Publicacions Matemàtiques, Vol 33 (1989), 147–152.

INVARIANTS OF ANALYTIC CURVES

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Abstract

In this article we introduce a complete system of geometric invariants for an analytic curve. No restrictions are imposed on the curve and the invariants can be easily computed.

1. In differential geometry the usual system of invariants of a curve $\vec{c}$ in $\mathbb{R}^n$ consists of the curvature functions ([2, p. 94]). Their definition involves the assumption that the derivatives $\vec{c}',\ldots,\vec{c}^{(n-1)}$ are linearly independent. This excludes curves (in $\mathbb{R}^3$) like straight lines or $(t+t^2,t^3,t^4)$. In the following we will show how to overcome these difficulties in the context of analytic curves.

The key lies in the observation that the linear dependence of successive derivaties $\vec{c}',\ldots,\vec{c}^{(m)}$ in a (non–degenerate) interval is equivalent to the fact that $\vec{c}$ is contained in a linear manifold of dimension at most m (Proposition 1). This statement is false for C^∞–curves as an example in section 5 shows; and this explains why we restrict our attention to analytic curves which include the vast majority of natural, regular curves.

The linear dependence can be determined using Gram–determinants, and these determinants (suitably normalized) turn out to furnish a complete set of invariants for *every* analytic curve (Theorem 1 and Theorem 2).

2. A curve $\vec{c}: I \to \mathbb{R}^n$ ($n \geq 2$, fixed) is called *analytic* if $I \subseteq \mathbb{R}$ is an interval with interior points, $\vec{c}$ can be expanded in a convergent power series at each point lying in some open $U \supseteq I$ and $\vec{c}' \neq \vec{0}$ on I.

Denote by $C(i_1,\ldots,i_j)$ the matrix $[\vec{c}^{(i_1)},\ldots,\vec{c}^{(i_j)}]$ where $i_1,\ldots,i_j \in \mathbb{N}$, and let

$$G_j(\vec{c}) = \det\,(C^T(1,\ldots,j)\cdot C(1,\ldots,j))/\,\|\,\vec{c}'\,\|^{j(j+1)} \text{ for } 1 \leq j \leq n-1 \text{ and}$$
$$G_n(\vec{c}) = \det\,(C(1,\ldots,n))/\,\|\,\vec{c}'\,\|^{n(n+1)/2}\,.$$

In the case of linearly independent derivatives $\vec{c}',\ldots,\vec{c}^{(n-1)}$ they are related to the usual invariants by the formulae

$$\kappa_j = \sqrt{G_{j+1}(\vec{c})G_{j-1}(\vec{c})}/G_j(\vec{c}) \qquad (1 \leq j \leq n-2) \text{ and}$$
$$\kappa_{n-1} = G_n(\vec{c})\sqrt{G_{n-2}(\vec{c})}/G_{n-1}(\vec{c}) \text{ with the convention } G_o \equiv 1$$

([2, p. 94]). E.g. in $\mathbf{R}^3$ we obtain the curvature $\kappa = \sqrt{G_2(\vec{c})}/G_1(\vec{c})$ and the torsion $\tau = G_3(\vec{c})\sqrt{G_1(\vec{c})}/G_2(\vec{c})$.

First we want to discuss how these expressions change if $\vec{c}$ is exposed to a motion or a change of parameter. Here a *motion* $M : \mathbf{R}^n \to \mathbf{R}^n$ is given by $M(\vec{x}) = R\vec{x} + \vec{m}$ with fixed $\vec{m} \in \mathbf{R}^n$ and an orthogonal matrix R with det $R = 1$; and a *change of parameter* is an analytic, bijective function $\phi : I' \to I$ with $\phi' > 0$ on I' (it respects orientation). Defining $\hat{\vec{c}} = M \circ \vec{c} \circ \phi$ one immediately verifies

Theorem 1. $G_j(\hat{\vec{c}}) = G_j(\vec{c}) \circ \phi$ $(1 \leq j \leq n)$, *i.e. these functions are invariant under motions and changes of parameter.*

In order to show that they in fact constitute a complete system of invariants we first exhibit some of their properties.

3. The *dimension* of the curve $\vec{c}$ is the dimension of the smallest linear manifold or flat ([1, p. 142]) containing $\{\vec{c}(t) : t \in I\}$. With the convention $G_{n+1} \equiv 0$ we obtain

Proposition 1. *The following statements are equivalent:*

(i) $\vec{c}$ *has dimension* m.

(ii) *The vectors* $\vec{c}^{(j)}(t)$ $(1 \leq j < \infty)$ *span an* m*–dimensional subspace of* $\mathbf{R}^n$ *for each* $t \in I$.

(iii) $G_m \not\equiv 0$, $G_{m+1} \equiv 0$.

Proof: (i) $\Rightarrow$ (ii) : $\vec{c}(t)$ can be written as $\vec{c}(a) + \Sigma_{\nu=1}^m a_\nu(t)\vec{b}_\nu$ where $a \in I$, $\vec{b}_\nu$ are a fixed orthonormal system and $a_\nu(t) = (\vec{c}(t) - \vec{c}(a)) \cdot \vec{b}_\nu$ are analytic on I. Thus the derivatives $\vec{c}^{(j)}(t)$ $(1 \leq j)$ lie in the space spanned by $\vec{b}_1, \ldots, \vec{b}_m$. Hence, their span has dimension $m' \leq m$.

Let $\vec{c}^{(i_1)}(a), \ldots, \vec{c}^{(i_{m'})}(a)$ be linearly independent then we obtain the formula $\vec{c}(t) = \vec{c}(a) + \Sigma_{\nu=1}^{m'} a_\nu(t)\vec{c}^{(i_\nu)}(a)$ from the Taylor expansion where the $a_\nu(t)$ are power series which converge near a. Integrating $\vec{c}'$ we find that $\vec{c}$ has dimension at most m'. Hence $m = m'$.

(ii) $\Rightarrow$ (iii): $\vec{c}', \ldots, \vec{c}^{(m+1)}$ are always linearly dependent. Thus $G_{m+1} \equiv 0$. Then we consider G_j $(j \leq m)$ near a fixed $t_0 \in I$. By [1, p. 329] we know that

$$\det\,(C^T(1, \ldots, j) \cdot C(1, \ldots, j)) = \Sigma_{\nu=1}^{\binom{n}{j}} D_\nu^2(1, \ldots, j)$$

for $j < n$ where $D_\nu(1, \ldots, j)$ is the determinant of some (j, j) – submatrix of $C(1, \ldots, j)$. We want to find the first derivative of G_j which does not vanish. Thus we first discuss the derivatives of $D_\nu(1, \ldots, j)$ for fixed

ν and j. Differentiation of a determinant leads to a sum of determinants each of which is obtained by differentiating one column. Since determinants with two equal columns vanish identically $D_\nu^{(p)}(1,\ldots,j)$ is a sum of terms $D_\nu(i_1,\ldots,i_j)$ with $1 \le i_1 < \cdots < i_j$ and $p = \Sigma_{k=1}^j (i_k - k) \in \mathbb{N}$. This suggests that we should look for linearly independent derivatives $\vec{c}^{(i_1)},\ldots,\vec{c}^{(i_j)}$ such that $\Sigma_{k=1}^j i_k$ is minimal. They can be obtained inductively by defining $i_1 = 1, i_k \in \mathbb{N}$ as the smallest number such that $\vec{c}^{(i_k)}(a)$ is linearly independent of $\vec{c}^{(i_1)}(a),\ldots,\vec{c}^{(i_{k-1})}(a)$ $(2 \le k \le m)$. That they minimize $\Sigma_{k=1}^j i_k$ for each $j(1 \le j \le m)$ can be seen by looking at a different sequence $1 \le i_1' < \cdots < i_j'$ where the vectors $\vec{c}^{(i_k')}$ are independent. If k is the first index with $i_k \ne i_k'$ then by construction $i_k < i_k'$ and we can replace one of the derivatives $\vec{c}^{(i_k')},\ldots,\vec{c}^{(i_j')}$ by $\vec{c}^{i_k}$ without destroying the linear independence [**1**, p. 102]. Hence $\Sigma_{k=1}^j i_k'$ was not minimal. Thus we conclude that $p_j = \Sigma_{k=1}^j (i_k - k)$ is the first number such that some $D_\nu^{p_j}$ does not vanish. Therefore the first non–vanishing derivatives of G_j are $G_j^{(2p_j)}$ as long as $j < n$ and $j \le m$ resp. $G_n^{p_n}$ for $j = m = n$. Especially $G_m \not\equiv 0$.

(iii) $\Rightarrow$ (i): Let $a \in I$ be chosen such that $G_m(a) \ne 0$. Then $\vec{c}'(t),\ldots,\vec{c}^{(m)}(t)$ are linearly independent near a and from them $\vec{c}^{(m+1)}(t)$ can be obtained as a linear combination. This is nothing but an m-th order linear differential equation for $\vec{c}'$. Thus $\vec{c}'(t)$ lies in an m–dimensional subspace of $\mathbb{R}^n$ for all t near a, and hence for all $t \in I$ since $\vec{c}'$ is analytic. This shows that the dimension of $\vec{c}$ is at most m. But it cannot be less since otherwise $G_m \equiv 0$. ■

During the proof we introduced the numbers $p_j(a) \in \mathbb{N}_0$ which denote half of the (resp. the full) multiplicity of a as zero of G_j for $j < n$ (resp. $j = n$) if $G_j \not\equiv 0$. We also use the convention $p_0 \equiv 0$.

Proposition 2. *The functions $G_j(\vec{c})$ have the following properties:*

(i) *They are analytic on I,*
(ii) $G_j \ge 0, G_1 \equiv 1$ $(1 \le j \le n-1)$,
(iii) $G_j(a) = 0 \Rightarrow G_k(a) = 0 \quad \forall k \ge j$,
(iv) $\Delta^2 p_j(a) = p_j(a) - 2p_{j-1}(a) + p_{j-2}(a) \ge 0$ *for* $2 \le j$ *if* $G_j \not\equiv 0$.

Proof: (i),(ii) and (iii) are obvious. And (iv) follows from $p_j(a) = \Sigma_{\nu=1}^j (i_\nu - \nu)$ as in the proof of Proposition 1 where the i_ν are strictly increasing. ■

4. Now we can show that the functions G_j are a complete system of invariants.

Theorem 2. *Let the functions $G_j : I \to \mathbb{R}^n (1 \le j \le n)$ have the properties stated in Proposition 2. Then there exists an analytic curve $\vec{c} : I \to \mathbb{R}^n$*

parametrized by arclength (i.e. $\| \vec{c}\,' \| \equiv 1$*) with* $G_j(\vec{c}) = G_j$ $(1 \le j \le n)$*. Any other such curve is obtained from it by a motion.*

Proof: Define a number $m \in \mathbb{N}$ by $G_m \not\equiv 0, G_{m+1} \equiv 0$ if $G_n \equiv 0$ resp. $m = n$ if $G_n \not\equiv 0$. Then m is well-defined because of properties (ii) and (iii) in Proposition 2. Fix $a \in I$ such that $G_j \neq 0$ at a for $j \le m$. Such a point exists since the zeros of analytic functions ($\not\equiv 0$) are isolated. Define $\varepsilon_j(t) = (-1)^{Z_j(t)}$ for $1 \le j \le m$ where $Z_j(t)$ stands for the number of zeros z of G_j in $[a, t]$ resp. $[t, a]$ with $\Delta^2 p_j(z)$ odd (for $j < n$) resp. with $p_{n-2}(z)$ odd (for $j = n$). Then we consider the functions $\kappa_j = \varepsilon_{j+1}\sqrt{G_{j+1}G_{j-1}}/G_j$ $(1 \le j \le n-2, j \le m-1)$, $\kappa_{n-1} = \varepsilon_n G_n \sqrt{G_{n-2}}/G_{n-1}$ (if $n = m$) and $\kappa_j \equiv 0$ (for $j \ge m$).

The power series of the functions G_j, property (iv) in Proposition 2 and the definitions of ε_j show that all κ_j are analytic on I. Thus the Frenet–Serret formulae

$$[\vec{e}\,'_1, \ldots, \vec{e}\,'_n] = [\vec{e}_1, \ldots, \vec{e}_n] \begin{bmatrix} 0 & -\kappa_1 & 0 & \cdot & \cdot & \cdot & 0 \\ \kappa_1 & & & & & & \cdot \\ 0 & & & & & & \cdot \\ 0 & & & & & & 0 \\ \vdots & & & & & & -\kappa_{n-1} \\ 0 & \cdot & \cdot & \cdot & 0 & \kappa_{n-1} & 0 \end{bmatrix}$$

have a unique solution on I with $\vec{e}_i(a)$ equal to the $i-th$ vector of the canonical basis [**2**, p. 96]. Furthermore $\vec{e}_1, \ldots, \vec{e}_n$ form a positive orthonormal system. Thus $\vec{c}(t) = \int_a^t \vec{e}_1(\tau)d\tau$ is an analytic curve with $\| \vec{c}\,' \| \equiv 1$. Moreover $\vec{c}^{(j)} = \kappa_1 \cdot \ldots \cdot \kappa_{j-1} \cdot \vec{e}_j + \vec{r}_j$ where $\vec{r}_j$ is a linear combination of $\vec{e}_1, \ldots, \vec{e}_{j-1}$ (The empty product is taken to be 1.). Then we compute every $G_j(\vec{c})$ whose value is unchanged if we add a multiple of one occuring derivative to another. This leads to $G_j(\vec{c}) = \Pi_{\nu=1}^{j-1} \kappa_\nu^{2(j-\nu)}$ which is easily seen to be $G_j (1 \le j \le n-1)$. Moreover $G_n(\vec{c}) = \Pi_{\nu=1}^{n-1} \kappa_\nu^{n-\nu}$ is analytic, has the same absolute value as G_n and the two coincide at a. Hence $G_n(\vec{c}) = G_n$ (notice that $G_n \neq 0$ at a unless $G_n \equiv 0$). Now consider another curve $\vec{c}_1$ with the same properties. By Proposition 1 both curves have dimension m and applying appropriate motions to them we may assume that they lie in $\mathbb{R}^m \times \{0\}^{n-m}$. If $m = 1$ the curves are subsets of straight lines where the nature of the subset (bounded, ray or full line resp. open, closed etc.) is completely determined by I. Therefore $\vec{c}_1$ can be obtained from $\vec{c}$ by a motion and we assume from now on $m \ge 2$. For the moment we restrict our attention to a neighborhood U of a where $G_j \neq 0$ (for all $j \le m$). Assume first that $m = n$. Then the κ_j are the usual curvature functions [**2**, p.93] and the theory of the Frenet–Serret formulae shows that $\vec{c}_1(t)$ is obtained from $\vec{c}(t)$ by a fixed motion for $t \in U$. But since both curves are analytic, this must hold consequently for all $t \in I$. If $m < n$ we try the same approach in the space $\mathbb{R}^m$ by disregarding the remaining $n-m$ coordinates (which are 0). The only problem is that the orientation of the vectors $\vec{c}^{(1)}, \ldots, \vec{c}^{(m)}$ (in $\mathbb{R}^m$) may lead to the conclusion that our κ_{m-1} has the wrong sign. Then we first apply the rotation R to $\vec{c}$ which is obtained from the identity-matrix by replacing the elements in the (1,1)-resp. $(m+1, m+1)$-position by -1's. If necessary we do the same to $\vec{c}_1$. Then we can use (in $\mathbb{R}^m$) the same reasoning as above. ■

This proof shows that, in principle, we continued the functions κ_i analytically to all of I. This was also done in [4] but only for curves in $\mathbb{R}^3$ by different methods. The disadvantages of these continued functions κ_i are that their computation is involved and that their signs depend on the orientation of the Frenet-vectors $\vec{e}_1, \ldots, \vec{e}_n$. This was incorporated in our choice of the functions $\varepsilon_j(t)$.

5. Our invariants G_j are not only easy to compute, but do not depend on arbitrary normalizations and do still furnish a complete system of invariants for analytic curves. Furthermore they occur in natural connection with the geometric concept of the dimension of the curve $\vec{c}$. E.g. plane curves are characterized by $G_m \equiv 0$ ($\forall m \geq 3$) and straight lines in $\mathbb{R}^3$ possess the invariants $G_1 \equiv 1$, $G_2 \equiv 0$, $G_3 \equiv 0$.

That our considerations cannot be transferred to C^∞-curves is demonstrated by the curves

$$\vec{c}_1 = \begin{cases} (t, e^{-1/t^2}, 0) & t \neq 0 \\ (0,0,0) & t = 0 \end{cases} \quad \text{and} \quad \vec{c}_2 = \begin{cases} \vec{c}_1 & t \geq 0 \\ (t, 0, e^{-1/t^2}) & t < 0. \end{cases}$$

They are not linked by a rigid motion, since $\vec{c}_1$ has dimension 2, but $\vec{c}_2$ has dimension 3. Yet the corresponding functions G_m coincide for all $m \in \mathbb{N}$ and $t \in \mathbb{R}$, and the same is true for κ and τ whose limits at $t = 0$ exist.

I would like to thank K. Nomizu whose stimulating article [3] roused my interest in these questions and W.B. Jurkat whose comments were of great help when I prepared this article.

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Rebut el 27 de juliol de 1988