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EXPLICIT SOLUTIONS FOR NON HOMOGENEOUS STURM LIOUVILLE OPERATOR PROBLEMS

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Abstract

In this paper we study existence and uniqueness conditions for the solutions of Sturm-Liouville operator problems related to the operator differential equation $X'' - QX = F(t)$. Explicit solutions of the problem in terms of a square root of the operator Q are given.

1. Introduction

Throught this paper H denotes a complex Hilbert space and $L(H)$ denotes the algebra of all bounded linear operators on H. If T lies in $L(H)$, its spectrum $\sigma(T)$ is the set of all complex numbers z such $zI - T$ is not invertible in $L(H)$.

Second order operator differential equations with constant operator coefficients appear in the theory of oscillatory and vibrating systems, [6,10,12]. Sturm-Liouville operator problems have been studied by several authors and with several techniques ([13,14,15,16]). For the scalar case these problems are completely studied [1,7]. In a recent paper [9] we studied the homogeneous Sturm-Liouville operator problem.

$$\begin{aligned} &X^{(2)}(t) - \lambda QX(t) = 0 \\ &E_1X(0) + E_2X^{(1)}(0) = 0 \\ (1.1)\qquad &F_1X(a) + F_2X^{(1)}(a) = 0 \quad 0 \le t \le a \end{aligned}$$

where λ is a complex parameter and $X(t)$, Q, E_i, F_i, for $i = 1, 2$, are in $L(H)$. The method proposed in [9] is based on the existence of square roots for the operator λQ. The existence of square roots of matrices is treated in [3] and for the infinite-dimensional case a recent paper [2] studies this problem. In [4] and [8] methods for obtaining solutions of more general equations of polynomial type are given.

By differentiation in (2.9) and considering (2.8) one gets

$$X^{(2)}(t) = X_0^2 \exp(tX_0)C(t) + X_0^2 \exp(-tX_0)D(t) + F(t)$$

Thus we have

$$\begin{aligned}(2.10)\quad & X^{(2)}(t) - \lambda QX(t) = \\ &= (X_0^2 - \lambda Q)\exp(tX_0)C(t) + (X_0^2 - \lambda Q)\exp(-tX_0)D(t) + F(t) = F(t)\end{aligned}$$

and $X(t)$ given by (2.3) is a solution of (2.2) if $C(t)$ and $D(t)$ satisfy (2.5). Considering the equality

$$\begin{bmatrix} I & I \\ X_0 & -X_0 \end{bmatrix}^{-1} = \begin{bmatrix} I/2 & X_0^{-1}/2 \\ I/2 & -X_0^{-1}/2 \end{bmatrix} \tag{2.11}$$

by integration in (2.5) and taking into account (2.6) one gets

$$\begin{aligned}\begin{bmatrix} C(t) \\ D(t) \end{bmatrix} &= \begin{bmatrix} C(0) \\ D(0) \end{bmatrix} + \int_0^t \begin{bmatrix} \exp(-sX_0) & 0 \\ 0 & \exp(sX_0) \end{bmatrix} \begin{bmatrix} I/2 & X_0^{-1}/2 \\ I/2 & -X_0^{-1}/2 \end{bmatrix} \begin{bmatrix} 0 \\ F(s) \end{bmatrix} ds \\ &= \begin{bmatrix} C(0) + \frac{1}{2}\int_0^t X_0^{-1} \exp(-sX_0)F(s)ds \\ D(0) - \frac{1}{2}\int_0^t X_0^{-1} \exp(sX_0)F(s)ds \end{bmatrix}\end{aligned}$$

From (2.3) and (2.9) the operators $C(0)$ and $D(0)$ must verify

$$C(0) + D(0) = C_0 \quad , \quad X_0C(0) - X_0D(0) = C_1 \tag{2.12}$$

Taking into account (2.11) and solving (2.12) we have

$$C(0) = (\frac{1}{2})(C_0 + X_0^{-1}C_1) \quad , \quad D(0) = (\frac{1}{2})(C_0 - X_0^{-1}C_1)$$

Hence, the result is proved.

Theorem 1 of [9] gives a sufficient condition for the existence of one unique solution of problem (1.1), the trivial one $X(t) = 0$ for all $t \in [0, a]$, when $\lambda \neq 0$ and Q is an invertible operator such that

$$\sigma(\lambda Q) = \{z\,; \quad z \quad \text{belongs to } \sigma(Q)\} \subset D_\alpha \tag{2.13}$$

for some $\alpha \in [0, 2\pi[$. If $X_0 = \exp(\log_\alpha(\lambda Q)/2)$, this condition is the invertiblity of the operator

$$S = \begin{bmatrix} E_1 + E_2X_0 & E_1 - E_2X_0 \\ (F_1 + F_2X_0)\exp(aX_0) & (F_1 - F_2X_0)\exp(-aX_0) \end{bmatrix} \tag{2.14}$$

Next result shows that the same condition ensures the existence of only one solution for the non-homogeneous problem (1.2). Also a sufficient condition for the existence of solutions for (1.2) and explicit expressions of them in terms of data are given.

Theorem 2. *Let Q an invertible operator such that condition (2.13) is satisfied and let $X_0 = \exp(\log_\alpha(\lambda Q)/2)$. Let us consider the operators P, Q and Y defined by the expressions*

$$P = -(\frac{1}{2}) \int_0^a \exp(sX_0)X_0^{-1}F(s)ds \quad ; \quad Q = (\frac{1}{2}) \int_0^a \exp(-sX_0)X_0^{-1}F(s)ds$$

$$Y = -(F_1 - F_2X_0) = \exp(-aX_0)P - (F_1 + F_2X_0)\exp(aX_0)Q \tag{2.15}$$

(i) Problem (1.2) is solvable, if and only if, the system

$$S\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ Y \end{bmatrix} \tag{2.16}$$

where S is given by (2.14) admits a solution $\begin{bmatrix} U \\ V \end{bmatrix}$. Under this hypothesis the solution set of problem (1.2) is given by the operator function (2.3) where $C(t)$ and $D(t)$ are defined by (2.4) and $U = C(0)$, $V = D(0)$.

(ii) If S is invertible, then problem (1.2) has only one solution given by the operator function $X(t)$ defined by (2.3) where $C(t)$, $D(t)$ are determined by (2.4) and

$$\begin{bmatrix} C(0) \\ D(0) \end{bmatrix} = S^{-1}\begin{bmatrix} 0 \\ Y \end{bmatrix} \tag{2.17}$$

Proof: From theorem 1, the general solution of the operator differential equation (2.1) is given by (2.3)-(2.4). Note that from the proof of th. 1, one gets

$$X^{(1)}(t) = X_0\exp(tX_0)C(t) - X_0\exp(-tX_0)D(t) \tag{2.18}$$

Thus, we have the following relationship between $C_0 = X(0)$, $C_1 = X^{(1)}(0)$ and the operators $C(0), D(0)$,

$$C_0 = C(0) + D(0)$$
$$C_1 = X_0C(0) - X_0D(0)$$

or

$$\begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = \begin{bmatrix} I & I \\ X_0 & -X_0 \end{bmatrix}\begin{bmatrix} C(0) \\ D(0) \end{bmatrix} \tag{2.19}$$

In order to find solutions of (1.2), we are going to impose to the operators $C(0), D(0)$, that $X(t) = \exp(tX_0)C(t) + \exp(-tX_0)D(t)$, satisfies the boundary value conditions of (1.2). From th. 1, we have

$$C(a) = C(0) - Q \quad , \quad D(a) = D(0) + P \tag{2.20}$$

Taking into account (2.18), the boundary value conditions of (1.2) take the form

$$\begin{aligned}(2.21)\qquad & E_1\big(C(0)+D(0)\big)+E_2X_0\big(C(0)-D(0)\big)=0\\ & F_1\big(\exp(aX_0)C(a)+\exp(-aX_0)D(a)\big)+\\ & +F_2\big(X_0\exp(aX_0)C(a)-X_0\exp(-aX_0)D(a)\big)=0\end{aligned}$$

From (2.20), second equation of (2.21) may be written

$$(2.22)\qquad (F_1+F_2X_0)\exp(aX_0)C(0)+(F_1-F_2X_0)\exp(-aX_0)D(0)=Y$$

where Y is given by (2.15). Thus the boundary value conditions of (1.2) are equivalent to obtain operators $C(0)$ and $D(0)$ such that

$$(2.23)\qquad \begin{bmatrix} E_1+E_2X_0 & E_1-E_2X_0\\ (F_1+F_2X_0)\exp(aX_0) & (F_1-F_2X_0)\exp(-aX_0)\end{bmatrix}\begin{bmatrix}C(0)\\ D(0)\end{bmatrix}=\begin{bmatrix}0\\ Y\end{bmatrix}$$

Hence and from (2.14) the proof of theorem 2 is concluded. ■

Remark 1. Note that given the operators $C(0), D(0)$ such that $X(t)=\exp(tX_0)C(t)+\exp(-tX_0)D(t)$ satisfies (1.2), the explicit expression of the operator functions $C(t)$ and $D(t)$ are given by (2.4), and from (2.19), it is equivalent to solve the Cauchy problem (2.2) with the initial conditions $C_0=C(0)+D(0)$ and $C_1=X_0\big(C(0)-D(0)\big)$. In order to compute the solution of problem (1.2), it is necessary to solve (2.16). For the finite dimensional case it is an easy matter; for the infinite-dimensional case, and under the invertibility hypothesis of S, an explicit expression of S^{-1} is given in Lemma 1 of [9].

Theorem 2 provides a sufficient condition for the existence of only one solution of problem (1.2) in terms of the invertibility of the operator matrix S given by (2.14). In order to obtain a more concrete condition, in terms of data and a square root of λQ, the following corollary is an easy consequence of the above theorem 2 and lemma 1 of [9].

Corollary 3. *Let us consider Problem (1.2) where $\lambda\neq 0$, Q is an invertible operator which satisfies the condition (2.13) and let S be defined by (2.14), $X_0=\exp\big(\log_\alpha(\lambda Q)/2\big)$. Then the following results hold:*

(i) If $F_1-F_2X_0$ is invertible in $L(H)$ and the operator

(2.24)

$$K=(E_1+E_2X_0)-(E_1-E_2X_0)\exp(aX_0)(F_1-F_2X_0)^{-1}(F_1+F_2X_0)\exp(aX_0)$$

is invertible in $L(H)$, then problem (1.2) has only one solution $X(t)$ given by (2.3)-(2.4), where

(2.25)

$$C(0)=-K^{-1}(E_1-E_2X_0)\exp(aX_0)(F_1-F_2X_0)^{-1}Y$$

$$\begin{aligned}D(0)=&\{\exp(aX_0)(F_1-F_2X_0)^{-1}\}\\ &\{I+(F_1+F_2X_0)\exp(aX_0)K^{-1}(E_1-E_2X_0)\exp(aX_0)(F_1-F_2X_0)^{-1}\}Y\end{aligned}$$

(ii) If $E_1 + E_2X_0$ is invertible in $L(H)$ and the operator

(2.26)
$$W = (F_1 - F_2X_0)\exp(-aX_0) - (F_1 + F_2X_0)\exp(aX_0)(E_1 + E_2X_0)^{-1}(E_1 - E_2X_0)$$

is invertible in $L(H)$, then problem (1.2) has only one solution $X(t)$ given by (2.3)-(2.4), where

$$C(0) = -(E_1 + E_2X_0)^{-1}(E_1 - E_2X_0)W^{-1}Y \qquad (2.27)$$
$$D(0) = W^{-1}Y$$

Next example provides a lot of cases where the uniqueness property and an explicit expression of the solution of problem (1.2) are available.

Example 1. Let us consider Problem (1.2) where $E_i = F_i = I$, $i = 0, 1$, $a = 1$ and H is a complex separable Hilbert space with orthonormal basis $\{e_n; n \geqq\}$ and let $\{u_n\}_{n \geq 1}$ be a sequence of complex numbers convergent to a complex number u, such that for $\lambda \neq 0$ the following properties are satisfied

(2.28)
$$(\lambda u_n)^{\frac{1}{2}} \neq k\pi i\,,\ (\lambda u)^{\frac{1}{2}} \neq k\pi i\,,\ (\lambda u)^{\frac{1}{2}} \neq \pm 1\,,\ (\lambda u_n)^{\frac{1}{2}} \neq \pm 1\,,\ k \text{ integer},\ n \geqq 1$$

Then the operator Q defined on H by $Q(e_j) = u_j e_j$, for $j \geqq 1$, is invertible because from (2.28), $u_n \neq 0$, and $u \neq 0$, for all $n \geqq 1$, and the spectrum of Q is the set

$$\sigma(Q) = \{u_n\,,\ n \geqq 1\} \cup \{u\} \qquad (2.29)$$

and a square root of λQ is defined by the diagonal operator X_0 defined by $X_0(e_j) = (\lambda u_j)^{\frac{1}{2}}$, for $j \geqq 1$. So, the operator $F_1 - F_2X_0 = I - X_0$ is invertible in $L(H)$ because from (2.8), the spectrum $\sigma(I - X_0) = \{1 - w\,,\ w \text{ belongs to } \sigma(X_0)\}$ does not contains to the complex number 0. Also, the operator K defined by (2.24) takes the form

$$K = (I + X_0) - (I - X_0)\exp(X_0)(I - X_0)^{-1}(I + X_0)\exp(X_0)$$

and from the commutativity between $I - X_0$, $(I - X_0)^{-1}$ and $\exp(X_0)$, it follows that

$$K = (I + X_0)(I - \exp(2X_0))$$

From the spectral mapping theorem, [5, p. 569], the spectrum $\sigma(X_0)$ is defined by

$$\sigma(K) = \{(1 + z)(1 - \exp(2z)); z \quad \text{belongs to } \sigma(X_0)\} =$$
$$= \{(1 + z)(1 - \exp(2z)); z = (\lambda w)^{\frac{1}{2}}, w \quad \text{belongs to } \sigma(Q)\}$$

and from the hypothesis (2.28) one concludes that $0 \notin \sigma(K)$. Thus K is invertible in $L(H)$ and from corollary 3, for any continuous function $F(t)$, Problem (1.2) has only one solution given by (2.3)-(2.4) and (2.25), considering the corresponding data of our problem.

3. The case $\lambda = 0$

We begin this section with an analogous result to theorem 1 of section 2, corresponding to the case $\lambda = 0$, It may be considered as a variation of the parameters method for the operator differential equation $X^{(2)}(t) = F(t)$.

Lemma 4. *Let us consider the Cauchy problem*

$$X^{(2)}(t) = F(t) \quad ; \quad X(0) = C_0 \quad , \quad X^{(1)}(0) = C_1 \tag{3.1}$$

where C_i, for $i = 0,1$, are operators in $L(H)$ and $F(t)$ is a continuous $L(H)$ valued operator function. Then the only solution of Problem (3.1) is given by the operator function

$$X(t) = C(t) + D(t) \tag{3.2}$$

(3.3)

$$C(t) = C(0) - \int_0^t sF(s)ds\,,\ C(0) = C_0\,,\ D(t) = D(0) - \int_s^t F(s)ds\,,\ D(0) = C_1$$

Proof: From the uniqueness for the solution of a Cauchy problem of the type (3.1) [11], it is sufficient to prove that $X(t)$ given by (3.2)-(3.3), satisfies the problem (3.1). Let us consider operator functions $C(t)$ and $D(t)$ satisfying the system

$$\begin{bmatrix} I & t \\ 0 & I \end{bmatrix} \begin{bmatrix} C^{(1)}(t) \\ D^{(1)}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ F(t) \end{bmatrix} \tag{3.4}$$

As $\begin{bmatrix} I & t \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -t \\ 0 & I \end{bmatrix}$, condition (3.4) implies that $C(t)\,,\ D(t)$ satisfy

$$\begin{bmatrix} C^{(1)}(t) \\ D^{(1)}(t) \end{bmatrix} = \begin{bmatrix} I & -t \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ F(t) \end{bmatrix} = \begin{bmatrix} -tF(t) \\ F(t) \end{bmatrix} \tag{3.5}$$

From (3.2) it follows that

$$C_0 = X(0) = C(0) \tag{3.6}$$

and by differentiation of (3.2), and taking into account that first equation of (3.4) implies $C^{(1)}(t) + tD^{(1)}(t) = 0$, one gets $X^{(1)}(t) = D(t)$, and taking $t = 0$,

$$C_1 = X^{(1)}(0) = D(0) \tag{3.7}$$

If we integrate in (3.5) and we consider (3.6)-(3.7), it follows that $C(t)\,,\ D(t)$ must be defined by (3.3). Note that by differentiation in (3.2) and taking into account (3.4), we have $X^{(1)}(t) = D(t)\,,\ X^{(2)}(t) = D^{(1)}(t) = F(t)$, thus $X(t)$ given by (3.2)-(3.3), is the solution of (3.1).

Next result concerns with the boundary value problem

(3.8)

$$X^{(2)}(t) = F(t)\,,\ E_1X(0)+E_2X^{(1)}(0) = 0\,,\ F_1X(a)+F_2X^{(1)}(a) = 0\,,\ 0 \leqq t \leqq a$$

where $F(t)$ is a continuous $L(H)$ valued operator function defined on $[0, a], a > 0$.

Theorem 5. *Let us consider Problem (3.8), let M, N, R be the operators defined by*

$$M = -\int_0^a sF(s)ds\,,\ N = \int_0^a F(s)ds\,,\ R = -F_1(aN + M) - F_2 N \tag{3.9}$$

and let T be the operator matrix

$$T = \begin{bmatrix} E_1 & E_2 \\ F_1 & aF_1 + F_2 \end{bmatrix} \tag{3.10}$$

(i) Problem (3.8) is solvable, if and only if, the system

$$T\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ R \end{bmatrix} \tag{3.11}$$

admits a solution $\begin{bmatrix} U \\ V \end{bmatrix}$. Under this hypothesis, the solution set of problem (3.8) is given by (3.2)-(3.3) where $U = C_0$ and $V = C_1$.

(ii) If the operator T given by (3.10) is invertible, then Problem (3.8) has one unique solution $X(t)$ given by (3.2)-(3.3) where

$$\begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = T^{-1}\begin{bmatrix} 0 \\ R \end{bmatrix} \tag{3.12}$$

Proof: From Lemma 4, the general solution of the operator differential equation $X^{(2)}(t) = F(t)$, is given by (3.2)-(3.3). Note that operators $C(a)$ and $D(a)$ and $C(0)\,D(0)$, are related by the expressions

$$C(a) = C(0) + M \qquad , \qquad D(a) = D(0) + N \tag{3.13}$$

where M and N are defined by (3.9). Taking into account that the operator functions $C(t)$ and $D(t)$ satisfy (3.4), and $X^{(1)}(t) = D(t)$, by impossing to the operator function $X(t)$ given by (3.2), that the boundary value conditions of (3.8) are satisfied, one gets

$$\begin{aligned} E_1C(0) + E_2D(0) &= 0 \\ F_1\big(C(a) + aD(a)\big) + F_2D(a) &= 0 \end{aligned} \tag{3.14}$$

Considering (3.13), second equation of (3.14) may be written

$$F_1C(0) + (aF_1 + F_2)D(0) = -F_1(aN + M) - F_2N = R \tag{3.15}$$

Thus Problem (3.8) is solvable, if and only if, there are operators $U = C(0)$, $V = D(0)$ such that

$$\begin{bmatrix} E_1 & E_2 \\ F_1 & aF_1 + F_2 \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ R \end{bmatrix} \tag{3.16}$$

Hence parts (i) and (ii) are established.

Next corollary provides sufficient conditions in terms of data, in order to obtain the uniqueness of solution for Problem (3.8), as well as an explicit expression of the solution.

Corollary 6. *Let us consider Problem (3.8) and let R be the operator in $L(H)$ defined by (3.9)*

(i) If E_1 is invertible in $L(H)$, then Problem (3.8) has only one solution, if the operator $V = (aF_1 + F_2) - F_1E_1^{-1}E_2$ is invertible in $L(H)$, Under this hypothesis the unique solution of the problem is given by (3.2)-(3.3), where

$$C(0) = -E_1^{-1}E_2V^{-1}R \quad , \quad D(0) = V^{-1}R \tag{3.17}$$

(ii) If the operator $aF_1 + F_2$ is invertible, then Problem (3.8) has only one solution if the operator $W = E_1 - E_2(aF_1 + F_2)^{-1}F_1$ is invertible in $L(H)$. Under this hypothesis, the unique solution is given by (3.2)-(3.3), where $C(0)$ and $D(0)$ are given by

$$\begin{gathered} C(0) = -W^{-1}E_2(aF_1 + F_2)^{-1}R,\ D(0) = \\ (aF_1 + F_2)^{-1}F_1W^{-1}E_2(F_1 + aF_2)^{-1} + IR \end{gathered} \tag{3.18}$$

Proof: It is an easy consequence of Lemma 1 of [9] and of the previous th. 5

References

1. E.A. CODDINGTON AND N. LEVINSON, Theory of Ordinary Differential Equations, *MacGraw-Hill* (1955).
2. J.B. CONWAY AND B.B. MORREL, Roots and Logarithms of Bounded Operators on Hilbert Space, *J. of Funct. Anal.* **70** (1) (1987), 171–193.
3. G.W. CROSS AND P. LANCASTER, Square Roots of Complex Matrices, *Linear and Mult. Algebra,* 1 (1974), 289–293.
4. J.E. DENNIS, JR, J.F. TRAUB AND R.P. WEBER, The Algebraic Theory of Matrix Polynomials, *SIAM J. Numer. Anal.* **14** (1976), 831–845.
5. N. DUNFORD AND J. SCHWARTZ, Linear Operators I, *Interscience* (1957).
6. I. GOHBERG, P. LANCASTER AND L. RODMAN, Matrix Polynomials, *Academic Press* (1982).
7. E.L. INCE, Ordinary Differential Equations, *Dover* (1927).
8. L. JÓDAR, Boundary value problems for second order operator differential equations, *Linear Algebra and Appl.* **83** (1986), 29–38.
9. L. JÓDAR, Explicit Expressions for Sturm-Liouville Operator Problems, *Proc. of the Edinburg Math. Soc.* **30** (1987), 301–309.
10. M.G. KREIN AND H. LANGER, On some mathematical principles in the linear theory of damped oscillations of continua, I, II, Integral Equation and Operator Theory 1, *Translation* (1978), 364–399, 539–566.
11. S.G. KREIN, Linear Differential Equations in Banach spaces, *Trans. Math. Monographs* **29** (1971).

12. P. LANCASTER, Lambda-matrices and Vibrating Systems, *Pergamon* (1966)
13. P.A. MISNAEVSKII, On the Green's function and the asymptotic behavior of eigenvalues of Sturm-Liouville operator problem, *Soviet. Math. Dokl.* **13** (1972), 465–469.
14. P.A. MISNAEVSKII, On the Spectral theory for the Sturm-Liouville equation with operator coefficient, *Math. U.S.S.R. Izv.* **10** (1976), 145–180.
15. V.A. MILHAILEC, The solvability and completeness of the system of eigenfunctions and associated functions of nonselfadjoint boundary value problems for a Sturm-Liouville operator equation, *Soviet. Math. Dokl.* **15** (1974), 1327–1330.
16. H.H. MURTAZIN, On the point spectrum of a class of differential operator, *Soviet. Math. Dokl.* **15** (1974), 1812–1814.

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