# THE LATTICE R-tors FOR PERFECT RINGS

HUGO ALBERTO RINCÓN-MEJÍA

#### Abstract .

We define  $\sim_{\mathbf{F}}$  in *R*-tors by  $\tau \sim_{\mathbf{F}} \sigma$  iff the class of  $\tau$ -codivisible modules coincides with the class of  $\sigma$ -codivisible modules. We prove that if *R* is left perfect ring (resp. semiperfect ring) then every  $[\tau]_F \in R$ -tors/ $\sim_{\mathbf{F}}$ (resp.  $[\chi]_{\mathbf{F}}$  and  $[\xi]_{\mathbf{F}}$ ) is a complete sublattice of *R*-tors. We describe the largest element in  $[\tau]$  as  $\chi(\operatorname{Rad} R/t_{\tau}(\operatorname{Rad} R))$  and the least element of  $[\tau]$ as  $\xi(t_{\tau}(\operatorname{Rad} R))$ .

Using these results we give a necessary and sufficient condition for the central splitting of Goldman torsion theory when R is semiperfect.

We prove that for a QF ring R the least element of  $\chi_{r}$  is the Goldie torsion theory. This can be used to prove that for a QF ring  $\sim_{\mathbf{F}}$  and  $\sim_{\mathbf{T}}$  are equal, where  $\tau \sim_{\mathbf{T}} \sigma$  iff the class of  $\tau$ -injective modules coincides with the class of  $\sigma$ -injective modules.

## 0. Introduction

Throughout this work R will denote an associative unital ring; R-tors will denote the complete brouwerian lattice of all left hereditary torsion theories;  $\chi$  (resp.  $\xi$ ) will denote the largest (resp. the smallest) element of R-tors.

If  $\{M_{\alpha}\}_{\alpha \in X}$  is a family of left *R*-modules, then  $\chi(\{M_{\alpha}\})$  will denote the largest torsion theory respect to which every  $M_{\alpha}$  is torsion free.  $\xi(\{M_{\alpha}\})$  will denote the smallest torsion theory respect to which every  $M_{\alpha}$  is torsion. We consider a torsion theory  $\tau$  as an ordered pair  $\tau = (T_{\tau}, F_{\tau})$ , where  $T_{\tau}$  denotes the class of  $\tau$ -torsion modules, and  $F_{\tau}$  denotes the class of  $\tau$ -torsion free modules. Also remember that the order in *R*-tors is given by:  $\tau \leq \sigma$  iff  $T_{\tau} \subseteq T_{\sigma}$ .

Remember that a left module M is  $\tau$ -codivisible iff  $\operatorname{Ext}_R(M, K) = (0) \forall K \in F_{\tau}$ . Let us denote  $\mathsf{P}_{\tau}$  the class of  $\tau$ -codivisible modules. We define  $\sim_{\mathsf{F}}$  in R-tors by  $\tau \sim_{\mathsf{F}} \sigma$  iff  $\mathsf{P}_{\tau} = \mathsf{P}_{\sigma}$ . Obviously this is an equivalence relation in R-tors. Our aim in this work is to study R-tors by looking at the equivalence classes  $[\tau] \in R$ -tors/ $\sim_{\mathsf{F}}$ . In case R is a left perfect ring, these equivalence classes are complete sublattices of R-tors. So, in  $[\tau]$  there must exist a largest element (resp. a smallest element) which will be denote  $\tau^*$  (resp.  $\tau_*$ ). We describe  $\tau^* = \chi(\operatorname{Rad} R/t_{\tau}(\operatorname{Rad} R))$  (resp.  $\tau_* = \xi(t_{\tau}(\operatorname{Rad} R))$ ), where  $\operatorname{Rad} R$  denotes the Jacobson radical of  $\mathbb{R}$ .

We also obtain some generalizations of some results of Bland (see 3).

We also prove that for a QF-ring R the smallest element of  $[\chi]_{\sim_F}$  (which exists, since R is left perfect) is Goldie's torsion theory. In fact, it can be proved that for a QF-ring R the equivalence relations  $\sim_F$  and  $\sim_T$  coincide, where we define  $\tau \sim_T \sigma$  iff the class of  $\tau$ -injective modules coincides with the class of  $\sigma$ -injective modules.

The partition R-tors/ $\sim_T$  has been studied by Raggi & Ríos (see [12] and [13]).

We will denote by  $S_{\tau}$  the class of all short exact sequences  $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$  in *R*-mod such that  $K \in \mathsf{F}_{\tau}$ , where  $\tau \in R$ -tors.

We will denote  $P_{\tau}$  the class of R-modules that are projective with respect to each sequence in  $S_{\tau}$ .

We will denote  $\mathcal{A}_{\tau}$  the proper class of short exact sequences in *R*-mod which make projective each element of  $\mathbf{P}_{\tau}$ .

We should observe that  $_{R}P$  is projective with respect to each short exact sequence in  $S_{\tau} \iff P$  is projective with respect to each element of  $\mathcal{A}_{\tau}$ .

### Remarks.

1) (Ohtake [10], Bican, Nemec, Kepka [2]). If  $\tau = (\mathsf{T},\mathsf{F}) \in R$ -tors and  $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$  is a short exact sequence in R-mod such that P is projective an  $K \in \mathsf{T}$ , then  $M \in \mathsf{P}_{\tau}$ .

2) R-mod has enough  $\mathcal{A}_r$ -projectives (this means that  $\forall_R M \in R - mod \exists 0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0 \in \mathcal{A}_r$  with P projective with respect to  $\mathcal{A}_r$ .

3) Let  $_RM \in R$ -mod. Then:  $M \in P_\tau \iff M$  is a direct summand of a module of the form P/T, where P is projective and  $T \in T_\tau$ .

We should observe that in the above remark we can replace "projective" by "free".

Definition 1. ( $\tau$ -codivisible cover, Bland [3]). An  $\mathcal{A}_{\tau}$ -projective cover of <sub>R</sub>M is an exact sequence  $0 \longrightarrow L \longrightarrow P \longrightarrow M \longrightarrow 0$ , such that

i)  $L \in F_{\tau}$ .

ii) P is  $\tau$ -codivisible (i.e.  $A_{\tau}$ -projective).

iii) i(L) is small in P(i(L) << (P).

The fact of that  $\tau$ -codivisible covers are unique except for isomorphic copies is a known result [3].

We will denote by  $0 \longrightarrow K_{\tau}(M) \longrightarrow P_{\tau}(M) \longrightarrow M \longrightarrow 0$  the  $\tau$ -codivisible cover of M, when it exists, and by  $0 \longrightarrow K(M) \longrightarrow P(M) \longrightarrow M \longrightarrow 0$  the projective cover of M, when it exists.

Definition 2. We define  $\sim_{\rm F}$  in R-tors by:  $\sigma \sim_{\rm F} \tau$  iff  $\mathcal{A}_{\sigma} = \mathcal{A}_{\tau}$  (or equivalently, if  $\mathbf{P}_{\sigma} = \mathbf{P}_{\tau}$ , i.e. if the class of  $\sigma$ -codivisible modules coincides with the class of  $\tau$ -codivisible covers).

The relation defined above is, obviously, an equivalence relation. Under

appropriate conditions the corresponding equivalence classes  $[\tau]_{\sim_F}$ , are complete sublattices of *R*-tors. This is the case when *R* is a left perfect ring.

Theorem 1. If  $0 \longrightarrow K_{\tau}(M) \longrightarrow P_{\tau}(M) \longrightarrow M \longrightarrow 0$  is a  $\tau$ -codivisible cover of M and if  $0 \longrightarrow K(M) \longrightarrow P(M) \longrightarrow M \longrightarrow 0$  is a projective cover of M, then ker $(P(M) \longrightarrow P_{\tau}(M))$  is  $\tau$ -torsion.

Lemma 1. Let  $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$  be a projective cover. Let us suppose  $\tau \sim_{\mathsf{F}} \sigma$ , then  $K \in \mathsf{T}_{\tau} \iff K \in \mathsf{T}_{\sigma}$ .

Proof: Straightforward.

Theorem 2. Suppose that  $0 \longrightarrow K(M) \longrightarrow P(M) \longrightarrow M \longrightarrow 0$  is a projective cover. Then  $0 \longrightarrow K(M)/t_{\tau}(K(M)) \longrightarrow P(M)/t_{\tau}(K(M)) \longrightarrow M \longrightarrow 0$  (\*) is a  $\sigma$ -codivisible cover  $\forall \sigma \in [\tau]_{\mathbf{F}}$ .

Proof: Direct from the definitions.

Note that the above theorem implies that if  $0 \longrightarrow K_{\tau}(M) \longrightarrow P(M) \longrightarrow M \longrightarrow 0$  is a  $\tau$ -codivisible cover, then  $K_{\tau}(M) \in \mathsf{F}_{v[\tau]}^{\sigma}$ . This is because  $K_{\tau}(M) \in \bigcap_{[\tau]} \mathsf{F}_{\sigma} = \mathsf{F}_{v[\tau]}^{\sigma}$ .

Let us also note that the following implications hold for  $\sigma, \tau \in R$ -tors:

 $\tau \leq \sigma \Longleftrightarrow \mathsf{F}_{\tau} \supseteq \mathsf{F}_{\sigma} \Longrightarrow \mathcal{A}_{\tau} \supseteq \mathcal{A}_{\sigma} \Longleftrightarrow \mathsf{P}_{\tau} \subseteq \mathsf{P}_{\sigma}.$ 

**Remarks.** For a proper class  $\mathcal{A}$  we have:

i)  $\mathcal{A} = \mathcal{A}_{\xi} \iff \mathcal{A}$  is the class of all short exact sequences in R-mod  $\iff \mathsf{P}_{\mathcal{A}} = \mathsf{P}_{\xi}$ .

Also note that  $P_{\xi}$ , the class of  $\xi$ -codivisible modules is precisely the class of all projective modules.

ii) $\mathcal{A} = \mathcal{A}_{\xi} \iff S_{\mathcal{A}} = \{0 \longrightarrow 0 \longrightarrow M \longrightarrow M \longrightarrow 0 : M \in R \text{-mod}\} \iff R \text{-mod} = \mathsf{P}_{\mathcal{A}}$ , the class of all projective modules.

Also note  $\mathcal{A}_{\chi}$  is the class of all splitting short exact sequences in *R*-mod.

iii)  $\tau \in R$ -tors faithful  $\implies \tau \in [\xi]$ : for if P is  $\tau$ -codivisible, then P is a direct summand of a module  $R^{(X)}/T$ , where T is a  $\tau$ -torsion submodule of  $R^{(X)}$ , which is in  $\mathsf{F}_{\tau}$  (being R in  $\mathsf{F}_{\tau}$ , by hypothesis). Then T = 0, and hence P is a direct summand of a free module; i.e., P is projective. So  $\mathsf{P}_{\xi} = \mathsf{P}$ , and we conclude by using i).

iv) If R is a domain (e.g. **Z**) every  $\chi \neq \tau \in R$ -tors is faithful and hence is in  $[\xi]_{\mathsf{F}}$ . So R-tors/ $\sim_{\mathsf{F}}$  has only the two elements  $[\chi]_{\mathsf{F}} = \{\chi\}$ , and  $[\xi]_{\mathsf{F}} = R$ -tors\ $\{\chi\}$ .

Moreover [ $\xi$ ] has a maximal member:  $\chi(R) = \tau_L$ , Lambek's torsion theory.

- v) For a stable torsion theory  $\tau$  the following statements are equivalent:
  - a)  $R \cong t_{\tau}(R) \times S$ , where S is semisimple artinian.
  - b)  $\tau \in [\chi]_{\mathsf{F}}$ .

c)  $\forall N \in \mathsf{F}_{\tau}$ , N is an injective semisimple module.

*Proof:* a)  $\iff$  b) (See [11]), b)  $\iff$  c) follows from Theorem 3.

- vi) For a left semiartinian ring are equivalent
  - a)  $\tau_G \in [\chi]$  ( $\tau_G$  denotes Goldie's torsion theory).
  - b)  $R \cong \tau_G(R) \times S$ , where S is semisimple artinian.
  - c)  $\tau_G$  centrally splits.
  - d)  $\tau_0$  is stable. Here  $\tau_0$  denotes Goldman's torsion theory; i.e., the torsion theory generated by the projective semisimple modules.

Proof: b)  $\iff$  c)  $\iff$  d) (See [11]). a)  $\iff$  b) follows from Remark v).

- vii) If R is right perfect ring, then the above conditions are also equivalent to:
  - e)  $\operatorname{soc}_p(\operatorname{Rad} R) = 0$  (See Theorem 18). Here  $\operatorname{soc}_p$  denotes the projective socle, and  $\operatorname{Rad} R$  denotes the Jacobson radical.

The following is an easy generalization of a Theorem of Bland, in our context.

**Theorem 3.** Are equivalent for  $\tau \in R$ -tors:

i) τ ∈ [χ].
ii) P<sub>τ</sub> = P<sub>χ</sub> = R-mod.
iii) A<sub>τ</sub> = class of all splitting short exact sequences.
iv) ∀<sub>R</sub>N ∈ F<sub>τ</sub>, N is semisimple and injective.
v) The ring R/t<sub>τ</sub>(R) is semisimple.
vi) All cyclic modules are A<sub>τ</sub>-projective.

(Bland in (3) shows the equivalence of ii), iv) and v), the equivalence of the others follows directly from the definitions).

Corollary 1. R is semisimple  $\iff R$ -tors/ $\sim_{\mathbf{F}} = \{[\xi]\}(\iff \xi \sim_{\mathbf{F}} \chi).$ 

Proof:  $\implies$ ) If R is semisimple, then  $\forall \tau \in R$ -tors,  $R/t_{\tau}(R)$  is semisimple; so by v)  $\implies$  i) in Theorem 3 we get  $\tau \in [\chi]_{\mathbf{F}}$ . Hence  $[\xi] = [\chi] = R$ -tors.

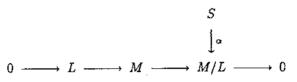
 $\Leftarrow$  ) If R-tors/ $\sim_{\mathsf{F}} = \{[\xi]\}$ . In particular  $\xi \in [\chi] = [\xi]$ . So by using i)  $\Leftrightarrow$  iv) in the above theorem, we get N is semisimple  $\forall_R N \in \mathsf{F}_{\xi}$  (but  $\mathsf{F}_{\xi} = R$ -mod). Then R is semisimple.

From the preceeding corollary, we obtain immediately the following result.

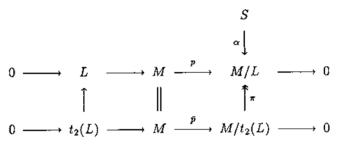
Corollary 2. (Bland [3], Corollary 3.4 proves the "if" part). R is semisimple  $\iff \exists \tau \in [\chi]$ , faithful.

*Proof:* ⇒ ) If *R* is semisimple, then  $\xi$  has the required properties. ⇐ ) If  $\tau \in [\chi]$  is faithful, then we get that  $\tau \in [\xi]$  (see remark iii), after Theorem 2). Thus  $\tau \in [\xi] \cap [\chi]$ . Hence  $[\xi] = [\chi]$ . Theorem 4. Let  $\tau$  be an element of R-tors. Then  $[\tau]_{\mathsf{F}}$  is closed under finite meets.

Proof: Let us suppose that  $\tau_1 \sim_{\mathbf{F}} \tau_2 \sim_{\mathbf{F}} \tau$ . By the observation after Theorem 2 we have that  $\mathcal{A}_{\tau_1} \subseteq \mathcal{A}_{\tau_1 \wedge \tau_2}$   $(\tau_1 \wedge \tau_2 \leq \tau_2)$ . Now, let us consider the diagram

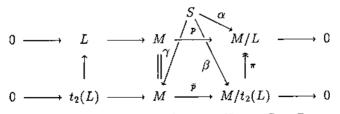


with  $L \in F_{\tau_1 \wedge \tau_2}$ ,  $S \in P_{\tau_1}$ , and remember that S is  $\mathcal{A}_{\tau}$ -projective iff S is projective with respect to each exact sequence of the form  $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$  with  $L \in F_{\tau}$ . Let us extend the above diagram to



where  $\pi$  is the natural epimorphism. Now  $M/t_2(L) \in \mathbb{F}_{r_2}$ ; so  $0 \longrightarrow \ker \pi \longrightarrow M/t_2(L) \xrightarrow{\pi} M/L \longrightarrow 0 \in \mathcal{A}_{r_2} = \mathcal{A}_{r_1}$ . Inasmuch as S is in  $\mathbb{P}_{r_1} = \mathbb{P}_{r_2}$ , we have that  $\exists \beta: S \longrightarrow M/t_2(L)$ , such that  $\pi \circ \beta = \alpha$ . Now let us observe that  $t_1(t_2(L)) \in \mathbb{T}_{r_2} \cap \mathbb{T}_{r_2} = \mathbb{T}_{r_1 \wedge r_2}$ .

But in the other hand,  $t_1(t_2(L)) \subseteq L \in \mathsf{F}_{\tau_1 \wedge \tau_2}$ ; hence  $t_1(t_2(L)) = 0$ . So  $t_2(L) \in \mathsf{F}_{\tau_1}$ , which implies that  $0 \longrightarrow t_2(L) \longrightarrow M \longrightarrow M/t_2(L) \longrightarrow 0$  belongs to  $\mathcal{A}_{\tau_1}$ . Hence  $\exists \gamma \colon S \longrightarrow M$  such that  $\bar{p} \circ \gamma = \beta$ ; so the following diagram is commutative:



But then  $\gamma \circ p = \pi \circ \overline{p} \circ \gamma = \pi \circ \beta = \alpha$ . Hence  $S \in \mathsf{P}_{r_1 \wedge r_2}$ , and then  $\mathsf{P}_{r_1} \subseteq \mathsf{P}_{\tau_1 \wedge \tau_2}$ , and from this we get  $\mathcal{A}_{r_1 \wedge \tau_2} \subseteq \mathcal{A}_{\tau_1}$ , (see the observation after Theorem 2).

Hence  $\mathcal{A}_{\tau_1 \wedge \tau_2} = \mathcal{A}_{\tau_1}$ , and so  $\tau_1 \wedge \tau_2 \sim_{\mathsf{F}} \tau_1 \sim_{\mathsf{F}} \tau$ .

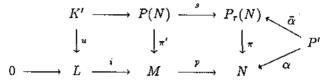
If the ring R is left perfect we can prove much more.

**Theorem 5.** If R is a left perfect ring, then  $[\tau]$  is closed under taking arbitrary meets,  $\forall \tau \in R$ -tors.

Proof: Let  $P' \in \mathbf{P}_{\tau}$  and let

$$\begin{array}{c} P' \\ & \downarrow^{\alpha} \\ 0 & \longrightarrow L \xrightarrow{i} M \xrightarrow{p} N & \longrightarrow 0 \end{array}$$

be a diagram with  $L \in \mathsf{F}_{\wedge[\tau]}$ . Let  $0 \longrightarrow K(N) \longrightarrow P(N) \longrightarrow N \longrightarrow 0$  and  $0 \longrightarrow K_{\tau}(N) \longrightarrow P_{\tau}(N) \longrightarrow N \longrightarrow 0$  be a projective and  $\tau$ -codivisible covers, respectively. Then  $\exists \alpha \colon P' \longrightarrow P_{\tau}(N)$  such that



commutes (because P' is  $\tau$ -codivisible and  $0 \longrightarrow K_{\tau}(N) \longrightarrow P_{\tau}(N) \longrightarrow N \longrightarrow 0 \in \mathcal{A}_{\tau}$ ), where  $\pi'$  is the epimorphism provided by the projectivity of P(N), and u is the morphism obtained from the universal property of kernels.

Moreover, by Theorem 1, we have that  $K' \in T_{\sigma} \quad \forall \sigma \in [\tau]$ . Hence we get  $K' \in T_{\Lambda_{[\tau]}\sigma}$ . As  $L \in F_{\Lambda_{[\tau]}\sigma}$ , we get u = 0. But then, given the commutativity in the first square, we get that  $\exists \beta \colon P_{\tau}(N) \longrightarrow M$  such that  $\beta \circ s = \pi'$ .

So we have that in the diagram

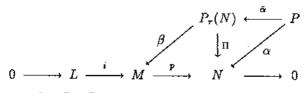
$$P(N) \xrightarrow{s} P_{\tau}(N)$$

$$\downarrow_{\pi'} \xrightarrow{\beta} \qquad \downarrow_{\pi}$$

$$M \xrightarrow{p} N$$

the square and the top triangle commute; i.e.,  $\pi \circ s = p \circ \pi' = p \circ \beta \circ s$ . But as s is epi, we have that  $\pi = p \circ \beta$ ; i.e. the bottom triangle is also commutative.

Summarizing, we have the following commutative diagram



from which we get that  $P \in \mathbf{P}_{\wedge[\tau]}$ . Hence  $\mathbf{P}_{\tau} \subset \mathbf{P}_{\wedge[\tau]}$  and then  $\mathcal{A}_{\wedge[\tau]} \subset \mathcal{A}_{\tau}$ . But  $\wedge[\tau] \leq \tau \implies \mathcal{A}_{\wedge[\tau]} \subseteq \mathcal{A}_{\tau}$  (observation after Theorem 2). Hence  $\mathcal{A}_{\wedge[\tau]} = \mathcal{A}_{\tau}$  and so  $\wedge_{[\tau]} \sigma \sim_{\mathbf{F}} \tau$ .

So we have proved  $\wedge[\tau] \in [\tau]$  and this is sufficient for seeing that  $[\tau]$  is closed taking under arbitrary meets  $(\{\tau_{\alpha}\} \subseteq [\tau] \Longrightarrow \wedge[\tau] \leq \wedge\{\tau_{\alpha}\} \leq \tau_{\alpha}$  and hence  $\mathcal{A}_{\tau_{\alpha}} \subseteq \mathcal{A}_{\wedge\{\tau_{\alpha}\}} \subseteq \mathcal{A}_{\wedge[\tau]} = \mathcal{A}_{\tau_{\alpha}}$ .

**Theorem 6.** If R is a left perfect ring, then  $[\tau]$  is closed under arbitrary joins.

*Proof:* It's enough to prove that  $\forall [\tau] \in [\tau]$ . Let

where the row is a  $\tau$ -codivisible cover of M and where P' is a  $V[\tau]$ -codivisible module. By Theorem 2 we have that  $L \in F_{\sigma}, \forall \sigma \in [\tau]$ ; hence  $L \in \cap_{[\tau]} F_{\sigma} = F_{v[\tau]}$ . So, (\*) belongs to  $\mathcal{A}_{v[\tau]}$ , and consequently  $\exists \bar{\alpha} \colon P' \longrightarrow P_{\tau}$  such that  $p \circ \bar{\alpha} = \alpha$ . Hence  $P' \in P_{\tau}$  and so  $P_{v[\tau]} \subseteq P_{\tau}$ , which is equivalent to saying that  $\mathcal{A}_{\tau} \subseteq \mathcal{A}_{v[\tau]}$ .

On the other hand,  $\tau \leq \forall [\tau] \iff \mathcal{A}_{\tau} \supseteq \mathcal{A}_{\forall [\tau]}$ . Then  $\mathcal{A}_{\tau} = \mathcal{A}_{\forall [\tau]}$  and so  $\forall [\tau] \in [\tau]$ .

From the two preceeding theorems we get at once:

Theorem 7. R Left perfect  $\implies [\tau]$  is a complete sublattice of R-tors,  $\forall \tau \in R$ -tors.

By the preceeding theorem, we know that if R is a left perfect ring, then  $[\tau]$  is closed under taking arbitrary joins and meets. Consequently, in  $[\tau]$  must exist a largest and a smallest element, which will be denoted  $\tau^*$  and  $\tau_*$ , respectively. The following theorem gives us a useful description of each of them.

Theorem 8. If R is a left perfect ring, then:

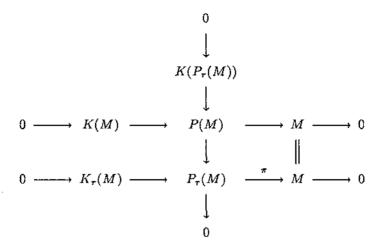
i)  $\tau^* = \chi \{K_\tau(M)|0 \longrightarrow K_\tau(M) \longrightarrow P_\tau(M) \longrightarrow M \longrightarrow 0$  is an  $\mathcal{A}_\tau$ -codivisible cover,  $M \in R$ -mod  $\}$ .

ii)  $\tau_* = \xi\{K(P_r(M))|0 \longrightarrow K(P_r(M)) \longrightarrow P(M) \longrightarrow P_r(M) \longrightarrow 0$ is a projective cover of  $P_r(M)$ , where  $P_r(M)$  is a  $\tau$ -codivisible cover of M,  $M \in R \cdot mod\}$ .

*Proof:* First, let us observe that the sequence

$$0 \longrightarrow K(P_{\tau}(M)) \longrightarrow P(M) \longrightarrow P_{\tau}(M) \longrightarrow 0$$

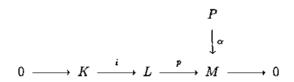
in ii) comes from the diagram



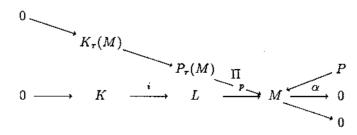
where the rows and the column are exact, the rows are the projective and the  $\tau$ -codivisible covers of M, respectively, and the R-morphism  $P(M) \longrightarrow P_{\tau}(M)$  is given by the projectivity of P(M).

i) By the note after Theorem 2, we have that  $K_{\tau}(M) \in \mathbb{F}_{\sigma} \quad \forall \sigma \in [\tau]$ ; so  $\chi\{K_{\tau}(M)|M \in R \text{-mod}\} \geq \tau^*$ . It would be enough to see that  $\chi\{K_{\tau}(M)|M \in R \text{-mod}\} \in [\tau]$  and for this it would be enough to see that  $P_{\chi\{K_{\tau}(M)|M \in R \text{-mod}\}} \subseteq \mathbb{P}_{\tau^*}$ .

But if  $P \in \mathbb{P}_{\chi\{K_r(M)|M \in R \text{-mod}\}}$  and if the diagram



is such that  $K \in F_{\tau}$ , then by taking a  $\tau$ -codivisible cover of M we get the diagram



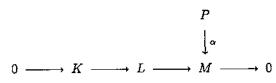
Since  $K_r(M) \in \mathbb{F}_{\chi\{K_r(M)|M \in R \text{-mod}\}}, \exists \bar{\alpha} : P \longrightarrow P_r(M)$  such that  $\pi \circ \bar{\alpha} = \alpha$ . Inasmuch as  $K \in \mathbb{F}_r, \subseteq \mathbb{F}_r, \exists \bar{\alpha} : P_r(M) \longrightarrow L$  such that  $p \circ \bar{\alpha} = \pi$ , hence  $p \circ (\bar{\alpha} \circ \bar{\alpha}) = \alpha$  and then  $P \in P_{\tau^*}$ . So  $P_{\chi\{K_{\tau}(M)|M \in R \text{-mod}\}} \subseteq P_{\tau^*}$ . Hence  $\tau^* \leq \chi\{K_{\tau}(M)|M \in R \text{-mod}\}$  and hence  $\tau^* = \chi\{K_{\tau}(M)|M \in R \text{-mod}\}$ .

ii) By Lemma 1, we have that  $K(P_{\tau}(M)) \in \mathsf{T}_{\wedge_{[\tau]}\sigma}$ , hence  $\xi\{K(P_{\tau}(M)|M \in R\text{-mod}\} \leq \tau_* = \wedge_{[\tau]}$ .

To get the converse inclusion, it is enough to see that

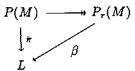
$$\mathsf{P}_{\tau^*} \subseteq \mathsf{P}_{\xi\{K(P_\tau(M)|M \in R \text{-mod}\}}.$$

So, let  $P \in P_{\tau}$  and

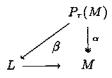


be a diagram such that  $K \in F_{\xi\{K(P_r(M)|M \in R \mod \})}$ . Let us take  $0 \longrightarrow K(P_r(M)) \longrightarrow P(M) \longrightarrow P_r(M) \longrightarrow 0$  as in the statement. Then  $K_r(P_r(M)) \in T_{\Lambda[r]}$ . In the diagram

where  $\bar{\pi}$  is given by projectivity of P(M), and  $\beta$  is the restriction of  $\bar{\pi}$  to  $K_r(P_r(M))$ , we have that  $\beta = 0$ , inasmuch  $K \in F_{\xi\{K(P_r(M)|M \in R \text{-mod}\}}$ . Then, by the universal property of cokernels, we have that  $\exists \beta \colon P_r(M) \colon \longrightarrow L$  such that



commutes. But as  $P(M) \longrightarrow P_{\tau}(M)$  is epic, we have that

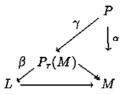


is commutative, too.

Now,

$$\begin{array}{cccc} & & P \\ & & \downarrow \alpha \\ 0 & \longrightarrow & K_{\tau}(M) & \longrightarrow & P_{\tau}(M) & \stackrel{\pi}{\longrightarrow} & M & \longrightarrow & 0 \end{array}$$

with  $P \in \mathsf{P}_{\tau^*}$  and  $K_{\tau}(M) \in \mathsf{F}_{\sigma}$   $(\forall \sigma \in [\tau])$  imply that  $K_{\tau}(M) \in \mathsf{F}_{\tau^*}$ , and so  $\exists \gamma \colon P \longrightarrow P_{\tau}(M)$  such that  $\pi \circ \gamma = \alpha$ . But then



commutes.

Hence  $P \in \mathsf{P}_{\xi\{K(P_{\tau}(M))|M \in R \text{-mod}\}}$  Thus,  $\mathsf{P}_{\tau^*} \subseteq \mathsf{P}_{\xi\{K(P_{\tau}(M))|M \in R \text{-mod}\}}$ . So we get  $\tau^* = \xi\{K(P_{\tau}(M))|M \in R \text{-mod}\}$ .

For the particular cases when  $\tau \in \{\xi, \chi\}$  and when the ring R is left perfect, we give descriptions of  $\tau^*$  and  $\tau_*$  by using the Jacobson radical of R, which we will extend to arbitrary torsion theories and for semiperfect rings.

**Theorem 9.** For left perfect R we have that  $(\mathcal{J}(R))$ 

i) 
$$\xi^* = \chi(\mathcal{J}(R))$$
  
ii)  $\chi_* = \xi(\mathcal{J}(R)),$ 

where  $\mathcal{J}(R)$  denotes the Jacobson radical of R.

Proof: i) By Theorem 8,

$$\xi^* = \chi\{K_{\xi}(M) \mid 0 \longrightarrow K_{\xi}(M) \longrightarrow P_{\xi}(M) \longrightarrow M \longrightarrow 0$$
  
is a  $\xi$ -codivisible cover,  $M \in R$ -mod}
$$= \chi\{K(M) \mid 0 \longrightarrow K(M) \longrightarrow P(M) \longrightarrow M \longrightarrow 0$$
  
is a projective cover,  $M \in R$ -mod 
$$\}$$
$$= \chi\{K \mid K \ll P \text{ and } _RP \text{ is projective }\}.$$

As R is left perfect,  $\operatorname{Rad}(P) = \mathcal{J}(R)P$  (see Anderson-Fuller, [1], Remark 28.5.(3)); so  $K \ll P \iff K \subseteq \mathcal{J}(R)P \subseteq \mathcal{J}(R)R^{(X)}$  for some set X. Hence  $K \ll P \iff \exists K \rightarrowtail \mathcal{J}(R)^{(X)} \iff K \in \mathsf{F}_{\chi(\mathcal{J}(R))}$ . Thus  $\xi^* \ge \chi(\mathcal{J}(R))$ .

On the other hand,  $\mathcal{J}(R) \ll R$  so we have that  $0 \longrightarrow \mathcal{J}(R) \longrightarrow R \longrightarrow R/\mathcal{J}(R) \longrightarrow 0$  is a projective cover (=  $\xi$ -codivisible cover). Therefore  $\mathcal{J}(R) \in \mathbb{F}_{\xi^*}$  (since  $\mathcal{J}(R)$  is one of the modules cogenerating the torsion theory  $\xi^*$ , see the above description of  $\xi^*$ ). Hence  $\xi^* \geq \chi(\mathcal{J}(R))$ . And therefore  $\xi^* = \chi(\mathcal{J}(R))$ .

$$\chi_{*} = \xi \left\{ K_{\chi}(P_{\chi}(M)) \middle| \begin{array}{c} 0 \longrightarrow K_{\chi}(P_{\chi}(M)) \longrightarrow P(M) \longrightarrow P_{\chi}(M) \longrightarrow 0 \\ P(M) \longrightarrow P_{\chi}(M) \\ \text{is induced by } \downarrow_{\pi} \qquad \qquad \downarrow_{\pi'} \\ M \longrightarrow M \\ \text{where $\pi$ and $\pi'$ are projective and} \\ \tau\text{-codivisible cover, respectively.} \end{array} \right\}$$

Now  $0 \longrightarrow K_{\chi}(M) \longrightarrow P_{\chi}(M) \longrightarrow M \longrightarrow 0$  is a  $\chi$ -codivisible cover but  $0 \longrightarrow 0 \longrightarrow M \longrightarrow M \longrightarrow 0$  is another (every left *R*-module is  $\chi$ -codivisible). Thus we have that

$$0 \longrightarrow K_{\mathbf{X}}(P_{\mathbf{X}}(M)) \longrightarrow P(M) \longrightarrow P_{\mathbf{X}}(M) \longrightarrow 0$$

is a projective cover of  $_{R}M$ . We have then that

$$\chi_* = \xi \{ K \mid K \ll P, {}_RP \quad \text{projective} \}.$$

Again,  $K \ll P$ ,  $_RP$  projective  $\iff K \subseteq \mathcal{J}(R)^{(X)}$  for some set X. Therefore  $K \ll P$ , P projective  $\implies K \in \xi(\mathcal{J}(R))$ . Hence  $\chi_* \leq \xi(\mathcal{J}(R))$ .

On the other hand,  $0 \longrightarrow \mathcal{J}(R) \longrightarrow R \longrightarrow R/\mathcal{J}(R) \longrightarrow 0$  is a projective cover. Therefore  $\mathcal{J}(R) \in T_{\xi \{K_X P_X(M) \mid M \in R \mod \}}$  (is one of the generators of the above torsion theory). Therefore  $\xi(\mathcal{J}(R)) \leq \chi_*$  and hence  $\chi_* = \xi(\mathcal{J}(R))$ .

We give now more "concrete" descriptions of  $\tau^*$  and  $\tau_*$ , in case R is left perfect.

**Theorem 10.** If R is left perfect, then i)  $\tau^* = \chi(\mathcal{J}(R)/t_\tau(\mathcal{J}(R)))$ ii)  $\tau_* = \xi(t_\tau(\mathcal{J}(R)))$ , Where  $\mathcal{J}(R)$  denotes the Jacobson's radical of R.

Proof: i)  $0 \longrightarrow \mathcal{J}(R)/t_{\tau}(\mathcal{J}(R)) \longrightarrow R/t_{\tau}(\mathcal{J}(R)) \longrightarrow R/\mathcal{J}(R) \longrightarrow 0$  is a projective cover, since: a)  $\mathcal{J}(R)/t_{\tau}(\mathcal{J}(R)) \ll R/t_{\tau}(\mathcal{J}(R))$ , b)  $R/t_{\tau}(\mathcal{J}(R))$  is  $\tau$ -codivisible (by Remark 3, before Definition 1) and c)  $\mathcal{J}(R)/t_{\tau}(\mathcal{J}(R)) \in \mathbb{F}_{\tau}$ . Thus, by the note after Theorem 2,  $\mathcal{J}(R)/t_{\tau}(\mathcal{J}(R)) \in \mathbb{F}_{\tau}$ ; therefore  $\tau \leq \tau^* \leq \chi(\mathcal{J}(R)/t_{\tau}(\mathcal{J}(R)))$ .

If  $\tau^* \leq \chi(\mathcal{J}(R)/t_{\tau}(\mathcal{J}(R)))$  then  $\exists 0 \neq {}_{R}M \in \mathsf{T}_{\chi(\mathcal{J}(R)/t_{\tau}(\mathcal{J}(R)))} \cap \mathsf{F}_{\tau^*}$ . ( $\exists 0 \neq M$  that is  $\chi(\mathcal{J}(R)/t_{\tau}(\mathcal{J}(R))$ -torsion but not  $\tau^*$ -torsion, and by taking  $M/t_{\tau^*}(M)$  if it would be necessary, we can suppose, without loss generality, that  $M \in \mathsf{F}_{\tau^*}$ ).

By Theorem 8,  $\tau^* = \chi \{ K_\tau(M) \mid M \in R \text{-mod} \}$ , so if  $M \in F_{\tau^*}$ , then M is cogenerated by  $\{ E(K_\tau(M) \mid M \in R \text{-mod} \}$  (i.e.,  $\exists M \gg \prod_{N \in R \text{-mod}} E(K_\tau(N))$ ). Therefore,  $\forall 0 \neq x \in M$ ,  $\exists f_x : M \longrightarrow E(K_\tau(N))$  such that  $f_x(x) \neq 0$ ([15]. Prop.VI.3.39). Therefore  $0 \neq f_x(x) \in E(K_\tau(N))$ . Because  $K_\tau(N) <_e E(K_\tau(N))$  we have that  $f_x(M) \cap K_\tau(N) \neq 0$ . Hence  $\exists 0 \neq y \in M$  such that  $0 \neq f_x(y) \in K_\tau(N)$ . Consequently,  $Ry \stackrel{(f_x|Ry)}{\longrightarrow} K_\tau(N)$  is well defined. Now, thanks to Theorem 2, we have that the following diagram is commutative:

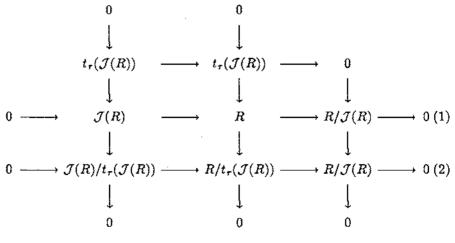
(Here we assume that  $0 \longrightarrow K(N) \longrightarrow P(N) \longrightarrow N \longrightarrow 0$  is a projective cover of N). Thus  $K(N) \ll P(N)$  and then we have that  $K(N) \leq \mathcal{J}(P(N)) = \mathcal{J}(R)P(N) \leq \mathcal{J}(R)R^{(Z)} = \mathcal{J}(R)^{(Z)}$  for some set  $Z(\mathcal{J}(P(N)) = \mathcal{J}(R)P(N)$  since P(N) is projective).

Therefore we have the following situation:

$$\begin{array}{cccc} Ry & \stackrel{\subseteq}{\longrightarrow} & M \\ & & \downarrow_{f_{x}} \\ K_{\tau}(N) & \stackrel{\stackrel{\alpha}{\longrightarrow}}{\longrightarrow} & K(N)/t_{\tau}(K(N)) & \stackrel{i}{\longrightarrow} & \mathcal{J}(R)^{(Z)}/t_{\tau}(K(N)) & \stackrel{\ast}{\longrightarrow} \\ & & - & \longrightarrow & \mathcal{J}(R)^{(Z)}/t_{\tau}(\mathcal{J}(R)^{(Z)}) \cong [\mathcal{J}(R)/t_{\tau}(\mathcal{J}(R))]^{(Z)}. \end{array}$$

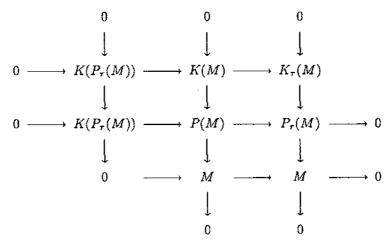
As we that  $\operatorname{Hom}_R(M, \mathcal{J}(R)/t_\tau(\mathcal{J}(R))) = 0$ , we also have that  $\operatorname{Hom}_R(Ry, \mathcal{J}(R)/t_\tau(\mathcal{J}(R))) = 0$  which implies that  $i \circ \alpha(f_x(y)) \in t_\tau(\mathcal{J}(R^{(Z)}))$ . Therefore  $\exists I \in \mathcal{F}_\tau$  such that  $I \ i \circ \alpha(f_x(y)) = 0$ . But as i is a monomorphism, then  $I(f_x(y)) = 0$ ; hence  $0 \neq f_x(y) \in t_\tau(K_\tau(N)) = 0$ , which is a contradiction  $(K_\tau(N) \cong K(N)/t_\tau(K(N)) \in \mathbb{F}_\tau)$ . Therefore  $\tau^* = \chi(\mathcal{J}(R)/t_\tau(\mathcal{J}(R)))$  (here  $\mathcal{F}_\tau$  denotes the idempotent filter corresponding to  $\tau$ ).

ii) If we consider the diagram



the fact that (1) and (2) are projective and  $\tau$ -codivisible covers, respectively, tells us that ker  $\pi$  in Column (3) is one of the modules generating the torsion theory  $\tau_*$  (see Theorem 8). Therefore  $t_\tau(\mathcal{J}(R)) \in T_{\tau^*}$  and  $\xi(t_\tau(\mathcal{J}(R))) \leq \tau_*$ .

Now, if  $K(P_{\tau}(M))$  is one of the generators of  $\tau_*$ ; i.e., if  $0 \longrightarrow K(P_{\tau}(M)) \longrightarrow P(M) \longrightarrow P_{\tau}(M) \longrightarrow 0$  can be extended to a diagram



where the two last rows are projective and  $\tau$ -codivisible covers, respectively, then we have that  $K(P_{\tau}(M)) \ll K(M) \ll P(M)$ .

By Theorem 2,  $K(P_{\tau}(M)) = t_{\tau}(K(M))$ ; therefore  $K(P_{\tau}(M)) \leq \operatorname{Rad}(P(M))$ =  $\mathcal{J}(R)P(M) \xrightarrow{\subseteq} \mathcal{J}(R)R^{(X)} = \operatorname{Rad} R^{(X)}$  and moreover  $K(P_{\tau}(M)) \xrightarrow{\subseteq} t_{\tau}(\mathcal{J}(R)^{(X)}) = (t_{\tau}(\mathcal{J}(R)))^{(X)}$ . Therefore  $K(P_{\tau}(M)) \in \mathsf{T}_{\xi(t_{\tau}(\mathcal{J}(R)))} \forall M \in R$ -mod. Hence  $\tau_{\star} = \xi\{K(P_{\tau}(M)) \mid M \in R \text{-mod}\} \leq \xi(t_{\tau}(\mathcal{J}(R)))$  and so  $\tau^{\star} = \xi(t_{\tau}(\mathcal{J}(R)))$ .

**Corollary 3.** If R is a left perfect ring, then  $\tau \leq \sigma \implies \tau_* \leq \sigma_*$ .

Proof: Straightforward.

Theorem 10 is extended in [14] to the case of local rings. In that situation each  $[\tau] \in R$ -tors/ $\sim_{\mathsf{F}}$  is closed under taking joins and meets and moreover the biggest element in  $[\tau], \tau^*$  is given by  $\tau^* = \chi(\mathcal{J}(R)/t_{\tau}(\mathcal{J}(R)))$  and also  $\tau_* = \xi(t_{\tau}(\mathcal{J}(R))).$ 

However, a ring may have the property of having each  $[\sigma]_{\mathsf{F}}$  closed under arbitrary joins and meets without being semiperfect. Moreover, the elements  $\sigma^*$  and  $\sigma_*$  are not given by  $\chi(\mathcal{J}(R)/t_{\sigma}(\mathcal{J}(R)))$  and by  $\xi(t_{\sigma}(\mathcal{J}(R)))$ , in general. As we see in the following examples.

**Examples.** In view of Remark 3 before Definition 1, is easy to see that if R is a domain, then R-tors admits the following partition:

$$\{ [\xi] = [\chi(R)], \quad [\chi] = \{\chi\} \}.$$

It is clear that each equivalence class in R-tors/ $\sim_F$  admits a largest and a least element.

In particular this is the situation for  $\mathbf{Z}$ , the ring of integers, which is not a perfect ring.

Moreover, let us note that for  $\mathbb{Z}$ , in spite of the fact that each element in R-tors/ $\sim_{\mathbb{F}}$  has a largest and a least element, they are not given as in Theorem 10. Explicity,  $\mathcal{J}(\mathbb{Z}) = 0$ , but we have that  $[\chi] = \{\chi\}$ , and so  $\chi_* = \chi = \chi^*$ . Nevertheless  $\chi_* \neq \xi(t_{\chi}(\mathcal{J}(\mathbb{Z}))) = \xi(t_{\chi}(0)) = \xi(0) = \xi$ .

On the other hand  $[\xi] = [\tau_G = \tau_L]$  and  $\xi^* = \tau_L$ , but  $\xi^* \neq \chi(\mathcal{J}(\mathbb{Z})/t_{\xi}(\mathcal{J}(\mathbb{Z})) = \chi(0/0) = \chi(0) = \chi$  (here  $\tau_G$  denotes Goldie's torsion theory and  $\tau_L$  denotes Lambek's torsion theory).

Lemma 2. The following statements are equivalent for a left perfect ring: i)  $\xi^* \vee \tau = \tau^* \forall \tau \in R$ -tors. ii)  $[\tau] \xrightarrow{-\Lambda \xi^*} [\xi]$  is a lattice monomorphism with left inverse  $[\xi] \xrightarrow{-\vee \tau^*} [\tau]$ . iii)  $\sigma \leq \tau \implies [\tau] \xrightarrow{-\Lambda \sigma^*} [\sigma]$  is a lattice monomorphism with left inverse  $[\sigma] \xrightarrow{-\vee \tau_*} [\tau]$ . iv)  $\sigma \leq \tau \implies \tau \vee \sigma^* = \tau^*$ . v)  $\forall \sigma, \tau \in R$ -tors  $\tau \vee \sigma^* = (\tau \vee \sigma)^* = \tau^* \vee \sigma$ .

Proof: Straightforward.

**Theorem 11.** If R is a left perfect ring, all of whose torsion free classes  $F_{\tau}$  are also torsion classes (i.e. each  $F_{\tau}$  is closed under taking factors), then R enjoys the properties of Lemma 2.

Proof: We will prove that  $\xi^* \vee \tau = \tau^*$ ,  $\forall \tau \in R$ -tors. As  $\xi^* \leq \tau^*$ , we have that  $\xi^* \vee \tau \leq \tau^*$  (by Theorem 9 we have that  $\xi^* = \chi(\operatorname{Rad} R)$ ;  $\tau^* = \chi(\operatorname{Rad} R/t_{\tau}(\operatorname{Rad} R))$ . The hypothesis that  $\mathsf{F}_{\tau}$  is closed under factors  $\Longrightarrow$   $\operatorname{Rad} R/t_{\tau}(\operatorname{Rad} R) \in \mathsf{F}_{\xi^*}$ ; hence  $\tau^* \geq \xi^*$ ).

It remains to prove that  $\xi^* \vee \tau$  cannot be different from  $\tau^*$ . If it was, then  $\exists 0 \neq M \in T_{\tau^*} \cap F_{\xi^* \vee \tau} = T_{\tau^*} \cap F_{\xi^*} \cap F_{\tau}$ . And as  $\tau^* = \chi(\operatorname{Rad} R/t_{\tau}(\operatorname{Rad} R))$ (Theorem 10) we have that  $\operatorname{Hom}_R(M, E(\operatorname{Rad} R/t_{\tau}(\operatorname{Rad} R)) = 0$  (\*)

But as  $M \in F_{\xi^*}$  and  $\xi^* = \chi(\operatorname{Rad} R)$  (Theorem 9) we have that  $\exists u: M \mapsto (E(\operatorname{Rad} R))^X$ , monomorphism for some set X. Hence  $\exists x \in X$  such that  $p_x u(M) \neq 0$ , where  $p_x: (E(\operatorname{Rad} R))^X \longrightarrow E(\operatorname{Rad} R)$  is the canonical projection. Hence, in view of (\*), we have that  $u(M) \subseteq (t_r(E(\operatorname{Rad} R)))^X$ . For if this were not true,  $\exists y \in X$  such that  $p_y(u(M)) \notin t_r(E(\operatorname{Rad} R))$  and hence

$$M \xrightarrow{\mu_{\Psi}} E(\operatorname{Rad} R)/t_{\tau}(E(\operatorname{Rad} R))$$

is not the zero morphism. But  $E(\operatorname{Rad} R)/t_{\tau}(E(\operatorname{Rad} R)) \in F_{\tau}$  and  $M \in T_{\tau}$  and so  $\operatorname{Hom}_R(M, E(\operatorname{Rad} R)/t_{\tau}(E(\operatorname{Rad} R)) = 0$ . This is a contradiction.

Now as  $u(M) \subseteq (t_{\tau}(E(\operatorname{Rad} R)))^X$ , we have that  $p_x(u(m)) \subseteq t_{\tau}(E(\operatorname{Rad} R)) \in T_{\tau}$ , but being also a factor of  $M \in F_{\tau}$ , it belongs to  $F_{\tau}$ . Hence  $0 \neq u(m) \in T_{\tau} \cap F_{\tau}$ . This is a contradiction. Hence  $\xi^* \vee \tau = \tau^*$ .

The rings such that every torsion free class is closed under factors have been charaterized by Teply [16] and by Bronowitz and Teply [5]. We will call these rings BT-rings.

It is clear that for a BT-ring we have that:

$$\tau \leq \sigma \Longrightarrow t_{\tau}(\operatorname{Rad} R) \leq t_{\sigma}(\operatorname{Rad} R)$$
  
$$\Longrightarrow \operatorname{Rad} R/t_{\tau}(\operatorname{Rad} R) \twoheadrightarrow \operatorname{Rad} R/t_{\sigma}(\operatorname{Rad} R)$$
  
$$\Longrightarrow \operatorname{Rad} R/t_{\sigma}(\operatorname{Rad} R) \in \mathsf{F}_{\chi}(\operatorname{Rad} R/t_{\tau}(\operatorname{Rad} R)) = \mathsf{F}_{\tau^*}$$
  
$$\Longrightarrow [\sigma^* = \chi(\operatorname{Rad} R/t_{\sigma}(\operatorname{Rad} R) \geq \tau^*]$$
  
$$\Longrightarrow \tau^* \geq \sigma^*.$$

Moreover, for a *BT*-ring, we have that  $\xi^* \vee \tau = \tau^*$ , since it is clear from the preceeding that  $\xi^* \vee \tau \leq \tau^*$ . And we would have, if the above inequality was estrict, that  $F_{\tau^*} \subsetneq F_{\xi^* \vee \tau} = F_{\xi^*} \cap F_{\tau}$ .

Hence  $\exists 0 \neq M \in (\mathsf{F}_{\xi^*} \cap \mathsf{F}_r) \setminus \mathsf{F}_{r^*}$ , and we can assume (changing M by  $t_{r^*}(M) \neq 0$  if it was necessary), that  $M \in \mathsf{T}_{r^*} \cap \mathsf{F}_{\xi^*} \cap \mathsf{F}_r$   $(t_{r^*}(M) \neq 0$  because  $M \notin \mathsf{F}_{r^*}$ ).

Inasmuch as  $M \in \mathsf{F}_{\xi^*}$ ,  $\exists 0 \neq f \in \operatorname{Hom}_R(M, E(\operatorname{Rad} R))$ ; hence  $\exists 0 \neq m \in M$ such that  $\operatorname{Hom}_R(Rm, \operatorname{Rad} R) \neq 0$ . But as  $M \in \mathsf{T}_{\tau^*}$ , we have that  $\operatorname{Hom}_R(Rm, \operatorname{Rad} R/t_\tau(\operatorname{Rad} R)) = 0$   $(Rm \subseteq M \in \mathsf{T}_{\tau^*})$ . So, if we take  $0 \neq g \in$  $\operatorname{Hom}_R(Rm, \operatorname{Rad} R)$ , then we would have that  $0 \neq g(Rm) \subseteq t_\tau(\operatorname{Rad} R) \in \mathsf{T}_{\tau}$ . But on the other hand, g(Rm) is a factor of  $Rm \subseteq M \in \mathsf{F}_{\tau}$ , and we have  $\mathsf{F}_{\tau}$ closed under taking factors by hypothesis. So we get that  $0 \neq g(Rm) \in \mathsf{T}_{\tau} \cap \mathsf{F}_{\tau}$ ; which is a contradiction. So, we conclude that  $\xi^* \vee \tau = \tau^*$ .

So, for a *BT*-ring we have that Lemma 2 applies to give a nice partition of *R*-tors via the equivalence relation  $\sim_{\mathbf{F}}$ , because the equivalence class  $[\xi]_{\mathbf{F}}$ contains an isomorphic copy or every other  $[\tau]_{\mathbf{F}} \in R$ -tors/ $\sim_{\mathbf{F}}$ . So, we will have *R*-tors completely determined as a lattice if we know the lattice structure of the sublattice  $[\xi]_{\mathbf{F}}$ .

Theorem 12. (Bland [3, Theorem 2.8]). If R is a semiperfect ring, then

$$\tau \sim_{\mathsf{F}} \chi \iff \operatorname{Rad} R \in \mathsf{T}_{\tau}.$$

Bland's theorem is equivalent to the following result.

Theorem 13. If R is a semiperfect ring, then  $[\chi]$  contains a smallest element  $\chi_* = \xi(\operatorname{Rad} R)$ .

Proof:  $\Longrightarrow$ ) Since  $0 \longrightarrow \operatorname{Rad} R \longrightarrow R \longrightarrow R/\operatorname{Rad} R \longrightarrow 0$  is a projective cover with  $\operatorname{Rad} R \in T_{\chi} = R$ -mod, we have, using Bland's Theorem, that  $\xi(\operatorname{Rad} R) \in [\chi]_{\mathsf{F}}$ . Therefore  $\xi(\operatorname{Rad} R)$  is the least element of  $[\chi]_{\mathsf{F}}$ .

 $\iff ) \text{ Let us suppose that } \chi_* = \xi(\operatorname{Rad} R). \text{ Now we have, for } \tau \in R\text{-tors,} \\ \tau \in [\chi] \iff \tau \ge \xi(\operatorname{Rad} R) \iff \operatorname{Rad} R \in \mathsf{T}_{\tau}. \blacksquare$ 

The following two results can be proved (Rincón-Mejía [14]).

Theorem 14. If R is a semiperfect ring, then  $\xi^* = \chi(\operatorname{Rad} R)$ , where  $\xi^*$  is the biggest element of  $[\xi]_{\mathbf{F}}$ .

Theorem 15. Rincón-Mejía (14).

If R is a local ring, then  $\forall [\tau] \in R$ -tors/ $\sim_F$ , we have that  $[\tau]_F$  has a biggest element,  $\tau^*$ , given by  $\tau^* = \chi(\operatorname{Rad} R/t_\tau(\operatorname{Rad} R))$ , and a smallest element given by  $\tau_* = \xi(t_\tau(\operatorname{Rad} R))$ .

**Theorem 16.** Let R be a semiperfect ring, then Goldman's torsion theory centrally splits  $\iff \operatorname{soc}_p(\operatorname{Rad} R) = 0.$ 

(Remember that M is a Goldman torsion module iff  $M = \operatorname{soc}_{P}(M)$ , where  $\operatorname{soc}_{p}(M)$ , where  $\operatorname{soc}_{p}(M)$  denotes the projective socle of M).

Proof:  $\iff$  If  $\operatorname{soc}_p(\operatorname{Rad} R) = (0)$ , then every projective simple module  ${}_RS$  is injective: for if  ${}_RS$  is a simple projective module, then  $S \in \mathsf{T}_{\xi(\operatorname{Rad} R)} \cup \mathsf{F}_{\xi(\operatorname{Rad} R)}$ , since S is simple. But  $S \in \mathsf{T}_{\xi(\operatorname{Rad} R)} \Longrightarrow \exists 0 \neq f: \operatorname{Rad} R \longrightarrow E(S)$ . As  $S \leq_e E(S)$ , we have that  $S \leq \inf f$ , so we have the diagram

Rad R  

$$\int f^{-1}(S) \xrightarrow{f|_{f^{-1}(S)}} S$$

where  $f|_{f^{-1}(S)}$  is an epimorphism with codomain being a projective module. Therefore S is isomorphic to a submodule of  $f^{-1}(S)$ , which is a submodule of the projective socle of Rad R; this is contradiction.

Thus we have, that if  ${}_RS$  is a projective simple module, then  $S \in \mathsf{F}_{\xi(\operatorname{Rad} R)}$ . But  $\xi(\operatorname{Rad} R) = \chi_*$ , by Bland's Theorem, from which we get that if M is a direct sum of projective simple modules, then  $M \in \mathsf{F}_{\chi_*}$  and hence M is injective (by Theorem 3).

Thus we have that  $\forall N \in R$ -mod,  $\operatorname{soc}_p(N)$  is an injective submodule of N and hence it is also a direct summand of N; i.e., Goldman's torsion theory splits. In particular  $R = \operatorname{soc}_p(R) \oplus_R K$ . But now, since R is semiperfect, R is semiartinian and therefore  $\operatorname{soc}(R) \leq_e R$ . In particular  $\operatorname{soc}(K) \leq_e K$ . Let us note that every left simple submodule of K is singular (since a left simple module is either singular or projective, but  $\operatorname{soc}_p(K) = \operatorname{soc}_p(R) \cap K = 0$ )). Thus we have that  $\operatorname{soc}(K)$  is a Goldie's torsion-module. Hence K is a Goldie's torsion-module, too (Goldie's torsion theory is closed under taking essential extentions). Thus,  $K \leq t_G(R) = t_G(\operatorname{soc}_p(R)) \oplus t_G(K)$ , but each simple summand of  $\operatorname{soc}_p(R)$  is non singular (being projective). So,  $K = t_G(R)$  and so we have that K is a bilateral ideal of R. As a result,  $R = \operatorname{soc}_p(R) \oplus K$  (ring direct sum); i.e., Goldman's torsion theory centrally splits.

⇒ ) If  $\operatorname{soc}_p(\operatorname{Rad} R) \neq 0$  then  $0 \longrightarrow \operatorname{soc}_p(R) \longrightarrow R \longrightarrow R/\operatorname{soc}_p(R) \longrightarrow 0$ does not split. For if it split, then taking a simple submodule S of Rad R we have that the monomorphisms  $S \xrightarrow{\subseteq} \operatorname{soc}_p(\operatorname{Rad} R)$ ,  $\operatorname{soc}_p(\operatorname{Rad} R) \xrightarrow{\subseteq} \operatorname{soc}_p(R)$ and  $\operatorname{soc}_p(R) \xrightarrow{\subseteq} R$  are splitting; so its composition also splits. So we would have that  $R = S \oplus K$ , where  $_RK$  is a maximal ideal of R, but this is impossible  $(S \leq \operatorname{Rad} R \leq K \Longrightarrow S \cap K = S \neq 0)$ . Hence Goldman's torsion theory does not split, and a fortiori, does not centrally split. ■

Corollary 4. If R is a commutative perfect ring, then Goldman's torsion theory centrally splits.

Proof: Raggi & Ríos ([17], Corolario 2.9) have proved in the general situation that  $\operatorname{soc}_p(M) = \operatorname{soc}_p(R)M \forall M \in R$ -mod. In our particular case we have that  $\operatorname{soc}_p(\operatorname{Rad} R) = \operatorname{soc}_p(R)$  Rad R = 0, since the Jacobson radical annihilates every simple module.

We should note that the preceeding proof does not apply for non commutative right perfect rings, because  $\operatorname{soc}_p(\operatorname{Rad} R)$  is not necessarily a right semisimple module.

From Theorem 3.1 of Raggi & Ríos [11], we have that for a right perfect ring, Goldie's torsion theory  $\tau_G$  is a *TTF* torsion theory generated by the left singular simple modules and cogenerated by the left projective simple modules (in fact the preceeding statements hold when *R* is left semiartinian ring).

In the following theorem we will denote  $S_I$  the class of the left injective simple modules and by  $S_P$  the class of left projective simple modules.

**Theorem 17.** If R is a right perfect ring satisfying  $soc_p(\operatorname{Rad} R) = (0)$ , then are equivalent:

i)  $\chi_* = \tau_G$ , where  $\chi_*$  denotes the least element of  $[\tau] \in R$ -tors/ $\sim_F$ . ii)  $S_I = S_P$ .

Proof: i)  $\Longrightarrow$  ii)  $S_P \subseteq S_I$  follows from the part  $\Leftarrow$  ) of the proof of Theorem 16. Let  $_RS$  be a left injective simple module. We want to prove that it is projective. Let us observe that since R is right perfect, then  $R/\operatorname{Rad} R$  is semisimple, so that  $_RM$  is semisimple iff  $\operatorname{Rad} R M = 0$ . Therefore every direct product of simple modules is semisimple. As a consequence, using Theorem 18, we get that  $\chi(S)$  belongs to  $[\chi]_{\mathsf{F}}$ . For if  $M \in \mathsf{F}_{\chi(S)}$ , then  $\exists_M \mapsto S^r$  for some set X, and as  $S^X$  is a semisimple module. But on the other hand, M is injective, as it is isomorphic to a direct summand of the injective module  $S^X$ .

Thus,  $\chi(S) \in [\chi]_F$ , and therefore  $\chi(S) \ge \chi_* = \tau_G$ . Then we have that S is Goldie torsion free, which is cogenerated by the left projective simple modules.

Hence  $\exists 0 \neq f: S \longrightarrow U$ , where U is a left projective simple module. Since f must be an isomorphism, we have that S is a projective module. Therefore  $S_I \subseteq S_P$ , and hence  $S_I = S_P$ .

ii)  $\Longrightarrow$  i) Since  $\tau_G$  is cogenerated by the left projective simple modules, we have that every  $\tau_G$ -torsion free module is semisimple, since it is (isomorphic to) a submodule of a direct product of simple modules (this product is annihilated by Rad R). But a  $\tau_G$ -torsion free module is an injective module, since it is a direct summand of a product of projective simple modules, and such a product is injective by the hypothesis that all projective simple modules are injective modules. Since every  $\tau_G$ -torsion free module is injective,  $\tau_G \in [\chi]_F$  by Theorem 3.

Analogously, if  $\tau \in [\chi]_F$  let us take E an injective module which cogenerates  $\tau$ ; i.e.,  $\tau = \chi(E)$ . By another use of Theorem 3, we get that E is semisimple. Now, if  $_RS$  is a simple submodule of E, it has to be injective. Because S is an injective module, S is also projective by hypothesis. Therefore it is  $\tau_G$ -torsion free. So,  $E \in F_G$ , since E is a direct sum of  $\tau_G$ -torsion free modules. But  $E \in F_G \implies \tau = \chi(E) \geq \tau_G$ ; so we have that  $\tau_G = \chi_*$ .

Corollary 5. If R is a quasifrobenius ring (QF-ring), then  $\chi_* = \tau_G$ .

Proof: R is right perfect and the class of projective modules coincides with the class of injective modules. Moreover,  $\operatorname{soc}_p(\operatorname{Rad} R) = 0$ : if  $_RS \leq \operatorname{Rad} R$  was a projective simple module, then as S had to be injective, S would be a direct summand of R. Consequently,  $S = Re \leq \operatorname{Rad} R$ , with  $e = e^2$ , this is impossible. We conclude using Theorem 17.

Acknowledgements. I thank Dr. Francisco Raggi Cárdenas and M.C. José Ríos Montes for their advice, suggestions and encouragement, for without them I would not have been capable to do this work.

### References

- 1. ANDERSON F. AND FULLER K., Rings and categories of modules "Springer Verlag," 1973.
- 2. BICAN L., KEPKA T. AND NEMEC P., Torsion theories and Homological dimensions, Journal of Algebra 35 (1975), 99-122.
- 3. BLAND P.E., Divisible and codivisible modules, Math. Scand. 34 (1974), 153-161.
- 4. BLAND P.E., Perfect torsion theories, Proceedings of the American Mathematical Society 41 (1973), 349-355.
- 5. BRONOWITZ R. AND TEPLY M., Torsion theories of simple type, Journal of Pure and Applied Algebra 3 (1973), 329-336.

- 6. EILENBERG S. AND MOORE J. C., Foundations of Relative Homological Algebra, Memoirs of the American Mathematical Society 55.
- EILENBERG S., Algebra Homológica, Anales del Instituto de Matemáticas U.N.A.M. 1 (1961), 117-145.
- 8. GOLAN J., Localization of noncommutative rings, Marcel Dekker (1975).
- 9. GOLAN J., Structure Sheaves over noncommutative rings, Marcel Dekker (1980).
- 10. OHTAKE K., Colocalization and Localization, Journal of Pure and Applied Algebra (1977), 217-241.
- RAGGI F. AND RÍOS J., Algunas relaciones entre anillos semiartinianos y la teoría de torsión de Goldie, Anales del Instituto de Matemáticas, U.N.A.M. 23 (1983), 41-54.
- 12. RAGGI F. AND RÍOS J., Proper classes associated to torsion theories, Comm. in Algebra 15 (1987), 575-588.
- 13. RAGGI F. AND RÍOS J., Sublattices of *R*-tors associated to proper classes,

Comm. in Algebra 15 (1987), 555-574.

- 14. RINCÓN-MEJÍA H., Un estudio de subretículas de R-tors para anillos perfectos, Tesis, Facultad de Ciencias Universidad Nacional Autónoma de México (1986).
- 15. STENSTROM B., Rings of Quotients, "Springer Verlag," 1975.
- 16. TEPLY M., Homological dimension and splitting torsion theories, Pacific Journal of Mathematics 34 (1970), 193-205.
- 17. TEPLY M., Codivisible and projective covers, Communications in Algebra 1 (1974), 23-38.

Departamento de Matemáticas de la Facultad de Ciencias de la Universidad Nacional Autónoma de México. Ciudad Universitaria, D.F., C.P. 04510 MEXICO

Rebut el 12 de gener de 1988