^n \in \Gamma$ and note that this carries $[b_1, b_0]$ diffeomorphically onto itself, fixing b_1 and $a_0 = a$. Since the holonomy relative to X of semiproper leaves in X is compactly supported (4.1), we can assume that k is so large that $g'_0(b_1) = 1 = g'_0(a_0)$. Thus, the closed set C_0 of g_0-fixed points $y \in X \cap [b_1, a_0]$ with $g'_0(y) = 1$ is nonempty. Since g_0 moves some points of $X \cap [b_1, a_0]$, there is a connected component $]p, q[$ of $[b_1, a_0] \setminus C_0$ such that $X \cap]p, q[\neq \emptyset$. This proves that $g_0 \in \mathcal{G}$ and we can apply (5.9).

If property (1) of (5.9) holds, then, for a suitable choice of an element $h = g^{\pm 1} \in G'_{\natural}$ and a corresponding choice of $h_0 = g_0^{\pm 1}$, we obtain the contradiction that $h'(f^n(x)) = h'_0(x) < r_0 < r$, for some $f^n(x) \in f^n\,]p, q[\subset [0, b_k]$.

Thus, either property (2) or property (3) of (5.9) holds. We consider the first case, the second being entirely similar. Let $x \in]p, q[$ be such that inequalities $p < y = h_0(x) < x$ and $(x - p)/(x - y) \leq B$ hold. Set

$$\begin{aligned} x_n &= f^n(x) \\ y_n &= f^n(y) = h(x_n) \\ p_n &= f^n(p) = h(p_n). \end{aligned}$$

By a standard calculation (cf. [Sa 2]),

$$\frac{x_n - p_n}{x_n - y_n} = \frac{(f^n)'(\xi)}{(f^n)'(\eta)} \cdot \left(\frac{x - p}{x - y}\right) \leq \exp(\theta\epsilon) \cdot \left(\frac{x - p}{x - y}\right) \leq \exp(\theta\epsilon) \cdot B.$$

Then

$$1 - r < (\exp(\theta\epsilon) \cdot B)^{-1} \leq \frac{x_n - y_n}{x_n - p_n} = 1 - \frac{y_n - p_n}{x_n - p_n} = 1 - h'(\zeta),$$

so $h'(\zeta) < r$ for suitable $\zeta \in [p_n, x_n] \subset [0, b_k]$, again a contradiction. ■

The proof of Theorem 3 is complete.

6. Geometry of leaves

Let M be a closed, orientable 3-manifold, $\mathcal{F}$ a transversely orientable C^2 foliation of M by surfaces.

6.1. Conjecture. *If $X \subset M$ is a Markov LMS of $\mathcal{F}$, and if $\overline{X}$ contains no toral leaf, then there is a Riemannian metric on M relative to which every leaf of $\mathcal{F}|X$ has constant curvature -1.*

Indeed, refinements of the methods of this paper, together with standard facts about Teichmüller space, should yield a proof of (6.1) for the case of

Markov minimal sets. At higher levels, further techniques from [C-C 4] will also be needed.

If X is an arbitrary exceptional LMS of $\mathcal{F}$, it seems likely that a sequence $\{\mathcal{F}_n\}_{n=1}^{\infty}$ of foliations of M can be found, converging to $\mathcal{F}$ in the C^2-topology, each with a Markov LMS X_n, such that X is uniformly well C^2-approximated by X_n as $n \to \infty$. This should lead to the following.

6.2. Conjecture. *The previous conjecture is valid for an arbitrary exceptional LMS that does not approach a toral leaf.*

This, together with the methods of [C-C 4], would give the following.

6.3. Conjecture. *If no leaf of $\mathcal{F}$ is a torus or a sphere, and if no leaf is somewhere dense in M, then there is a Riemannian metric on M relative to which every leaf of $\mathcal{F}$ has constant curvature -1.*

Indeed, we believe the following.

6.4. Conjecture. *If no leaf of $\mathcal{F}$ is a torus, there exists a Riemannian metric on M and a choice of $r \in \{-1, 0, 1\}$ such that every leaf of $\mathcal{F}$ has constant curvature r.*

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