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THE LATTICE R-tors FOR PERFECT RINGS

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Abstract

We define $\sim_F$ in R-tors by $\tau \sim_F \sigma$ iff the class of τ-codivisible modules coincides with the class of σ-codivisible modules. We prove that if R is left perfect ring (resp. semiperfect ring) then every $[\tau]_F \in R\text{-tors}/\sim_F$ (resp. $[\chi]_F$ and $[\xi]_F$) is a complete sublattice of R-tors. We describe the largest element in $[\tau]$ as $\chi(\operatorname{Rad} R/t_\tau(\operatorname{Rad} R))$ and the least element of $[\tau]$ as $\xi(t_\tau(\operatorname{Rad} R))$.

Using these results we give a necessary and sufficient condition for the central splitting of Goldman torsion theory when R is semiperfect.

We prove that for a QF ring R the least element of $[\chi]_{\sim_F}$ is the Goldie torsion theory. This can be used to prove that for a QF ring $\sim_F$ and $\sim_T$ are equal, where $\tau \sim_T \sigma$ iff the class of τ-injective modules coincides with the class of σ-injective modules.

0. Introduction

Throughout this work R will denote an associative unital ring; R-tors will denote the complete brouwerian lattice of all left hereditary torsion theories; χ (resp. ξ) will denote the largest (resp. the smallest) element of R-tors.

If $\{M_\alpha\}_{\alpha\in X}$ is a family of left R-modules, then $\chi(\{M_\alpha\})$ will denote the largest torsion theory respect to which every M_α is torsion free. $\xi(\{M_\alpha\})$ will denote the smallest torsion theory respect to which every M_α is torsion. We consider a torsion theory τ as an ordered pair $\tau = (\mathsf{T}_\tau, \mathsf{F}_\tau)$, where T_τ denotes the class of τ-torsion modules, and F_τ denotes the class of τ-torsion free modules. Also remember that the order in R-tors is given by: $\tau \leq \sigma$ iff $\mathsf{T}_\tau \subseteq \mathsf{T}_\sigma$.

Remember that a left module M is τ-codivisible iff $\operatorname{Ext}_R(M, K) = (0)\ \forall K \in \mathsf{F}_\tau$. Let us denote P_τ the class of τ-codivisible modules. We define $\sim_F$ in R-tors by $\tau \sim_F \sigma$ iff $\mathsf{P}_\tau = \mathsf{P}_\sigma$. Obviously this is an equivalence relation in R-tors. Our aim in this work is to study R-tors by looking at the equivalence classes $[\tau] \in R\text{-tors}/\sim_F$. In case R is a left perfect ring, these equivalence classes are complete sublattices of R-tors. So, in $[\tau]$ there must exist a largest element (resp. a smallest element) which will be denote τ^* (resp. τ_*). We describe $\tau^* = \chi(\operatorname{Rad} R/t_\tau(\operatorname{Rad} R))$ (resp. $\tau_* = \xi(t_\tau(\operatorname{Rad} R))$), where $\operatorname{Rad} R$ denotes the Jacobson radical of R.

We also obtain some generalizations of some results of Bland (see 3).

We also prove that for a QF-ring R the smallest element of $[\chi]_{\sim_F}$ (which exists, since R is left perfect) is Goldie's torsion theory. In fact, it can be proved that for a QF-ring R the equivalence relations $\sim_F$ and $\sim_T$ coincide, where we define $\tau \sim_T \sigma$ iff the class of τ-injective modules coincides with the class of σ-injective modules.

The partition R-tors$/\sim_T$ has been studied by Raggi & Ríos (see [12] and [13]).

We will denote by $\mathcal{S}_\tau$ the class of all short exact sequences $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ in R-mod such that $K \in \mathsf{F}_\tau$, where $\tau \in R$-tors.

We will denote P_τ the class of R-modules that are projective with respect to each sequence in $\mathcal{S}_\tau$.

We will denote $\mathcal{A}_\tau$ the proper class of short exact sequences in R-mod which make projective each element of P_τ.

We should observe that ${}_RP$ is projective with respect to each short exact sequence in $\mathcal{S}_\tau \Longleftrightarrow P$ is projective with respect to each element of $\mathcal{A}_\tau$.

Remarks.

1) (Ohtake [10], Bican, Nemec, Kepka [2]). If $\tau = (\mathsf{T}, \mathsf{F}) \in R$-tors and $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ is a short exact sequence in R-mod such that P is projective an $K \in \mathsf{T}$, then $M \in \mathsf{P}_\tau$.

2) R-mod has enough $\mathcal{A}_\tau$-projectives (this means that $\forall_R M \in R - mod$ $\exists\, 0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0 \in \mathcal{A}_\tau$ with P projective with respect to $\mathcal{A}_\tau$.

3) Let ${}_RM \in R$-mod. Then: $M \in P_\tau \Longleftrightarrow M$ is a direct summand of a module of the form P/T, where P is projective and $T \in T_\tau$.

We should observe that in the above remark we can replace "projective" by "free".

Definition 1. (τ-codivisible cover, Bland [3]). *An $\mathcal{A}_\tau$-projective cover of ${}_RM$ is an exact sequence $0 \longrightarrow L \longrightarrow P \longrightarrow M \longrightarrow 0$, such that*

i) $L \in \mathsf{F}_\tau$.

ii) P is τ-codivisible (i.e. $\mathcal{A}_\tau$-projective).

iii) $i(L)$ is small in P ($i(L) << (P)$.

The fact of that τ-codivisible covers are unique except for isomorphic copies is a known result [3].

We will denote by $0 \longrightarrow K_\tau(M) \longrightarrow P_\tau(M) \longrightarrow M \longrightarrow 0$ the τ-codivisible cover of M, when it exists, and by $0 \longrightarrow K(M) \longrightarrow P(M) \longrightarrow M \longrightarrow 0$ the projective cover of M, when it exists.

Definition 2. *We define $\sim_F$ in R-tors by: $\sigma \sim_F \tau$ iff $\mathcal{A}_\sigma = \mathcal{A}_\tau$ (or equivalently, if $\mathsf{P}_\sigma = \mathsf{P}_\tau$, i.e. if the class of σ-codivisible modules coincides with the class of τ-codivisible covers).*

The relation defined above is, obviously, an equivalence relation. Under appropiate conditions the corresponding equivalence classes $[\tau]_{\sim_F}$, are complete sublattices of R-tors. This is the case when R is a left perfect ring.

Theorem 1. *If* $0 \longrightarrow K_\tau(M) \longrightarrow P_\tau(M) \longrightarrow M \longrightarrow 0$ *is a* τ*-codivisible cover of* M *and if* $0 \longrightarrow K(M) \longrightarrow P(M) \longrightarrow M \longrightarrow 0$ *is a projective cover of* M*, then* $\ker(P(M) \longrightarrow P_\tau(M))$ *is* τ*-torsion.*

Lemma 1. *Let* $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ *be a projective cover. Let us suppose* $\tau \sim_{\mathsf{F}} \sigma$*, then* $K \in \mathsf{T}_\tau \iff K \in \mathsf{T}_\sigma$*.*

Proof: Straightforward. ■

Theorem 2. *Suppose that* $0 \longrightarrow K(M) \longrightarrow P(M) \longrightarrow M \longrightarrow 0$ *is a projective cover. Then* $0 \longrightarrow K(M)/t_\tau(K(M)) \longrightarrow P(M)/t_\tau(K(M)) \longrightarrow M \longrightarrow 0$ *(*) is a* σ*-codivisible cover* $\forall\sigma \in [\tau]_{\mathsf{F}}$*.*

Proof: Direct from the definitions. ■

Note that the above theorem implies that if $0 \longrightarrow K_\tau(M) \longrightarrow P(M) \longrightarrow M \longrightarrow 0$ is a τ-codivisible cover, then $K_\tau(M) \in \mathsf{F}_{\bigvee_{[\tau]}\sigma}$. This is because $K_\tau(M) \in \bigcap_{[\tau]}\mathsf{F}_\sigma = \mathsf{F}_{\bigvee_{[\tau]}\sigma}$.

Let us also note that the following implications hold for $\sigma, \tau \in R$-tors:

$$\tau \leq \sigma \iff \mathsf{F}_\tau \supseteq \mathsf{F}_\sigma \Longrightarrow \mathcal{A}_\tau \supseteq \mathcal{A}_\sigma \iff \mathsf{P}_\tau \subseteq \mathsf{P}_\sigma.$$

Remarks. For a proper class $\mathcal{A}$ we have:

i) $\mathcal{A} = \mathcal{A}_\xi \iff \mathcal{A}$ is the class of all short exact sequences in R-mod $\iff \mathsf{P}_{\mathcal{A}} = \mathsf{P}_\xi$.

Also note that P_ξ, the class of ξ-codivisible modules is precisely the class of all projective modules.

ii)$\mathcal{A} = \mathcal{A}_\xi \iff S_{\mathcal{A}} = \{0 \longrightarrow 0 \longrightarrow M \longrightarrow M \longrightarrow 0 : M \in R\text{-mod}\} \iff R$-mod $= \mathsf{P}_{\mathcal{A}}$, the class of all projective modules.

Also note $\mathcal{A}_\chi$ is the class of all splitting short exact sequences in R-mod.

iii) $\tau \in R$-tors faithful $\Longrightarrow \tau \in [\xi]$: for if P is τ-codivisible, then P is a direct summand of a module $R^{(X)}/T$, where T is a τ-torsion submodule of $R^{(X)}$, which is in F_τ (being R in F_τ, by hypothesis). Then $T = 0$, and hence P is a direct summand of a free module; i.e., P is projective. So $\mathsf{P}_\xi = \mathsf{P}$, and we conclude by using i).

iv) If R is a domain (e.g. $\mathbb{Z}$) every $\chi \neq \tau \in R$-tors is faithful and hence is in $[\xi]_{\mathsf{F}}$. So R-tors$/\sim_{\mathsf{F}}$ has only the two elements $[\chi]_{\mathsf{F}} = \{\chi\}$, and $[\xi]_{\mathsf{F}} = R$-tors$\setminus\{\chi\}$.

Moreover $[\xi]$ has a maximal member: $\chi(R) = \tau_L$, Lambek's torsion theory.

v) For a stable torsion theory τ the following statements are equivalent:

a) $R \cong t_\tau(R) \times S$, where S is semisimple artinian.

b) $\tau \in [\chi]_{\mathsf{F}}$.

c) $\forall N \in \mathsf{F}_\tau$, N is an injective semisimple module.

Proof: a) $\Longleftrightarrow$ b) (See [11]), b) $\Longleftrightarrow$ c) follows from Theorem 3. ■

vi) For a left semiartinian ring are equivalent

a) $\tau_G \in [\chi]$ (τ_G denotes Goldie's torsion theory).

b) $R \cong \tau_G(R) \times S$, where S is semisimple artinian.

c) τ_G centrally splits.

d) τ_O is stable. Here τ_O denotes Goldman's torsion theory; i.e., the torsion theory generated by the projective semisimple modules.

Proof: b) $\Longleftrightarrow$ c) $\Longleftrightarrow$ d) (See [11]). a)$\Longleftrightarrow$ b) follows from Remark v).

vii) If R is right perfect ring, then the above conditions are also equivalent to:

e) $\mathrm{soc}_p(\mathrm{Rad}\, R) = 0$ (See Theorem 18). Here soc_p denotes the projective socle, and $\mathrm{Rad}\, R$ denotes the Jacobson radical. ■

The following is an easy generalization of a Theorem of Bland, in our context.

Theorem 3. *Are equivalent for $\tau \in R$-tors:*

i) $\tau \in [\chi]$.

ii) $\mathsf{P}_\tau = \mathsf{P}_\chi = R$-mod.

iii) $\mathcal{A}_\tau$ = class of all splitting short exact sequences.

iv) $\forall_R N \in \mathsf{F}_\tau$, N is semisimple and injective.

v) The ring $R/t_\tau(R)$ is semisimple.

vi) All cyclic modules are $\mathcal{A}_\tau$-projective.

(Bland in (3) shows the equivalence of ii), iv) and v), the equivalence of the others follows directly from the definitions).

Corollary 1. *R is semisimple $\Longleftrightarrow$ R-tors$/\sim_{\mathsf{F}} = \{[\xi]\}(\Longleftrightarrow \xi \sim_{\mathsf{F}} \chi)$.*

Proof: $\Longrightarrow$) If R is semisimple, then $\forall \tau \in R$-tors, $R/t_\tau(R)$ is semisimple; so by v) $\Longrightarrow$ i) in Theorem 3 we get $\tau \in [\chi]_{\mathsf{F}}$. Hence $[\xi] = [\chi] = R$-tors.

$\Longleftarrow$) If R-tors$/\sim_{\mathsf{F}} = \{[\xi]\}$. In particular $\xi \in [\chi] = [\xi]$. So by using i) $\Longleftrightarrow$ iv) in the above theorem, we get N is semisimple $\forall_R N \in \mathsf{F}_\xi$ (but $\mathsf{F}_\xi = R$-mod). Then R is semisimple. ■

From the preceeding corollary, we obtain immediately the following result.

Corollary 2. *(Bland [3], Corollary 3.4 proves the "if" part). R is semisimple $\Longleftrightarrow \exists \tau \in [\chi]$, faithful.*

Proof: $\Longrightarrow$) If R is semisimple, then ξ has the required properties.

$\Longleftarrow$) If $\tau \in [\chi]$ is faithful, then we get that $\tau \in [\xi]$ (see remark iii), after Theorem 2). Thus $\tau \in [\xi] \cap [\chi]$. Hence $[\xi] = [\chi]$. ■

Theorem 4. *Let τ be an element of R-tors. Then $[\tau]_{\mathsf{F}}$ is closed under finite meets.*

Proof: Let us suppose that $\tau_1 \sim_{\mathsf{F}} \tau_2 \sim_{\mathsf{F}} \tau$. By the observation after Theorem 2 we have that $\mathcal{A}_{\tau_1} \subseteq \mathcal{A}_{\tau_1 \wedge \tau_2}$ $(\tau_1 \wedge \tau_2 \leq \tau_2)$. Now, let us consider the diagram

$$\begin{tikzcd}
 & & & S \arrow[d, "\alpha"] & \\
0 \arrow[r] & L \arrow[r] & M \arrow[r] & M/L \arrow[r] & 0
\end{tikzcd}$$

with $L \in \mathsf{F}_{\tau_1 \wedge \tau_2}$, $S \in \mathsf{P}_{\tau_1}$, and remember that S is $\mathcal{A}_\tau$-projective iff S is projective with respect to each exact sequence of the form $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ with $L \in \mathsf{F}_\tau$. Let us extend the above diagram to

$$\begin{tikzcd}
 & & & S \arrow[d, "\alpha"'] & \\
0 \arrow[r] & L \arrow[r] & M \arrow[r, "p"] & M/L \arrow[r] & 0 \\
0 \arrow[r] & t_2(L) \arrow[r] \arrow[u] & M \arrow[r, "\bar{p}"] \arrow[u, equal] & M/t_2(L) \arrow[r] \arrow[u, "\pi"'] & 0
\end{tikzcd}$$

where π is the natural epimorphism. Now $M/t_2(L) \in \mathsf{F}_{\tau_2}$; so $0 \longrightarrow \ker \pi \longrightarrow M/t_2(L) \xrightarrow{\pi} M/L \longrightarrow 0 \in \mathcal{A}_{\tau_2} = \mathcal{A}_{\tau_1}$. Inasmuch as S is in $\mathsf{P}_{\tau_1} = \mathsf{P}_{\tau_2}$, we have that $\exists \beta \colon S \longrightarrow M/t_2(L)$, such that $\pi \circ \beta = \alpha$. Now let us observe that $t_1(t_2(L)) \in \mathsf{T}_{\tau_1} \cap \mathsf{T}_{\tau_2} = \mathsf{T}_{\tau_1 \wedge \tau_2}$.

But in the other hand, $t_1(t_2(L)) \subseteq L \in \mathsf{F}_{\tau_1 \wedge \tau_2}$; hence $t_1(t_2(L)) = 0$. So $t_2(L) \in \mathsf{F}_{\tau_1}$, which implies that $0 \longrightarrow t_2(L) \longrightarrow M \longrightarrow M/t_2(L) \longrightarrow 0$ belongs to $\mathcal{A}_{\tau_1}$. Hence $\exists \gamma : S \longrightarrow M$ such that $\bar{p} \circ \gamma = \beta$; so the following diagram is commutative:

$$\begin{tikzcd}
 & & & S \arrow[dl, "\gamma"'] \arrow[d, "\beta"] \arrow[r, "\alpha"] & M/L \arrow[r] & 0 \\
0 \arrow[r] & L \arrow[r] & M \arrow[urr, "p"] \arrow[d, equal] & \; & & \\
0 \arrow[r] & t_2(L) \arrow[r] \arrow[u] & M \arrow[r, "\bar{p}"] & M/t_2(L) \arrow[r] \arrow[uu, "\pi"'] & 0 &
\end{tikzcd}$$

But then $\gamma \circ p = \pi \circ \bar{p} \circ \gamma = \pi \circ \beta = \alpha$. Hence $S \in \mathsf{P}_{\tau_1 \wedge \tau_2}$, and then $\mathsf{P}_{\tau_1} \subseteq \mathsf{P}_{\tau_1 \wedge \tau_2}$, and from this we get $\mathcal{A}_{\tau_1 \wedge \tau_2} \subseteq \mathcal{A}_{\tau_1}$, (see the observation after Theorem 2).

Hence $\mathcal{A}_{\tau_1 \wedge \tau_2} = \mathcal{A}_{\tau_1}$, and so $\tau_1 \wedge \tau_2 \sim_{\mathsf{F}} \tau_1 \sim_{\mathsf{F}} \tau$. ■

If the ring R is left perfect we can prove much more.

Theorem 5. *If R is a left perfect ring, then $[\tau]$ is closed under taking arbitrary meets, $\forall \tau \in R$-tors.*

Proof: Let $P' \in \mathsf{P}_\tau$ and let

$$\begin{tikzcd} & & & P' \arrow[d, "\alpha"] & \\ 0 \arrow[r] & L \arrow[r, "i"] & M \arrow[r, "p"] & N \arrow[r] & 0 \end{tikzcd}$$

be a diagram with $L \in \mathsf{F}_{\wedge[\tau]}$. Let $0 \longrightarrow K(N) \longrightarrow P(N) \longrightarrow N \longrightarrow 0$ and $0 \longrightarrow K_\tau(N) \longrightarrow P_\tau(N) \longrightarrow N \longrightarrow 0$ be a projective and τ-codivisible covers, respectively. Then $\exists\, \alpha : P' \longrightarrow P_\tau(N)$ such that

$$\begin{tikzcd} & K' \arrow[r] \arrow[d, "u"] & P(N) \arrow[r, "s"] \arrow[d, "\pi'"] & P_\tau(N) \arrow[d, "\pi"] & \\ 0 \arrow[r] & L \arrow[r, "i"] & M \arrow[r, "p"] & N & P' \arrow[ul, "\bar{\alpha}"'] \arrow[l, "\alpha"] \end{tikzcd}$$

commutes (because P' is τ-codivisible and $0 \longrightarrow K_\tau(N) \longrightarrow P_\tau(N) \longrightarrow N \longrightarrow 0 \in \mathcal{A}_\tau$), where π' is the epimorphism provided by the projectivity of $P(N)$, and u is the morphism obtained from the universal property of kernels.

Moreover, by Theorem 1, we have that $K' \in \mathsf{T}_\sigma$ $\forall \sigma \in [\tau]$. Hence we get $K' \in \mathsf{T}_{\wedge_{[\tau]}\sigma}$. As $L \in \mathsf{F}_{\wedge_{[\tau]}\sigma}$, we get $u = 0$. But then, given the commutativity in the first square, we get that $\exists\, \beta : P_\tau(N) \longrightarrow M$ such that $\beta \circ s = \pi'$.

So we have that in the diagram

$$\begin{tikzcd} P(N) \arrow[r, two heads, "s"] \arrow[d, "\pi'"] & P_\tau(N) \arrow[d, "\pi"] \arrow[dl, "\beta"'] \\ M \arrow[r, "p"] & N \end{tikzcd}$$

the square and the top triangle commute; i.e., $\pi \circ s = p \circ \pi' = p \circ \beta \circ s$. But as s is epi, we have that $\pi = p \circ \beta$; i.e. the bottom triangle is also commutative.

Summarizing, we have the following commutative diagram

$$\begin{tikzcd} & & & P_\tau(N) \arrow[dl, "\beta"'] \arrow[d, "\Pi"] & P \arrow[l, "\bar{\alpha}"'] \arrow[dl, "\alpha"] \\ 0 \arrow[r] & L \arrow[r, "i"] & M \arrow[r, "p"] & N \arrow[r] & 0 \end{tikzcd}$$

from which we get that $P \in \mathbf{P}_{\wedge_{[\tau]}}$. Hence $\mathbf{P}_\tau \subset \mathbf{P}_{\wedge_{[\tau]}}$ and then $\mathcal{A}_{\wedge_{[\tau]}} \subset \mathcal{A}_\tau$. But $\wedge[\tau] \leq \tau \implies \mathcal{A}_{\wedge_{[\tau]}} \subseteq \mathcal{A}_\tau$ (observation after Theorem 2). Hence $\mathcal{A}_{\wedge_{[\tau]}} = \mathcal{A}_\tau$ and so $\wedge_{[\tau]}\sigma \sim_{\mathsf{F}} \tau$.

So we have proved $\wedge[\tau] \in [\tau]$ and this is sufficient for seeing that $[\tau]$ is closed taking under arbitrary meets ($\{\tau_\alpha\} \subseteq [\tau] \Longrightarrow \wedge[\tau] \leq \wedge\{\tau_\alpha\} \leq \tau_\alpha$ and hence $\mathcal{A}_{\tau_\alpha} \subseteq \mathcal{A}_{\wedge\{\tau_\alpha\}} \subseteq \mathcal{A}_{\wedge[\tau]} = \mathcal{A}_{\tau_\alpha}$). ■

Theorem 6. *If R is a left perfect ring, then $[\tau]$ is closed under arbitrary joins.*

Proof: It's enough to prove that $\vee[\tau] \in [\tau]$. Let

$$(*) \qquad \begin{array}{ccccccccc} & & & & & & P' & & \\ & & & & & & \big\downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & L_\tau & \xrightarrow{i} & P_\tau & \xrightarrow{p} & M & \longrightarrow & 0 \end{array}$$

where the row is a τ-codivisible cover of M and where P' is a $\vee[\tau]$-codivisible module. By Theorem 2 we have that $L \in \mathsf{F}_\sigma, \forall\sigma \in [\tau]$; hence $L \in \cap_{[\tau]}\mathsf{F}_\sigma = \mathsf{F}_{\vee[\tau]}$. So, (*) belongs to $\mathcal{A}_{\vee[\tau]}$, and consequently $\exists\bar{\alpha}$: $P' \longrightarrow P_\tau$ such that $p \circ \bar{\alpha} = \alpha$. Hence $P' \in \mathsf{P}_\tau$ and so $\mathsf{P}_{\vee[\tau]} \subseteq \mathsf{P}_\tau$, which is equivalent to saying that $\mathcal{A}_\tau \subseteq \mathcal{A}_{\vee[\tau]}$.

On the other hand, $\tau \leq \vee[\tau] \Longleftrightarrow \mathcal{A}_\tau \supseteq \mathcal{A}_{\vee[\tau]}$. Then $\mathcal{A}_\tau = \mathcal{A}_{\vee[\tau]}$ and so $\vee[\tau] \in [\tau]$. ■

From the two preceeding theorems we get at once:

Theorem 7. *R Left perfect $\Longrightarrow$ $[\tau]$ is a complete sublattice of R-tors, $\forall\tau \in R$-tors.*

By the preceeding theorem, we know that if R is a left perfect ring, then $[\tau]$ is closed under taking arbitrary joins and meets. Consequently, in $[\tau]$ must exist a largest and a smallest element, which will be denoted τ^* and τ_*, respectively. The following theorem gives us a useful description of each of them.

Theorem 8. *If R is a left perfect ring, then:*

i) $\tau^* = \chi\ \{K_\tau(M) | 0 \longrightarrow K_\tau(M) \longrightarrow P_\tau(M) \longrightarrow M \longrightarrow 0$ is an $\mathcal{A}_\tau$-codivisible cover, $M \in R$-mod $\}$.

ii) $\tau_* = \xi\{K(P_\tau(M)) | 0 \longrightarrow K(P_\tau(M)) \longrightarrow P(M) \longrightarrow P_\tau(M) \longrightarrow 0$ is a projective cover of $P_\tau(M)$, where $P_\tau(M)$ is a τ-codivisible cover of M, $M \in R$-mod$\}$.

Proof: First, let us observe that the sequence

$$0 \longrightarrow K(P_\tau(M)) \longrightarrow P(M) \longrightarrow P_\tau(M) \longrightarrow 0$$

in ii) comes from the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & K(P_\tau(M)) \arrow[d] & & \\
0 \arrow[r] & K(M) \arrow[r] & P(M) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_\tau(M) \arrow[r] & P_\tau(M) \arrow[r, "\pi"] \arrow[d] & M \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

where the rows and the column are exact, the rows are the projective and the τ-codivisible covers of M, respectively, and the R-morphism $P(M) \longrightarrow P_\tau(M)$ is given by the projectivity of $P(M)$.

i) By the note after Theorem 2, we have that $K_\tau(M) \in \mathsf{F}_\sigma$ $\forall \sigma \in [\tau]$; so $\chi\{K_\tau(M) | M \in R\text{-mod}\} \geq \tau^*$. Hence $\chi\{K_\tau(M) | M \in R\text{-mod}\} \geq \tau^*$. It would be enough to see that $\chi\{K_\tau(M) | M \in R\text{-mod}\} \in [\tau]$ and for this it would be enough to see that $\mathsf{P}_{\chi\{K_\tau(M)|M \in R\text{-mod}\}} \subseteq \mathsf{P}_{\tau^*}$.

But if $P \in \mathsf{P}_{\chi\{K_\tau(M)|M \in R\text{-mod}\}}$ and if the diagram

$$\begin{tikzcd}
 & & & P \arrow[d, "\alpha"] & \\
0 \arrow[r] & K \arrow[r, "i"] & L \arrow[r, "p"] & M \arrow[r] & 0
\end{tikzcd}$$

is such that $K \in \mathsf{F}_{\tau^*}$, then by taking a τ-codivisible cover of M we get the diagram

$$\begin{tikzcd}
0 \arrow[dr] & & & & \\
 & K_\tau(M) \arrow[dr] & & & \\
 & & P_\tau(M) \arrow[dr, "\Pi"] & & P \arrow[dl, "\alpha"'] \\
0 \arrow[r] & K \arrow[r, "i"] & L \arrow[r, "p"] & M \arrow[r] \arrow[dr] & 0 \\
 & & & & 0
\end{tikzcd}$$

Since $K_\tau(M) \in \mathsf{F}_{\chi\{K_\tau(M)|M \in R\text{-mod}\}}$, $\exists \bar{\alpha} : P \longrightarrow P_\tau(M)$ such that $\pi \circ \bar{\alpha} = \alpha$. Inasmuch as $K \in \mathsf{F}_{\tau^*} \subseteq \mathsf{F}_\tau$, $\exists\, \bar{\bar{\alpha}} : P_\tau(M) \longrightarrow L$ such that $p \circ \bar{\bar{\alpha}} = \pi$, hence

$p \circ (\bar{\bar{\alpha}} \circ \bar{\alpha}) = \alpha$ and then $P \in \mathsf{P}_{\tau^*}$. So $\mathsf{P}_{\chi\{K_\tau(M)|M \in R\text{-mod}\}} \subseteq \mathsf{P}_{\tau^*}$. Hence $\tau^* \leq \chi\{K_\tau(M)|M \in R\text{-mod}\}$ and hence $\tau^* = \chi\{K_\tau(M)|M \in R\text{-mod}\}$.

ii) By Lemma 1, we have that $K(P_\tau(M)) \in \mathsf{T}_{\wedge_{[\tau]}\sigma}$, hence $\xi\{K(P_\tau(M)|M \in R\text{-mod}\} \leq \tau_* = \wedge[\tau]$.

To get the converse inclusion, it is enough to see that

$$\mathsf{P}_{\tau^*} \subseteq \mathsf{P}_{\xi\{K(P_\tau(M))|M \in R\text{-mod}\}}.$$

So, let $P \in \mathsf{P}_{\tau^*}$ and

$$\begin{tikzcd} & & & P \arrow[d, "\alpha"] & \\ 0 \arrow[r] & K \arrow[r] & L \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

be a diagram such that $K \in \mathsf{F}_{\xi\{K(P_\tau(M))|M \in R\text{-mod}\}}$. Let us take $0 \longrightarrow K(P_\tau(M)) \longrightarrow P(M) \longrightarrow P_\tau(M) \longrightarrow 0$ as in the statement. Then $K_\tau(P_\tau(M)) \in T_{\wedge[\tau]}$. In the diagram

$$\begin{tikzcd} 0 \arrow[r] & K_\tau(P_\tau(M)) \arrow[r] \arrow[d, "\beta"] & P(M) \arrow[r] \arrow[d, "\bar{\pi}"] & P_\tau(M) \arrow[r] \arrow[d, "\pi"] & 0 \\ 0 \arrow[r] & K \arrow[r] & L \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

where $\bar{\pi}$ is given by projectivity of $P(M)$, and β is the restriction of $\bar{\pi}$ to $K_\tau(P_\tau(M))$, we have that $\beta = 0$, inasmuch $K \in \mathsf{F}_{\xi\{K(P_\tau(M))|M \in R\text{-mod}\}}$. Then, by the universal property of cokernels, we have that $\exists\, \beta\colon P_\tau(M)\colon \longrightarrow L$ such that

$$\begin{tikzcd} P(M) \arrow[r, two heads] \arrow[d, "\bar{\pi}"] & P_\tau(M) \arrow[dl, "\beta"] \\ L & \end{tikzcd}$$

commutes. But as $P(M) \longrightarrow P_\tau(M)$ is epic, we have that

$$\begin{tikzcd} & P_\tau(M) \arrow[dl, "\beta"'] \arrow[d, "\alpha"] \\ L \arrow[r] & M \end{tikzcd}$$

is commutative, too.

Now,

$$\begin{tikzcd} & & & P \arrow[d, "\alpha"] & \\ 0 \arrow[r] & K_\tau(M) \arrow[r] & P_\tau(M) \arrow[r, "\pi"] & M \arrow[r] & 0 \end{tikzcd}$$

with $P \in \mathsf{P}_{\tau^*}$ and $K_\tau(M) \in \mathsf{F}_\sigma$ ($\forall \sigma \in [\tau]$) imply that $K_\tau(M) \in \mathsf{F}_{\tau^*}$, and so $\exists \gamma\colon P \longrightarrow P_\tau(M)$ such that $\pi \circ \gamma = \alpha$. But then

$$\begin{tikzcd} & & P \arrow[dl, "\gamma"'] \arrow[d, "\alpha"] \\ & P_\tau(M) \arrow[dl, "\beta"'] \arrow[dr] & \\ L \arrow[rr] & & M \end{tikzcd}$$

commutes.

Hence $P \in \mathsf{P}_{\xi\{K(P_\tau(M))|M\in R\text{-mod}\}}$ Thus, $\mathsf{P}_{\tau^*} \subsetneq \mathsf{P}_{\xi\{K(P_\tau(M))|M\in R\text{-mod}\}}$. So we get $\tau^* = \xi\{K(P_\tau(M))|M \in R\text{-mod}\}$. ■

For the particular cases when $\tau \in \{\xi, \chi\}$ and when the ring R is left perfect, we give descriptions of τ^* and τ_* by using the Jacobson radical of R, which we will extend to arbitrary torsion theories and for semiperfect rings.

Theorem 9. *For left perfect R we have that*
i) $\xi^* = \chi(\mathcal{J}(R))$
ii) $\chi_* = \xi(\mathcal{J}(R))$,
where $\mathcal{J}(R)$ denotes the Jacobson radical of R.

Proof: i) By Theorem 8,

$$\begin{aligned} \xi^* &= \chi\{K_\xi(M)\,|\,0 \longrightarrow K_\xi(M) \longrightarrow P_\xi(M) \longrightarrow M \longrightarrow 0 \\ &\qquad\qquad \text{is a } \xi\text{-codivisible cover, } M \in R\text{-mod}\} \\ &= \chi\{K(M)\,|\,0 \longrightarrow K(M) \longrightarrow P(M) \longrightarrow M \longrightarrow 0 \\ &\qquad\qquad \text{is a projective cover, } M \in R\text{-mod}\ \} \\ &= \chi\big\{\, K \,\big|\, K \ll P \text{ and } {}_RP \text{ is projective}\,\big\}. \end{aligned}$$

As R is left perfect, $\mathrm{Rad}(P) = \mathcal{J}(R)P$ (see Anderson-Fuller, [1], Remark 28.5.(3)); so $K \ll P \iff K \subseteq \mathcal{J}(R)P \subseteq \mathcal{J}(R)R^{(X)}$ for some set X. Hence $K \ll P \iff \exists K \rightarrowtail \mathcal{J}(R)^{(X)} \iff K \in \mathsf{F}_{\chi(\mathcal{J}(R))}$. Thus $\xi^* \geq \chi(\mathcal{J}(R))$.

On the other hand, $\mathcal{J}(R) \ll R$ so we have that $0 \longrightarrow \mathcal{J}(R) \longrightarrow R \longrightarrow R/\mathcal{J}(R) \longrightarrow 0$ is a projective cover ($= \xi$-codivisible cover). Therefore $\mathcal{J}(R) \in \mathsf{F}_{\xi^*}$ (since $\mathcal{J}(R)$ is one of the modules cogenerating the torsion theory ξ^*, see the above description of ξ^*). Hence $\xi^* \geq \chi(\mathcal{J}(R))$. And therefore $\xi^* = \chi(\mathcal{J}(R))$.

ii)

$$\chi_* = \xi\left\{ K_\chi(P_\chi(M)) \;\middle|\; \begin{array}{c} 0 \longrightarrow K_\chi(P_\chi(M)) \longrightarrow P(M) \longrightarrow P_\chi(M) \longrightarrow 0 \\[1ex] \text{is induced by } \begin{array}{ccc} P(M) & \longrightarrow & P_\chi(M) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ M & \longrightarrow & M \end{array} \\ \text{where } \pi \text{ and } \pi' \text{ are projective and} \\ \tau\text{-codivisible cover, respectively.} \end{array} \right\}$$

Now $0 \longrightarrow K_\chi(M) \longrightarrow P_\chi(M) \longrightarrow M \longrightarrow 0$ is a χ-codivisible cover but $0 \longrightarrow 0 \longrightarrow M \longrightarrow M \longrightarrow 0$ is another (every left R-module is χ-codivisible). Thus we have that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_\chi(P_\chi(M)) & \longrightarrow & P(M) & \longrightarrow & P_\chi(M) & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & M & & \end{array}$$

is a projective cover of ${}_RM$. We have then that

$$\chi_* = \xi\{\, K \mid K \ll P, {}_RP \quad \text{projective} \,\}.$$

Again, $K \ll P$, ${}_RP$ projective $\Longleftrightarrow K \subseteq \mathcal{J}(R)^{(X)}$ for some set X. Therefore $K \ll P$, P projective $\Longrightarrow K \in \xi(\mathcal{J}(R))$. Hence $\chi_* \leq \xi(\mathcal{J}(R))$.

On the other hand, $0 \longrightarrow \mathcal{J}(R) \longrightarrow R \longrightarrow R/\mathcal{J}(R) \longrightarrow 0$ is a projective cover. Therefore $\mathcal{J}(R) \in \mathsf{T}_{\xi\{\, K_\chi P_\chi(M) \mid M \in R\text{-mod} \,\}}$ (is one of the generators of the above torsion theory). Therefore $\xi(\mathcal{J}(R)) \leq \chi_*$ and hence $\chi_* = \xi(\mathcal{J}(R))$. ■

We give now more "concrete" descriptions of τ^* and τ_*, in case R is left perfect.

Theorem 10. *If R is left perfect, then*
i) $\tau^* = \chi(\mathcal{J}(R)/t_\tau(\mathcal{J}(R)))$
ii) $\tau_* = \xi(t_\tau(\mathcal{J}(R)))$,
Where $\mathcal{J}(R)$ denotes the Jacobson's radical of R.

Proof: i) $0 \longrightarrow \mathcal{J}(R)/t_\tau(\mathcal{J}(R)) \longrightarrow R/t_\tau(\mathcal{J}(R)) \longrightarrow R/\mathcal{J}(R) \longrightarrow 0$ is a projective cover, since: a) $\mathcal{J}(R)/t_\tau(\mathcal{J}(R)) \ll R/t_\tau(\mathcal{J}(R))$, b) $R/t_\tau(\mathcal{J}(R))$ is τ-codivisible (by Remark 3, before Definition 1) and c) $\mathcal{J}(R)/t_\tau(\mathcal{J}(R)) \in \mathsf{F}_\tau$. Thus, by the note after Theorem 2, $\mathcal{J}(R)/t_\tau(\mathcal{J}(R)) \in \mathsf{F}_{\tau^*}$; therefore $\tau \leq \tau^* \leq \chi(\mathcal{J}(R)/t_\tau(\mathcal{J}(R)))$.

If $\tau^* \lneqq \chi(\mathcal{J}(R)/t_\tau(\mathcal{J}(R)))$ then $\exists 0 \neq {}_RM \in \mathsf{T}_{\chi(\mathcal{J}(R)/t_\tau(\mathcal{J}(R)))} \cap \mathsf{F}_{\tau^*}$. ($\exists 0 \neq M$ that is $\chi(\mathcal{J}(R)/t_\tau(\mathcal{J}(R))$-torsion but not τ^*-torsion, and by taking $M/t_{\tau^*}(M)$ if it would be necessary, we can suppose, without loss generality, that $M \in \mathsf{F}_{\tau^*}$).

By Theorem 8, $\tau^* = \chi\{\, K_\tau(M) \mid M \in R\text{-mod} \,\}$, so if $M \in \mathsf{F}_{\tau^*}$, then M is cogenerated by $\{\, E(K_\tau(M) \mid M \in R\text{-mod} \,\}$ (i.e., $\exists M \rightarrowtail \Pi_{N \in R\text{-mod}} E(K_\tau(N))$. Therefore, $\forall 0 \neq x \in M$, $\exists f_x : M \longrightarrow E(K_\tau(N))$ such that $f_x(x) \neq 0$ ([15]. Prop.VI.3.39). Therefore $0 \neq f_x(x) \in E(K_\tau(N))$. Because $K_\tau(N) <_e E(K_\tau(N))$ we have that $f_x(M) \cap K_\tau(N) \neq 0$. Hence $\exists 0 \neq y \in M$ such that $0 \neq f_x(y) \in K_\tau(N)$. Consequently, $Ry \overset{(f_x|Ry)}{\longrightarrow} K_\tau(N)$ is well defined.

Now,thanks to Theorem 2, we have that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & K_\tau(N) \arrow[r] \arrow[d, leftrightarrow] & P_\tau(N) \arrow[r] \arrow[d, leftrightarrow] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K(N)/t_\tau(N) \arrow[r] & P(N)/t_\tau(N) \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

(Here we assume that $0 \longrightarrow K(N) \longrightarrow P(N) \longrightarrow N \longrightarrow 0$ is a projective cover of N). Thus $K(N) \ll P(N)$ and then we have that $K(N) \leq \mathcal{J}(P(N)) = \mathcal{J}(R)P(N) \leq \mathcal{J}(R)R^{(Z)} = \mathcal{J}(R)^{(Z)}$ for some set Z ($\mathcal{J}(P(N)) = \mathcal{J}(R)P(N)$ since $P(N)$ is projective).

Therefore we have the following situation:

$$\begin{tikzcd}
Ry \arrow[r, "\subseteq"] \arrow[d, "f_x"] & M & & \\
K_\tau(N) \arrow[r, "\overset{\alpha}{\cong}"] & K(N)/t_\tau(K(N)) \arrow[r, "i"] & \mathcal{J}(R)^{(Z)}/t_\tau(K(N)) \arrow[r, two heads] & {}
\end{tikzcd}$$

$$\longrightarrow\!\!\!\!\rightarrow \mathcal{J}(R)^{(Z)}/t_\tau(\mathcal{J}(R)^{(Z)}) \cong [\mathcal{J}(R)/t_\tau(\mathcal{J}(R))]^{(Z)}.$$

As we that $\operatorname{Hom}_R(M, \mathcal{J}(R)/t_\tau(\mathcal{J}(R))) = 0$, we also have that $\operatorname{Hom}_R(Ry, \mathcal{J}(R)/t_\tau(\mathcal{J}(R))) = 0$ which implies that $i \circ \alpha(f_x(y)) \in t_\tau(\mathcal{J}(R^{(Z)}))$. Therefore $\exists I \in \mathcal{F}_\tau$ such that $I\ i \circ \alpha(f_x(y)) = 0$. But as i is a monomorphism, then $I\ (f_x(y)) = 0$; hence $0 \neq f_x(y) \in t_\tau(K_\tau(N)) = 0$, which is a contradiction ($K_\tau(N) \cong K(N)/t_\tau(K(N)) \in \mathsf{F}_\tau$). Therefore $\tau^* = \chi(\mathcal{J}(R)/t_\tau(\mathcal{J}(R))$ (here $\mathcal{F}_\tau$ denotes the idempotent filter corresponding to τ).

ii) If we consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & t_\tau(\mathcal{J}(R)) \arrow[r] \arrow[d] & t_\tau(\mathcal{J}(R)) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{J}(R) \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & R/\mathcal{J}(R) \arrow[r] \arrow[d] & 0\ (1) \\
0 \arrow[r] & \mathcal{J}(R)/t_\tau(\mathcal{J}(R)) \arrow[r] \arrow[d] & R/t_\tau(\mathcal{J}(R)) \arrow[r] \arrow[d] & R/\mathcal{J}(R) \arrow[r] \arrow[d] & 0\ (2) \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

(3)

the fact that (1) and (2) are projective and τ-codivisible covers, respectively, tells us that $\ker\pi$ in Column (3) is one of the modules generating the torsion theory τ_* (see Theorem 8). Therefore $t_\tau(\mathcal{J}(R)) \in \mathbb{T}_{\tau^*}$ and $\xi(t_\tau(\mathcal{J}(R))) \leq \tau_*$.

Now, if $K(P_\tau(M))$ is one of the generators of τ_*; i.e., if $0 \longrightarrow K(P_\tau(M)) \longrightarrow P(M) \longrightarrow P_\tau(M) \longrightarrow 0$ can be extended to a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K(P_\tau(M)) \arrow[r] \arrow[d] & K(M) \arrow[r] \arrow[d] & K_\tau(M) \arrow[d] & \\
0 \arrow[r] & K(P_\tau(M)) \arrow[r] \arrow[d] & P(M) \arrow[r] \arrow[d] & P_\tau(M) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where the two last rows are projective and τ-codivisible covers, respectively, then we have that $K(P_\tau(M)) \ll K(M) \ll P(M)$.

By Theorem 2, $K(P_\tau(M)) = t_\tau(K(M))$; therefore $K(P_\tau(M)) \leq \mathrm{Rad}(P(M)) = \mathcal{J}(R)P(M) \xrightarrow{\subseteq} \mathcal{J}(R)R^{(X)} = \mathrm{Rad}\, R^{(X)}$ and moreover $K(P_\tau(M)) \xrightarrow{\subseteq} t_\tau(\mathcal{J}(R)^{(X)}) = (t_\tau(\mathcal{J}(R)))^{(X)}$. Therefore $K(P_\tau(M)) \in \mathbb{T}_{\xi(t_\tau(\mathcal{J}(R)))}$ $\forall M \in R$-mod. Hence $\tau_* = \xi\{\, K(P_\tau(M)) \mid M \in R\text{-mod}\,\} \leq \xi(t_\tau(\mathcal{J}(R)))$ and so $\tau^* = \xi(t_\tau(\mathcal{J}(R)))$. ∎

Corollary 3. *If R is a left perfect ring, then $\tau \leq \sigma \Longrightarrow \tau_* \leq \sigma_*$.*

Proof: Straightforward. ∎

Theorem 10 is extended in [14] to the case of local rings. In that situation each $[\tau] \in R\text{-tors}/\sim_{\mathsf{F}}$ is closed under taking joins and meets and moreover the biggest element in $[\tau], \tau^*$ is given by $\tau^* = \chi(\mathcal{J}(R)/t_\tau(\mathcal{J}(R)))$ and also $\tau_* = \xi(t_\tau(\mathcal{J}(R)))$.

However, a ring may have the property of having each $[\sigma]_{\mathsf{F}}$ closed under arbitrary joins and meets without being semiperfect. Moreover, the elements σ^* and σ_* are not given by $\chi(\mathcal{J}(R)/t_\sigma(\mathcal{J}(R)))$ and by $\xi(t_\sigma(\mathcal{J}(R)))$, in general. As we see in the following examples.

Examples. In view of Remark 3 before Definition 1, is easy to see that if R is a domain, then R-tors admits the following partition:

$$\{[\xi] = [\chi(R)], \quad [\chi] = \{\chi\}\}.$$

It is clear that each equivalence class in R-tors$/\sim_{\mathsf{F}}$ admits a largest and a least element.

In particular this is the situation for $\mathbf{Z}$, the ring of integers, which is not a perfect ring.

Moreover, let us note that for $\mathbf{Z}$, in spite of the fact that each element in R-tors$/\sim_{\mathsf{F}}$ has a largest and a least element, they are not given as in Theorem 10. Explicity, $\mathcal{J}(\mathbf{Z}) = 0$, but we have that $[\chi] = \{\chi\}$, and so $\chi_* = \chi = \chi^*$. Nevertheless $\chi_* \neq \xi(t_\chi(\mathcal{J}(\mathbf{Z}))) = \xi(t_\chi(0)) = \xi(0) = \xi$.

On the other hand $[\xi] = \{\tau_G = \tau_L\}$ and $\xi^* = \tau_L$, but $\xi^* \neq \chi(\mathcal{J}(\mathbf{Z})/t_\xi(\mathcal{J}(\mathbf{Z})) = \chi(0/0) = \chi(0) = \chi$ (here τ_G denotes Goldie's torsion theory and τ_L denotes Lambek's torsion theory).

Lemma 2. *The following statements are equivalent for a left perfect ring:*

i) $\xi^* \vee \tau = \tau^* \; \forall \tau \in R$*-tors.*

ii) $[\tau] \xrightarrow{-\wedge\xi^*} [\xi]$ *is a lattice monomorphism with left inverse* $[\xi] \xrightarrow{-\vee\tau^*} [\tau]$.

iii) $\sigma \leq \tau \Longrightarrow [\tau] \xrightarrow{-\wedge\sigma^*} [\sigma]$ *is a lattice monomorphism with left inverse* $[\sigma] \xrightarrow{-\vee\tau_*} [\tau]$.

iv) $\sigma \leq \tau \Longrightarrow \tau \vee \sigma^* = \tau^*$.

v) $\forall \sigma, \tau \in R$*-tors* $\quad \tau \vee \sigma^* = (\tau \vee \sigma)^* = \tau^* \vee \sigma$.

Proof: Straightforward. ■

Theorem 11. *If R is a left perfect ring, all of whose torsion free classes F_τ are also torsion classes (i.e. each F_τ is closed under taking factors), then R enjoys the properties of Lemma 2.*

Proof: We will prove that $\xi^* \vee \tau = \tau^*$, $\forall \tau \in R$-tors. As $\xi^* \leq \tau^*$, we have that $\xi^* \vee \tau \leq \tau^*$ (by Theorem 9 we have that $\xi^* = \chi(\operatorname{Rad} R)$; $\tau^* = \chi(\operatorname{Rad} R/t_\tau(\operatorname{Rad} R))$. The hypothesis that F_τ is closed under factors $\Longrightarrow$ $\operatorname{Rad} R/t_\tau(\operatorname{Rad} R) \in \mathsf{F}_{\xi^*}$; hence $\tau^* \geq \xi^*$).

It remains to prove that $\xi^* \vee \tau$ cannot be different from τ^*. If it was, then $\exists 0 \neq M \in \mathsf{T}_{\tau^*} \cap \mathsf{F}_{\xi^* \vee \tau} = \mathsf{T}_{\tau^*} \cap \mathsf{F}_{\xi^*} \cap \mathsf{F}_\tau$. And as $\tau^* = \chi(\operatorname{Rad} R/t_\tau(\operatorname{Rad} R))$ (Theorem 10) we have that $\operatorname{Hom}_R(M, E(\operatorname{Rad} R/t_\tau(\operatorname{Rad} R)) = 0$ (*)

But as $M \in \mathsf{F}_{\xi^*}$ and $\xi^* = \chi(\operatorname{Rad} R)$ (Theorem 9) we have that $\exists u: M \rightarrowtail (E(\operatorname{Rad} R))^X$, monomorphism for some set X. Hence $\exists\, x \in X$ such that $p_x u(M) \neq 0$, where $p_x: (E(\operatorname{Rad} R))^X \longrightarrow E(\operatorname{Rad} R)$ is the canonical projection. Hence, in view of (*), we have that $u(M) \subseteq (t_\tau(E(\operatorname{Rad} R)))^X$. For if this were not true, $\exists y \in X$ such that $p_y(u(M)) \not\subset t_\tau(E(\operatorname{Rad} R))$ and hence

$$M \xrightarrow{p_y u} E(\operatorname{Rad} R)/t_\tau(E(\operatorname{Rad} R))$$

is not the zero morphism. But $E(\operatorname{Rad} R)/t_\tau(E(\operatorname{Rad} R)) \in \mathsf{F}_{\tau^*}$ and $M \in \mathsf{T}_{\tau^*}$ and so $\operatorname{Hom}_R(M, E(\operatorname{Rad} R)/t_\tau(E(\operatorname{Rad} R)) = 0$. This is a contradiction.

Now as $u(M) \subseteq (t_\tau(E(\operatorname{Rad} R)))^X$, we have that $p_x(u(m)) \subseteq t_\tau(E(\operatorname{Rad} R)) \in \mathsf{T}_\tau$, but being also a factor of $M \in \mathsf{F}_\tau$, it belongs to F_τ. Hence $0 \neq u(m) \in \mathsf{T}_\tau \cap \mathsf{F}_\tau$. This is a contradiction. Hence $\xi^* \vee \tau = \tau^*$. ■

The rings such that every torsion free class is closed under factors have been charaterized by Teply [16] and by Bronowitz and Teply [5]. We will call these rings BT-rings.

It is clear that for a BT-ring we have that:

$$\begin{aligned}\tau \leq \sigma &\Longrightarrow t_\tau(\operatorname{Rad} R) \leq t_\sigma(\operatorname{Rad} R)\\ &\Longrightarrow \operatorname{Rad} R/t_\tau(\operatorname{Rad} R) \twoheadrightarrow \operatorname{Rad} R/t_\sigma(\operatorname{Rad} R)\\ &\Longrightarrow \operatorname{Rad} R/t_\sigma(\operatorname{Rad} R) \in \mathsf{F}_\chi(\operatorname{Rad} R/t_\tau(\operatorname{Rad} R)) = \mathsf{F}_{\tau^*}\\ &\Longrightarrow [\sigma^* = \chi(\operatorname{Rad} R/t_\sigma(\operatorname{Rad} R) \geq \tau^*]\\ &\Longrightarrow \tau^* \geq \sigma^*.\end{aligned}$$

Moreover, for a BT-ring, we have that $\xi^* \vee \tau = \tau^*$, since it is clear from the preceeding that $\xi^* \vee \tau \leq \tau^*$. And we would have, if the above inequality was estrict, that $\mathsf{F}_{\tau^*} \subsetneqq \mathsf{F}_{\xi^* \vee \tau} = \mathsf{F}_{\xi^*} \cap \mathsf{F}_\tau$.

Hence $\exists 0 \neq M \in (\mathsf{F}_{\xi^*} \cap \mathsf{F}_\tau) \backslash \mathsf{F}_{\tau^*}$, and we can assume (changing M by $t_{\tau^*}(M) \neq 0$ if it was necessary), that $M \in \mathsf{T}_{\tau^*} \cap \mathsf{F}_{\xi^*} \cap \mathsf{F}_\tau$ ($t_{\tau^*}(M) \neq 0$ because $M \notin \mathsf{F}_{\tau^*}$).

Inasmuch as $M \in \mathsf{F}_{\xi^*}$, $\exists 0 \neq f \in \operatorname{Hom}_R(M, E(\operatorname{Rad} R))$; hence $\exists 0 \neq m \in M$ such that $\operatorname{Hom}_R(Rm, \operatorname{Rad} R) \neq 0$. But as $M \in \mathsf{T}_{\tau^*}$, we have that $\operatorname{Hom}_R(Rm, \operatorname{Rad} R/t_\tau(\operatorname{Rad} R)) = 0$ ($Rm \subseteq M \in \mathsf{T}_{\tau^*}$). So, if we take $0 \neq g \in \operatorname{Hom}_R(Rm, \operatorname{Rad} R)$, then we would have that $0 \neq g(Rm) \subseteq t_\tau(\operatorname{Rad} R) \in \mathsf{T}_\tau$. But on the other hand, $g(Rm)$ is a factor of $Rm \subseteq M \in \mathsf{F}_\tau$, and we have F_τ closed under taking factors by hypothesis. So we get that $0 \neq g(Rm) \in \mathsf{T}_\tau \cap \mathsf{F}_\tau$; which is a contradiction. So, we conclude that $\xi^* \vee \tau = \tau^*$.

So, for a BT-ring we have that Lemma 2 applies to give a nice partition of R-tors via the equivalence relation $\sim_{\mathsf{F}}$, because the equivalence class $[\xi]_{\mathsf{F}}$ contains an isomorphic copy or every other $[\tau]_{\mathsf{F}} \in R\text{-tors}/{\sim_{\mathsf{F}}}$. So, we will have R-tors completely determined as a lattice if we know the lattice structure of the sublattice $[\xi]_{\mathsf{F}}$.

Theorem 12. *(Bland [3, Theorem 2.8]). If R is a semiperfect ring, then*

$$\tau \sim_{\mathsf{F}} \chi \Longleftrightarrow \operatorname{Rad} R \in \mathsf{T}_\tau.$$

Bland's theorem is equivalent to the following result.

Theorem 13. *If R is a semiperfect ring, then $[\chi]$ contains a smallest element $\chi_* = \xi(\operatorname{Rad} R)$.*

Proof: $\Longrightarrow$) Since $0 \longrightarrow \operatorname{Rad} R \longrightarrow R \longrightarrow R/\operatorname{Rad} R \longrightarrow 0$ is a projective cover with $\operatorname{Rad} R \in \mathsf{T}_\chi = R$-mod, we have, using Bland's Theorem, that $\xi(\operatorname{Rad} R) \in [\chi]_{\mathsf{F}}$. Therefore $\xi(\operatorname{Rad} R)$ is the least element of $[\chi]_{\mathsf{F}}$.

$\Longleftarrow$) Let us suppose that $\chi_* = \xi(\operatorname{Rad} R)$. Now we have, for $\tau \in R$-tors, $\tau \in [\chi] \Longleftrightarrow \tau \geq \xi(\operatorname{Rad} R) \Longleftrightarrow \operatorname{Rad} R \in \mathsf{T}_\tau$. ■

The following two results can be proved (Rincón-Mejía [14]).

Theorem 14. *If R is a semiperfect ring, then $\xi^* = \chi(\operatorname{Rad} R)$, where ξ^* is the biggest element of $[\xi]_{\mathsf{F}}$.*

Theorem 15. *Rincón-Mejía [14].*

If R is a local ring, then $\forall[\tau] \in R$-tors$/\!\sim_{\mathsf{F}}$, we have that $[\tau]_{\mathsf{F}}$ has a biggest element, τ^*, given by $\tau^* = \chi(\operatorname{Rad} R/t_\tau(\operatorname{Rad} R))$, and a smallest element given by $\tau_* = \xi(t_\tau(\operatorname{Rad} R))$.

Theorem 16. *Let R be a semiperfect ring, then Goldman's torsion theory centrally splits $\Longleftrightarrow$ $\operatorname{soc}_p(\operatorname{Rad} R) = 0$.*

(Remember that M is a Goldman torsion module iff $M = \operatorname{soc}_P(M)$, where $\operatorname{soc}_p(M)$, where $\operatorname{soc}_p(M)$ denotes the projective socle of M).

Proof: $\Longleftarrow$) If $\operatorname{soc}_p(\operatorname{Rad} R) = (0)$, then every projective simple module ${}_RS$ is injective: for if ${}_RS$ is a simple projective module, then $S \in \mathsf{T}_{\xi(\operatorname{Rad} R)} \cup \mathsf{F}_{\xi(\operatorname{Rad} R)}$, since S is simple. But $S \in \mathsf{T}_{\xi(\operatorname{Rad} R)} \Longrightarrow \exists 0 \neq f\colon \operatorname{Rad} R \longrightarrow E(S)$. As $S \leq_e E(S)$, we have that $S \leq \operatorname{im} f$, so we have the diagram

$$\begin{tikzcd} \operatorname{Rad} R \\ f^{-1}(S) \arrow[u] \arrow[r, "f|_{f^{-1}(S)}"] & S \end{tikzcd}$$

where $f|_{f^{-1}(S)}$ is an epimorphism with codomain being a projective module. Therefore S is isomorphic to a submodule of $f^{-1}(S)$, which is a submodule of the projective socle of $\operatorname{Rad} R$; this is contradiction.

Thus we have, that if ${}_RS$ is a projective simple module, then $S \in \mathsf{F}_{\xi(\operatorname{Rad} R)}$. But $\xi(\operatorname{Rad} R) = \chi_*$, by Bland's Theorem, from which we get that if M is a direct sum of projective simple modules, then $M \in \mathsf{F}_{\chi_*}$ and hence M is injective (by Theorem 3).

Thus we have that $\forall N \in R$-mod, $\operatorname{soc}_p(N)$ is an injective submodule of N and hence it is also a direct summand of N; i.e., Goldman's torsion theory splits. In particular $R = \operatorname{soc}_p(R) \oplus_R K$. But now, since R is semiperfect, R is semiartinian and therefore $\operatorname{soc}(R) \leq_e R$. In particular $\operatorname{soc}(K) \leq_e K$. Let us note that every left simple submodule of K is singular (since a left simple module is either singular or projective, but $\operatorname{soc}_p(K) = \operatorname{soc}_p(R) \cap K = 0$)). Thus we have that $\operatorname{soc}(K)$ is a Goldie's torsion-module. Hence K is a Goldie's torsion-module, too (Goldie's torsion theory is closed under taking essential extentions). Thus, $K \leq t_G(R) = t_G(\operatorname{soc}_p(R)) \oplus t_G(K)$, but each simple summand of $\operatorname{soc}_p(R)$ is non singular (being projective). So, $K = t_G(R)$ and so we have that K is a bilateral ideal of R. As a result, $R = \mathrm{soc}_p(R) \oplus K$ (ring direct sum); i.e., Goldman's torsion theory centrally splits.

$\Longrightarrow$) If $\mathrm{soc}_p(\mathrm{Rad}\, R) \neq 0$ then $0 \longrightarrow \mathrm{soc}_p(R) \longrightarrow R \longrightarrow R/\mathrm{soc}_p(R) \longrightarrow 0$ does not split. For if it split, then taking a simple submodule S of $\mathrm{Rad}\, R$ we have that the monomorphisms $S \xrightarrow{\subseteq} \mathrm{soc}_p(\mathrm{Rad}\, R)$, $\mathrm{soc}_p(\mathrm{Rad}\, R) \xrightarrow{\subseteq} \mathrm{soc}_p(R)$ and $\mathrm{soc}_p(R) \xrightarrow{\subseteq} R$ are splitting; so its composition also splits. So we would have that $R = S \oplus K$, where ${}_RK$ is a maximal ideal of R, but this is impossible ($S \leq \mathrm{Rad}\, R \leq K \Longrightarrow S \cap K = S \neq 0$). Hence Goldman's torsion theory does not split, and a fortiori, does not centrally split. ∎

Corollary 4. *If R is a commutative perfect ring, then Goldman's torsion theory centrally splits.*

Proof: Raggi & Ríos ([17], Corolario 2.9) have proved in the general situation that $\mathrm{soc}_p(M) = \mathrm{soc}_p(R)M$ $\forall M \in R$-mod. In our particular case we have that $\mathrm{soc}_p(\mathrm{Rad}\, R) = \mathrm{soc}_p(R)\ \mathrm{Rad}\, R = 0$, since the Jacobson radical annihilates every simple module. ∎

We should note that the preceeding proof does not apply for non commutative right perfect rings, because $\mathrm{soc}_p(\mathrm{Rad}\, R)$ is not necessarily a right semisimple module.

From Theorem 3.1 of Raggi & Ríos [11], we have that for a right perfect ring, Goldie's torsion theory τ_G is a TTF torsion theory generated by the left singular simple modules and cogenerated by the left projective simple modules (in fact the preceeding statements hold when R is left semiartinian ring).

In the following theorem we will denote $\mathcal{S}_I$ the class of the left injective simple modules and by $\mathcal{S}_P$ the class of left projective simple modules.

Theorem 17. *If R is a right perfect ring satisfying $\mathrm{soc}_p(\mathrm{Rad}\, R) = (0)$, then are equivalent:*

i) $\chi_* = \tau_G$, where χ_* denotes the least element of $[\tau] \in R$-tors$/\sim_{\mathsf{F}}$.

ii) $\mathcal{S}_I = \mathcal{S}_P$.

Proof: i)$\Longrightarrow$ ii) $\mathcal{S}_P \subseteq \mathcal{S}_I$ follows from the part $\Longleftarrow$) of the proof of Theorem 16. Let ${}_RS$ be a left injective simple module. We want to prove that it is projective. Let us observe that since R is right perfect, then $R/\mathrm{Rad}\, R$ is semisimple, so that ${}_RM$ is semisimple iff $\mathrm{Rad}\, R\ M = 0$. Therefore every direct product of simple modules is semisimple. As a consequence, using Theorem 18, we get that $\chi(S)$ belongs to $[\chi]_{\mathsf{F}}$. For if $M \in \mathsf{F}_{\chi(S)}$, then $\exists_M \rightarrowtail S^x$ for some set X, and as S^X is a semisimple module. But on the other hand, M is injective, as it is isomorphic to a direct summand of the injective module S^X.

Thus, $\chi(S) \in [\chi]_{\mathsf{F}}$, and therefore $\chi(S) \geq \chi_* = \tau_G$. Then we have that S is Goldie torsion free, which is cogenerated by the left projective simple modules.

Hence $\exists 0 \neq f: S \longrightarrow U$, where U is a left projective simple module. Since f must be an isomorphism, we have that S is a projective module. Therefore $\mathcal{S}_I \subseteq \mathcal{S}_P$, and hence $\mathcal{S}_I = \mathcal{S}_P$.

ii)$\Longrightarrow$ i) Since τ_G is cogenerated by the left projective simple modules, we have that every τ_G-torsion free module is semisimple, since it is (isomorphic to) a submodule of a direct product of simple modules (this product is annihilated by $\operatorname{Rad} R$). But a τ_G-torsion free module is an injective module, since it is a direct summand of a product of projective simple modules, and such a product is injective by the hypothesis that all projective simple modules are injective modules. Since every τ_G-torsion free module is injective, $\tau_G \in [\chi]_{\mathsf{F}}$ by Theorem 3.

Analogously, if $\tau \in [\chi]_{\mathsf{F}}$ let us take E an injective module which cogenerates τ; i.e., $\tau = \chi(E)$. By another use of Theorem 3, we get that E is semisimple. Now, if ${}_RS$ is a simple submodule of E, it has to be injective. Because S is an injective module, S is also projective by hypothesis. Therefore it is τ_G-torsion free. So, $E \in \mathsf{F}_G$, since E is a direct sum of τ_G-torsion free modules. But $E \in \mathsf{F}_G \Longrightarrow \tau = \chi(E) \geq \tau_G$; so we have that $\tau_G = \chi_*$. ■

Corollary 5. *If R is a quasifrobenius ring (QF-ring), then $\chi_* = \tau_G$.*

Proof: R is right perfect and the class of projective modules coincides with the class of injective modules. Moreover, $\operatorname{soc}_p(\operatorname{Rad} R) = 0$: if ${}_RS \leq \operatorname{Rad} R$ was a projective simple module, then as S had to be injective, S would be a direct summand of R. Consequently, $S = Re \leq \operatorname{Rad} R$, with $e = e^2$, this is impossible. We conclude using Theorem 17. ■

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