Publicacions Matemàtiques, Vol 33 (1989), 3–15.

ERGODIC RESULTS FOR CERTAIN CONTRACTIONS ON ORLICZ SPACES WITH FIXED POINTS

DIEGO GALLARDO

Abstract

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $L_\phi \equiv L_\phi(X, \mathcal{M}, \mu)$ an Orlicz space associated to an N-function ϕ and let $T: L_\phi \longrightarrow L_\phi$ be a linear operator with a fixed point $h \neq 0$ a.e., such that

$$\int_X \phi(|Tf|)d\mu \leq \int_X \phi(|f|)d\mu \quad (f \in L_\phi)$$

and it is either a $\|\ \|_1$-contraction in $L_\phi \cap L_1$ or a $\|\ \|_\infty$-contraction in $L_\phi \cap L_\infty$. The main result of this paper is that for a wide class of N-functions ϕ, the ergodic maximal operator associated to T is bounded in L_ϕ. Moreover, for every $f \in L_\phi$ we have the almost everywhere convergence and the norm convergence of certain weighted averages which include the Césàro averages.

1. Introduction and preliminaries

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and $L_\phi \equiv L_\phi(X, \mathcal{M}, \mu)$ and Orlicz space associated to an N-function ϕ (L_ϕ may be a complex Banach space). In this paper we will consider linear operators T such that

i) $\int_X \phi(|Tf|)d\mu \leq \int_X \phi(|f|)d\mu$, $f \in L_\phi$

ii) T has a fixed point h, $h \neq 0$ a.e.

iii) T is either a $\|\ \|_1$-contraction in $L_\phi \cap L_1$ or a $\|\ \|_\infty$-contraction in $L_\phi \cap L_\infty$.

The main aim of this paper is to prove that, for a wide class of N-functions ϕ, *the ergodic maximal operator* M_T defined by

$$M_T f = \sup_{n \geq 1} \Big|\frac{1}{n} \sum_{k=0}^{n-1} T^k f\Big| \tag{1.1}$$

Keywords: Almost everywhere convergence, Besicovitch sequences, Césàro-averages, Contractions, Δ_2-condition, Ergodic maximal operator, Ergodic theorems, Extrapolation theorems, Fixed points, N-functions, norm convergence, Orlicz spaces, weighted averages.

1980 Mathematics subject classifications: Primary: 47A35. Secondary: 46E30.

is bounded in L_ϕ (*dominated ergodic theorem*). Moreover, we shall prove that if $\{b_k\}$ is a *bounded Besicovitch sequence*, then for every $f \in L_\phi$ there exists $f^* \in L_\phi$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} b_k T^k f(x) = f^*(x) \quad \text{a.e.}, \qquad \lim_{n\to\infty} \|\frac{1}{n}\sum_{k=0}^{n-1} b_k T^k f - f^*\|_{(\phi)} = 0.$$

A sequence of complex numbers $\{b_k\}$ is called a Besicovitch sequence if for every $\varepsilon > 0$ there exists a trigonometric polynomial α_ε such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} |b_k - \alpha_\varepsilon(k)| < \varepsilon.$$

As a special case we obtain the almost everywhere convergence (*individual ergodic theorem*) and the norm convergence (*mean ergodic theorem*) of *the Césàro-averages* $n^{-1}(f + Tf + \cdots + T^{n-1}f)$.

In the real L_p-case, with $1 < p < \infty$, and $(X, \mathcal{M}, \mu)$ a finite measure space the corresponding dominated ergodic theorem is proved by A. de la Torre in [10]. R. Sato proved in [9] that the de la Torre's result may be extended to the case $(X, \mathcal{M}, \mu)$ σ-finite and a complex L_p-space. The ergodic result for an operator which only satisfies conditions i) and iii) is an open problem even in the L_p-case, $1 < p < \infty$.

The bounded Besicovitch sequences as weights in the averages were used by J.H. Olsen in [8].

In order to obtain the dominated ergodic theorem we first need some *extrapolation theorems* which extend the ones given by M.A. Akcoglu and R.V. Chacon in [1] and R. Sato in [9], for L_p, $1 < p < \infty$.

Now, we shall present the basic definitions and results concerning to N-functions and Orlicz spaces which will be used in this paper. The proofs of most of these results can be found in [5] or in II-13 of [7].

An N-function is a continuous and convex function $\phi\colon [0,\infty) \longrightarrow \mathbf{R}$ such that $\phi(s) > 0$, $s > 0$, $s^{-1}\phi(s) \longrightarrow 0$ as $s \longrightarrow 0$ and $s^{-1}\phi(s) \longrightarrow \infty$ as $s \longrightarrow \infty$.

The function ϕ is an N-function if and only if it has the representation $\phi(s) = \int_0^s \varphi$ where $\varphi\colon [0,\infty) \longrightarrow \mathbf{R}$ is continuous from the right, non decreasing such that $\varphi(s) > 0$, $s > 0$, $\varphi(0) = 0$ and $\varphi(s) \longrightarrow \infty$ as $s \longrightarrow \infty$. More precisely φ is the right derivate of ϕ and will be called the *density function of* ϕ.

Associated to φ we have the function $\rho\colon [0,\infty) \longrightarrow \mathbf{R}$ defined by $\rho(t) = \sup\{s\colon \varphi(s) \le t\}$ which has the same aforementioned properties of φ. We will call ρ *the generalized inverse of* φ.

The N-function ψ defined by $\psi(t) = \int_0^t \rho$ is called *the complementary N-function of* ϕ. Thus, if $\phi(s) = p^{-1}s^p$, $p > 1$, then $\psi(t) = q^{-1}t^q$ where $pq = p + q$.

Young's inequality asserts that $st \leq \phi(s) + \psi(t)$ for $s, t \geq 0$, equality holding if and only if $\varphi(s-) \leq t \leq \varphi(s)$ or else $\rho(t-) \leq s \leq \rho(t)$ (See [3]).

If ϕ_1 and ϕ_2 are N-functions with complementary N-functions given by ψ_1 and ψ_2 respectively, then, *the inequality for complementary functions* asserts that if $\phi_1(s) \leq \phi_2(s)$ for $s \geq s_0$, then $\psi_2(t) \leq \psi_1(t)$ for $t \geq \varphi_2(s_0)$, where φ_2 is the density function of ϕ_2.

An N-function ϕ is said to satisfy the Δ_2*-condition* in $[s_0, \infty)$, $s_0 \geq 0$, if there exists a constant α such that $\phi(2s) \leq \alpha\phi(s)$ for every $s \geq s_0$.

If φ is the density function of ϕ, then ϕ satisfies Δ_2 in $[s_0, \infty)$ if and only if there exists a constant $\alpha > 1$ such that $s\varphi(s) \leq \alpha\phi(s)$, $s \geq s_0$.

The Δ_2-condition for ϕ does not transfer necessarily to the complementary N-function.

If $(X, \mathcal{M}, \mu)$ is a σ-finite measure space we denote by $\mathbf{M} = \mathbf{M}(X, \mathcal{M}, \mu)$ the space of $\mathcal{M}$-measurable and μ-a.e. finite functions from X ro $\mathbf{R}$ or to $\mathbf{C}$. If ϕ is an N-function we consider *the Orlicz spaces* $L_\phi \equiv L_\phi(X, \mathcal{M}, \mu)$ and $L_\phi{}^* \equiv L_\phi{}^*(X, \mathcal{M}, \mu)$ defined by $L_\phi = \{f \in \mathbf{M} : \int_X \phi(|f|)d\mu < \infty\}$ and $L_\phi{}^* = \{f \in \mathbf{M} : fg \in L_1 \text{ for all } g \in L_\psi\}$ where ψ is the complementary N-function of ϕ. We have $L_\phi \subset L_\phi{}^*$ and if ϕ satisfies Δ_2 then $L_\phi = L_\phi{}^*$.

We have that $L_\phi{}^*$ is a linear space with the usual operations on which we may define the norms $\|f\|_\phi = \sup\{\int_X |fg|d\mu : g \in S_\psi\}$, where $S_\psi = \{g \in L_\psi : \int_X \psi(|g|)d\mu \leq 1\}$, and $\|f\|_{(\phi)} = \ inf\{\lambda > 0 : \int_X \phi(\lambda^{-1}|f|)d\mu \leq 1\}$ which are called *Orlicz norm and Luxemburg norm* respectively. Both norms are equivalent.

Holder's inequality asserts that for every $f \in L_\phi{}^*$ and every $g \in L_\psi{}^*$ we have $\|fg\|_1 \leq \|f\|_{(\phi)}\|g\|_\psi$ where ϕ and ψ are complementary N-functions.

If $\phi(s) = s^p$ with $p > 1$ then $L_\phi{}^* = L_\phi = L_p$, $\|f\|_{(\phi)} = \|f\|_p$ and $\|g\|_\psi = \|g\|_q$ where $pq = p + q$.

The convergence $f_n \longrightarrow f$ in $[L_\phi{}^*, \|\ \ \|_\phi]$ implies the mean convergence $\lim_{n\to\infty} \int_X(|f_n - f|)d\mu = 0$ but, in general, mean convergence only implies norm convergence when ϕ satisfies Δ_2. Then the set S of simple functions (with support of finite measure) is dense in $[L_\phi, \|\ \ \|_\phi]$ if ϕ satisfies Δ_2.

If ϕ verifies Δ_2, then for every continuous linear functional F over $[L_\phi, \|\ \ \|_{(\phi)}]$ there exists an unique function $g \in L_\psi{}^*$ such that $F(f) = \int_X fgd\mu$, $f \in L_\phi$, and moreover $\|F\|_{(\phi)} = \|g\|_\psi$, where ψ is the complementary N-function of ϕ, but if ϕ does not satisfy Δ_2 then there exist linear functionals on $L_\phi{}^*$ which are not represented by functions of $L_\psi{}^*$.

If ϕ and ψ satisfy Δ_2 then $[L_\phi, \|\ \ \|_{(\phi)}]$ is *reflexive*.

In the following, we shall always assume that $(X, \mathcal{M}, \mu)$ is a *σ-finite measure space* and ϕ, *together with its complementary N-function* ψ, *satisfy the* Δ_2*-condition in* $[0, \infty)$. The Δ_2-condition for ϕ is a very important condition that plays fundamental roles in many questions and the best known Orlicz spaces are associated to functions which satisfy Δ_2. The Δ_2-condition for ψ may seem to be a restrictive assumption. Some know Orlicz spaces as, for example, the Zygmund Orlicz space $L\operatorname{Log}L$ and the $L\operatorname{Log}^k L$ spaces, $k>0$, are associated to N-functions which satisfy Δ_2 but their complementary N-functions do not; but the above spaces do not satisfy our dominated ergodic result. In fact *the Δ_2-condition for the complementary N-function is necessary for such result.*

Precisely, let $([0,1],\mathcal{B},\lambda)$ be the Lebesgue-space and let τ an invertible λ-measure preserving transformation from $[0,1]$ into itself. In [2] B. Bru and H. Heinich characterize the Orlicz spaces, associated to Young's functions, for which the ergodic maximal operator associated to the operator T, defined by $Tf=f\circ\tau^{-1}$, is bounded in L_ϕ (*classical dominated ergodic theorem*) (the Young's functions in [2] are our N-functions). The characterizing condition given in [2] is the condition of comoderation on ϕ.

The function ϕ is said to be *comoderated* if there exist s_0, a and $b>1$ such that $\varphi(as)\geq b\varphi(s)$ for $s\geq s_0$, where φ is the density function of ϕ or, equivalently, if *there exist s_0, a and $b>1$ such that $\phi(as)\geq ab\phi(s)$ for $s\geq s_0$* (in [2] a function continuous from the left is taken as density function of ϕ whereas our density function is right continuous).

The paper [2] does not establisch the equivalence between the comoderation of ϕ and the *moderation* (Δ_2-condition in some $[t_0,\infty)$) of the complementary N-function ψ unless φ be continuous. However, we observe that the comoderation of ϕ is equivalent to the moderation of ψ. At the same time, we shall prove another characterization of the moderation of ψ, which is used in this paper, and which appear in [2], [5] and in the rest of the literature with more restrictive hypothesis. Exactly:

Proposition 1.2. *Let ϕ be an N-function and ψ the complementary N-function of ϕ. The following conditions are equivalent:*

a) ϕ is comoderated.

b) ψ is moderated.

c) There exist s_0 and $\beta>1$ such that $\beta\phi(s)\leq s\varphi(s)$ for $s\geq s_0$.

Proof: $a)\Longrightarrow b)$. If ϕ is comoderated then $\phi(s)\leq\phi_1(s)$ for $s\geq s_0$ where ϕ_1 is the N-function given by $\phi_1(s)=(ab)^{-1}\phi(as)$. The complementary function of ϕ_1 is given by $\psi_1(t)=(ab)^{-1}\psi(bt)$. Taking into account the inequality for complementary N-functions we obtain that $\psi(bt)\leq ab\psi(t)$ for $t\geq t_0=\varphi_1(s_0)$, where $b>1$, which equivales to condition Δ_2 of ψ for $t\geq t_0$.

$b)\Longrightarrow c)$. Let ρ be the generalized-inverse of φ. Since ψ is moderated there exist t_0 and $\alpha>1$ such that $t\rho(t)\leq\alpha\psi(t)$ for every $t\geq t_0$. On the other hand, it follows from the equality cases in Young's inequality that $t\rho(t)=\phi(\rho(t))+\psi(t)$ and therefore

$$\phi(\rho(t))\leq\alpha^{-1}(\alpha-1)t\rho(t)\,,\quad t\geq t_0.$$

Then, since $\rho(\varphi(s))\geq s$ and the function $u\longrightarrow u^{-1}\phi(u)$ increases for $u>0$ we obtain

$$s^{-1}\phi(s)\leq\phi(\rho(\varphi(s)))/\rho(\varphi(s))\leq\alpha^{-1}(\alpha-1)\varphi(s)\,,\quad s\geq\rho(t_0)$$

and thus we obtain c) with $s_0 = \rho(t_0)$ and $\beta = \alpha(\alpha - 1)^{-1} > 1$.

c) $\Longrightarrow$ *a)*. Condition c) implies that there exist s_0 and $\beta > 1$ such that the function $s \longrightarrow s^{-\beta}\phi(s)$ increases for $s \geq s_0$ (or for $s > s_0$ if $s_0 = 0$). Then, if $a > 1$ is such that $a^{\beta-1} \geq 2$ we have $\phi(as) \geq a^{\beta}\phi(s) \geq 2a\phi(s)$ for $s \geq s_0$ and thus we obtain the comoderation of ϕ.

Note. Since $\varphi(0) = \rho(0) = 0$, if some of the conditions of Proposition 1.2 is satisfied for every $s \geq 0$, then the others two conditions are also valids for every $s \geq 0$.

In this way, the moderation of ψ is necessary for the classical dominated ergodic result and, therefore, for our dominated ergodic result since that the operator T, defined by $Tf = f \circ \tau^{-1}$ satisfies conditions i), ii) and iii), whatever the N-function ϕ may be. On the other hand, the space $([0,1], \mathcal{B}, \lambda)$ is of finite measure and our spaces can be of infinite measure. For this reason we shall assume the Δ_2-condition in $[0, \infty)$, but un the case $\mu(X) < \infty$ the argument which we shall use can be adapted if only we suppose the Δ_2-condition in some $[s_0, \infty)$.

Our results are valid, for example, for the known $L^p \operatorname{Log}^k L$ spaces, with $p > 1$ and $k \geq 0$, since the N-functions of the form $\phi(s) = s^p \log^k(1+s)$ satisfy that $1 < p < \phi(s)/s\varphi(s) \leq p + b$ for every $s > 0$ and certain constant b.

2. Extrapolation Theorems

We first observe that the convexity theorem for positive operators given by M.A. Akcoglu and R.V. Chacon in [1] can be easily extended to Orlicz spaces, following the same type of arguments, as follows

Proposition 2.1. *Let ϕ be an N-function strictly convex in some interval and let T be a conservative positive contraction in L_1 such that*

$$\int_X \phi(|Tf|)d\mu \leq \int_X \phi(|f|)d\mu\,, \quad (f \in L_1 \cap L_\phi). \tag{2.2}$$

Then, $\|Tf\|_\infty \leq \|f\|_\infty$ for every $f \in L_1 \cap L_\infty$.

Proof: The operator T is said to be conservative when $\mu(D) = 0$, where D is the dissipative part of X with respect to T.

First assume that $\mu(X) < \infty$. It is enough to prove that $Tc \leq c$ almost everywhere for some constant $c \neq 0$.

We have that φ increases strictly in some interval I, where φ is the density function of ϕ. Let $c \in I$ with $c \neq 0$. Then, we get that

$$\phi(c+s) > \phi(c) + s\varphi(c) \quad (0 \neq s \geq -c). \tag{2.3}$$

Since T is conservative we have $\int_X Tf d\mu = \int_X f d\mu$ for every $f \in L_1$.

Let $Tc(x) = c + g(x)$; then $\int_X g d\mu = 0$ and therefore if $\mu\{x \in X : g(x) > 0\} > 0$ we have

$$\int_X \phi(|Tc|)d\mu > \int_X \phi(c)d\mu,$$

which contradicts (2.2). This proves that $Tc \le c$.

The general case follows from the preceding by a method similar to the one given in [1] using the following result:

Lemma 2.4. *Let ϕ be an N-function and T a positive contraction in L_1 satisfing (2.2). Then, for every $A \in \mathcal{M}$ there exists a linear operator*

$T_A : L_1(A) \longrightarrow L_1(A)$ such that

a) T_A is a positive contraction in $L_1(A)$ and

$$\int_X \phi(|T_A f|)d\mu \le \int_X \phi(|f|)d\mu\ ,\ (f \in L_1(A) \cap L_\phi(A)).$$

b) For every $f \in {L_1}^+(A)$ and every $n \ge 1$

$$\sum_{k=0}^{n} T^k f(x) \le \sum_{k=0}^{n} T_A^k f(x) \quad a.e. \ in\ A.$$

The proof of Lemma 2.4 can be obtained easily following the arguments of [1].

Remarks.

1. The conservative condition of T cannot be eliminated from the hypothesis of Proposition 2.1 since in $\mathbf{R}$ with Lebesgue-measure if $Tf(x) = \sqrt{2}f(2x)$ then T is a positive contraction in L_1, an isometry in L_2 but $\|Tf\|_\infty = \sqrt{2}\|f\|_\infty$.

2. There exist N-function which are strictly convex over no interval. An example is the following. We consider the dyadic intervals $I_n = [2^{n-1}, 2^n)$ and $J_n = [2^{-n}, 2^{-n+1})$ where n is a positive integer and let $\varphi : [0,\infty) \longrightarrow [0,\infty)$ be defined by $\varphi(0) = 0$, $\varphi(t) = 2^{-n}$ if $t \in J_n$ and $\varphi(t) = 2^{n-1}$ if $t \in I_n$. Then ϕ defined by $\phi(s) = \int_0^s \varphi$ is an N-function. Since $\phi(2s) = 4\phi(s)$ we have that ϕ, as well as its complementary N-function, satisfy the Δ_2-condition. However ϕ is not strictly convex over any interval. Furthemore there is no constant $c \ne 0$ such that (2.3) holds.

However most of N-functions are strictly convex in some interval.

In the following results the operators are not necessarily positive but they have a fixed point h with $h \ne 0$ a.e.

Theorem 2.5. *Let ϕ be an N-function, strictly convex in some interval and let $T: L_\phi \longrightarrow L_\phi$ be a linear operator such that*

i) $\int_X \phi(|Tf|)d\mu \leq \int_X \phi(|f|)d\mu$, $(f \in L_\phi)$.

ii) $\|Tf\|_1 \leq \|f\|_1$, $(f \in L_1 \cap L_\phi)$.

iii) There exists $h \in L_\phi$, $h \neq 0$ a.e., such that $Th = h$.

Then, $\|Tf\|_\infty \leq \|f\|_\infty$ for every $f \in L_1 \cap L_\infty$ and consequently for every $f \in L_\phi \cap L_\infty$.

Proof: In this proof we follows the idea given by Sato in [9].

Let k be such that $\phi(s) < s$ for $0 < s < k$. Given $f \in L_1 \cap L_\infty$ let $B = \{x \in X: |f(x)| \geq k\}$; then $\mu(B) < \infty$ and therefore $\int_X \phi(|f|)d\mu \leq \|f\|_1 + \mu(B)\phi(\|f\|) < \infty$. Consequently $L_1 \cap L_\infty \subset L_\phi$.

Let $\hat{T}: L_1 \longrightarrow L_1$ be the linear extension of $T: [L_1 \cap L_\phi, \| \ \|_1] \longrightarrow L_1$ and P *the linear modulus of* $\hat{T}$. (See Theorem 4.1.1 in [6]). We shall prove that P satisfies the hypotheses of Proposition 2.1 and therefore $\|Pf\|_\infty \leq \|f\|_\infty$ $f \in L_1 \cap L_\infty$; in this way, since $|\hat{T}f| \leq P|f|$, $f \in L_1$, and $L_1 \cap L_\infty \subset L_1 \cap L_\phi$ we obtain that $\|Tf\|_\infty \leq \|f\|_\infty$, $f \in L_1 \cap L_\infty$, and consequently for every $f \in L_\phi \cap L_\infty$ since $L_1 \cap L_\infty$ is dense in $L_\phi \cap L_\infty$ with the L_∞-norm.

Now, we show that P satisfies the conditions of Proposition 2.1. The Δ_2-condition implies that $L_1 \cap L_\phi$ is dense in $[L_\phi, \| \ \|_{(\phi)}]$. On the other hand, it follows from i) that $\|Tf\|_{(\phi)} \leq \|f\|_{(\phi)}$, $f \in L_\phi$, and consequently given $\varepsilon > 0$ there is $f_\varepsilon \in L_1 \cap L_\phi$ such that for every $n \geq 1$

$$\|h - \frac{1}{n}\sum_{k=0}^{n-1} T^k f_\varepsilon\|_{(\phi)} \leq \varepsilon/2. \tag{2.6}$$

If T is a power bounded linear operator in a reflexive Banach space V, that is, the powers T^k, $k \geq 0$, are uniformly bounded in V, then the Césàro-averages.

$$R_n f = \frac{1}{n}\sum_{k=0}^{n-1} T^k f$$

converge in norm to a T-invariant limit for all $f \in V$ (See Theorem 2.1.2 in [6]).

Lef $f_\varepsilon{}^*$ be the limit in $[L_\phi, \| \ \|_{(\phi)}]$ of $R_n f_\varepsilon$. It follows from (2.6) that for $0 < \varepsilon < 1$ we have $\|h - f_\varepsilon^*\|_{(\phi)} < \varepsilon$ and consequently

$$\int_X \phi(|h - f_\varepsilon{}^*|)d\mu < \varepsilon. \tag{2.7}$$

On the other hand, $f_\varepsilon{}^*(x) = 0$ for a.e. $x \in D$, where D is the dissipative part of X with respect to P, since (Theorem 3.1.6 in [6]) $\sum_{k\geq 0} P^k f(x) < \infty$

on D for all $f \in L_1{}^+$. Since $\phi(|h|) > 0$ a.e. (2.7) shows that $\mu(D) = 0$ and thus P is conservative.

Now, in order to prove that P satisfies condition (2.2) we consider the Akcoglu and Brunel's theorem related with the structure of $\hat{T}$ on the conservative part C of X with respect to P (see Theorem 4.1.10 in [6]). Let $\mathcal{F}$ be the family of P-absorbing subsets of C; there exists a set $\Gamma \in \mathcal{F}$ and a function $s \in L_\infty(\Gamma)$, with $|s| = 1$ on Γ, such that $\hat{T}f = \bar{s}P(sf)$ for any $f \in L_1(\Gamma)$, where $\bar{s}$ is the complex conjugate of s, and if $\Delta = C - \Gamma$ then $(I - T)L_1(\Delta)$ is dense in $L_1(\Delta)$.

We have that $\operatorname{supp} T(\chi_\Gamma h) \subset \Gamma$ and $\operatorname{supp} T(\chi_\Delta h) \subset \Delta$; therefore $Tg = g$ where $g = \chi_\Delta h$. Carryng out a similar reasoning to the used for h we have that for every $\varepsilon > 0$ there exist $f_\varepsilon \in L_1(\Delta) \cap L_\phi(\Delta)$ and $f_\varepsilon{}^* \in L_\phi(\Delta)$ such that $\|g - f_\varepsilon{}^*\|_{(\phi)} < \varepsilon$ and $\lim_{n\to\infty} \|R_n f_\varepsilon - f_\varepsilon{}^*\|_{(\phi)} = 0$.

Given $\eta > 0$ there is $u_\eta \in L_1(\Delta)$ such that $\|u_\eta - Tu_\eta - f_\varepsilon\|_1 < \eta/2$ and therefore for every $n \geq 1$ we have $\|n^{-1}(u_\eta - T^n u_\eta) - R_n f_\varepsilon\|_1 = \|R_n(u_\eta - Tu_\eta - f_\varepsilon)\|_1 < \eta/2$, which proves that $\lim_{n\to\infty} \|R_n f_\varepsilon\|_1 = 0$ and so $f_\varepsilon{}^*(x) = 0$ a.e. This shows that $\|g\|_{(\phi)} = 0$ and consequently $\mu(\Delta) = 0$. Then, we have $\hat{T}f = \bar{s}P(sf)$ for every $f \in L_1$ and therefore it follows from i) that $\int_X \phi(|Pf|)d\mu = \int_X \phi(|\bar{s}\hat{T}(\bar{s}f)|)d\mu \leq \int_X \phi(|f|)d\mu$ for every $f \in L_1 \cap L_\phi$ and this finishes the proof. ∎

Now, our aim is to prove that the roles of L_1 and L_∞ in Theorem 2.5 can be interchanged. For this we shall considerer the adjoint operator of T.

Let $T\colon L_\phi \longrightarrow L_\phi$ be a bounded linear operator; more precisely, we suppose that there is a constant C such that $\|Tf\|_{(\phi)} \leq C\|f\|_{(\phi)}$, $f \in L_\phi$. Then, if $g \in L_\psi{}^*$, where ψ is the complementary N-function of ϕ, the linear functional F over $[L_\phi, \|\|_{(\phi)}]$ defined by $F(f) = \int_X gTf d\mu$ is continuous since by Holder's inequality we have $|F(f)| \leq C\|g\|_\psi \|f\|_{(\phi)}$ and therefore, since ϕ satisfies Δ_2, there exists an unique function $g^* \in L_\psi{}^*$ such that $\int_X gTf d\mu = \int_X fg^* d\mu$, $f \in L_\phi$. Then, we can define the bounded linear operator $T^*\colon L_\psi{}^* \longrightarrow L_\psi{}^*$, $g \longrightarrow T^*g$, where T^*g is the function in $L_\psi{}^*$ such that

$$\int_x gTf d\mu = \int_X fT^*g d\mu\,, \quad f \in L_\phi.$$

We shall call T^* *the adjoint operator of* T. T^* satisfies $\|T^*g\|_\psi \leq C\|g\|_\psi$. In our case we have

Lemma 2.8. *Let $T\colon L_\phi \longrightarrow L_\phi$ be a linear operator such that*

$$\int_X \phi(|Tf|)d\mu \leq \int_X \phi(|f|)d\mu \quad (f \in L_\phi).$$

Then, its adjoint operator T^* satisfies

$$\int_X \psi(|T^*g|)d\mu \leq \int_X \psi(|g|)d\mu \quad (g \in L_\psi) \tag{2.9}$$

and moreover, if T admits an invariant function h with $h \neq 0$ a.e., then there exists $g \in L_\psi$ with $g \neq 0$ a.e., such that $T^*g = g$.

Proof: We write $\operatorname{sig} z$ for $z/|z|$ and by $\bar{u}$ we denote the complex conjugate of u. For $g \in {L_\phi}^+$ we have

$$\int_X f|T^*g|d\mu = |\int_X f(\operatorname{sig}\overline{T^*g})T^*g d\mu| \leq \int_X |T(f \operatorname{sig}\overline{T^*g})||g|d\mu \leq \tag{2.10}$$
$$\leq \int_X \phi(f)d\mu + \int_X \psi(|g|)d\mu.$$

Let φ be the density function of ϕ and ρ the generalized inverse of φ. Since ψ satisfies Δ_2 there exists $\alpha > 1$ such that $s\rho(s) \leq \alpha\psi(s)$ and therefore $\phi(\rho(s)) = s\rho(s) - \psi(s) \leq (\alpha - 1)\psi(s)$. Therefore, for every $g \in L_\psi$ the function $\rho(|T^*g|)$ belongs to ${L_\phi}^+$ and so (2.9) follows from (2.10) for $f = \rho(|T^*g|)$.

Now, let us assume that $Th = h$ with $h \neq 0$ a.e. If φ is not continuous then there exists an at most countable set of positive reals $s_1, s_2, \dots, s_n$ where φ is not continuous; in this situation, since $h \in L_\phi$, it is easy to see that $\{c > 0 : \mu\{x \in X : |s_i^{-1}h(x)| = c\} > 0\}$ is at most countable and therefore there exists $\lambda > 0$ such that for every s_i we have

$$\mu\{x \in X : |\lambda^{-1}h(x)| = s_i\} = 0. \tag{2.11}$$

In the case φ continuous (2.11) holds trivially with $\lambda = 1$.

Let $u = \lambda^{-1}h$ and $g = \varphi(|u|)\operatorname{sig}\bar{u}$. We have that $g \neq 0$ a.e. and $g \in L_\phi$ since ϕ satisfies Δ_2. It follows from (2.9) that

$$\int_X |u|\varphi(|u|)d\mu = |\int_X uT^*g d\mu| \leq \int_X |u||T^*g|d\mu \leq \int_X \phi(|u|)d\mu + \tag{2.12}$$
$$+ \int_X \psi(|T^*g|)d\mu \leq \int_X \phi(|u|)d\mu + \int_X \psi(\varphi(|u|))d\mu = \int_X |u|\varphi(|u|)d\mu$$

and therefore

$$\int_X |u||T^*g|d\mu = \int_X (\phi(|u|) + \psi(|T^*g|))d\mu.$$

Then, Young's inequality shows that

$$|uT^*g| = \phi(|u|) + \psi(|T^*g|) \quad \text{a.e.} \tag{2.13}$$

It follows from (2.11) and (2.13) that $|T^*g| = \varphi(|u|)$ a.e. On the other hand we obtain from (2.12) that $(\operatorname{sig}\bar{u})\operatorname{sig}\overline{T^*u} = 1$ and therefore $T^*g = g$ which finishes the proof of the Lemma.

Theorem 2.5 and Lemma 2.8 imply easy

Theorem 2.14. *Let ϕ be an N-function whose complementary N-function is strictly convex in some interval and let $T: L_\phi \longrightarrow L_\phi$ be a linear operator such that*

i) $\int_X \phi(|Tf|)d\mu \leq \int_X \phi(|f|)d\mu \quad , \quad (f \in L_\phi)$.

ii) $\|Tf\|_\infty \leq \|f\|_\infty \quad , \quad (f \in L_\infty \cap L_\phi)$.

iii) There exists $h \in L_\phi$, $h \neq 0$ a.e., such that $Th = h$.

Then, $\|Tf\|_1 \leq \|f\|_1$ for every $f \in L_1 \cap L_\phi$.

Proof: Let ψ be the complementary N-function of ϕ, T^* the adjoint operator of T and let $\{A_n\}$ be an increasing sequence of measurable sets with $\mu(A_n) < \infty$ and $X = \cup A_n$. Then, for every $g \in L_1 \cap L_\psi$ we have

$$\int_X |T^*g|d\mu = \lim_{n\to\infty} \Big| \int_X gT(\chi_{A_n} \operatorname{sig} \overline{T^*g})d\mu \Big| \leq \|g\|_1 .$$

Consequently, $\|T^*g\|_\infty \leq \|g\|_\infty$ for every $g \in L_\psi \cap L_\infty$ and therefore for any $f \in L_1 \cap L_\phi$ and $n \geq 1$ we get $\big| \int_X fT^*(\chi_{A_n} \operatorname{sig} \overline{Tf})d\mu \big| \leq \|f\|_1$ and thus $\|Tf\|_1 \leq \|f\|_1$.

3. Ergodic results

Theorem 3.1. *(Dominated, individual and mean weighted ergodic theorem). Let ϕ and T satisfy the hypotheses of the extrapolation theorem 2.5 or 2.14. Then*

a) The ergodic maximal operator M_T-defined by (1.1) is bounded in $[L_\phi, \| \ \|_{(\phi)}]$.

b) If $\{b_k\}$ is a bounded Besicovitch sequence, then for every $f \in L_\phi$ there exists $f^* \in L_\phi$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} b_k T^k f(x) = f^*(x) \quad a.e., \qquad \lim_{n\to\infty} \Big\| \frac{1}{n} \sum_{k=0}^{n-1} b_k T^k f - f^* \Big\|_{(\phi)} = 0.$$

Proof: Since $L_1 \cap L_\infty \subset L_\phi$ it follows from Theorem 2.5 or 2.14 that $T: L_1 \cap L_\phi \longrightarrow L_1$ admits an unique extension $\hat{T}: L_1 \longrightarrow L_1$ which is a Dunford-Schwartz operator, that is, $\|\hat{T}f\|_1 \leq \|f\|_1$, $f \in L_1$, and $\|\hat{T}f\|_\infty \leq \|f\|_\infty$, $f \in L_1 \cap L_\infty$. Therefore the linear modulus P of $\hat{T}$ is also a Dunfort-Schwartz operator.

Consequently, for every $f \in L_1$ and $\lambda > 0$ we have (see Theorem 2.3.2 in [4])

$$\mu\{x \in X : M_P f(x) > \lambda\} \leq \lambda^{-1} \int_X |f|d\mu ,$$

where M_P is the maximal operator associated to P. Moreover, trivially, $\|M_P f\|_\infty \leq \|f\|_\infty$ for $f \in L_1 \cap L_\infty$.

For $f \in L_1 \cap L_\phi$ set $f_\lambda = f\chi_{A(\lambda)}$ and $f^\lambda = f - f_\lambda$ where $A(\lambda) = \{x \in X : |f(x)| > \lambda/2\}$. We have $f_\lambda \in L_1$, $f^\lambda \in L_1 \cap L_\infty$ and therefore

$$(3.2) \quad \int_X \phi(M_P f)d\mu = \int_0^\infty \varphi(\lambda)\mu\{x \in X : M_P f(x) > \lambda\}d\lambda \leq$$

$$\leq 2\int_0^\infty \lambda^{-1}\varphi(\lambda)\Big(\int_X |f_\lambda|d\mu\Big)d\lambda = 2\int_X |f(x)|\Big(\int_0^{2|f(x)|} \lambda^{-1}\varphi(\lambda)d\lambda\Big)d\mu(x),$$

where φ is the density function of ϕ.

Integrating by parts, we obtain

$$(3.3) \qquad \int_0^s \lambda^{-1}\varphi(\lambda)d\lambda = s^{-1}\phi(s) + \int_0^s \lambda^{-2}\phi(\lambda)d\lambda \quad , \quad (s > 0).$$

Since the N-function complementary of ϕ satisfies Δ_2 there exists a constant $\beta > 1$ such that $\beta\phi(s) \leq s\varphi(s)$, $s \geq 0$; then, if $0 < \lambda < 1$ we have that $\lambda^{-2}\phi(\lambda) \leq \phi(1)\lambda^{\beta-2}$ and therefore $\int_{(0,s]} \lambda^{-2}\phi(\lambda)d\lambda < \infty$. Then, (3.3) shows that

$$\int_0^s \lambda^{-1}\varphi(\lambda)d\lambda < \beta(\beta-1)^{-1}s^{-1}\phi(s)\,, \quad (s > 0).$$

Hence, it follows from (3.2) that

$$(3.4) \qquad \int_X \phi(M_P f)d\mu \leq \alpha\beta(\beta-1)^{-1}\int_X \phi(|f|)d\mu \quad (f \in L_1 \cap L_\phi),$$

where α is a constant in the Δ_2-condition for ϕ.

Since $|\hat{T}f| \leq P|f|$ for $f \in L_1$, (3.4) shows that there exists a constant $C_1 > 0$ such that $\int_X \phi(M_T f)d\mu \leq C_1 \int_X \phi(|f|)d\mu$, $f \in L_1 \cap L_\phi$, which proves that $\|M_T f\|_{(\phi)} \leq C\|f\|_{(\phi)}$, $f \in L_1 \cap L_\phi$, where $C = \max(1, C_1)$. Since $L_1 \cap L_\phi$ is a dense linear subspace of $[L_\phi, \|\ \|_{(\phi)}]$ it follows that $\|M_T f\|_{(\phi)} \leq C\|f\|_{(\phi)}$ for every $f \in L_\phi$, which proves a).

Now, let $\{b_k\}$ be a bounded Besicovitch sequence; then a) and the Banach principle show that for almost everywhere convergence it is enough to prove that the weighted averages

$$T_n f = \frac{1}{n}\sum_{k=0}^{n-1} b_k T^k f$$

converges a.e. for all f in a dense subset of $[L_\phi, \|\ \|_{(\phi)}]$.

Let $m \in \mathbb{N}$ and $S : L_\phi \longrightarrow L_\phi$ defined by $Sf = e^{im}Tf$. Since L_ϕ is reflexive and the powers S^k, $k \geq 0$, are uniformly bounded, exactly $\|S^k f\|_{(\phi)} \leq \|f\|_{(\phi)}$ for every $f \in L_\phi$ and $k \geq 0$, then , the Césàro averages $R_n f = n^{-1}(f + Sf + \ldots S^{n-1})$ converge in norm for every $f \in L_\phi$. Therefore L_ϕ is the closure of the direct sum of the set of fixed points of S and the space $(I-S)L_\phi$ (see 2.1 in [6]).

On the other hand, given $\beta > 1$ such that $\beta\phi(s) \leq s\varphi(s)$, $s \geq 0$, the function $s \longrightarrow s^{-\beta}\phi(s)$ increases for $s > 0$ and consequently $\phi(st) \leq s^\beta \phi(t)$ for $0 \leq s \leq 1$ and $t \geq 0$. Therefore, if $g \in L_\phi$ we have

$$\int_X \sum_{n=1}^{\infty} \phi(|n^{-1}S^n g|)d\mu \leq \sum_{n=1}^{\infty} n^{-\beta} \int_X \phi(|S^n g|)d\mu \leq$$

$$\leq \int_X \phi(|g|)d\mu \sum_{n=1}^{\infty} n^{-\beta} < \infty.$$

Hence $n^{-1}S^n g(x) \longrightarrow 0$ a.e. as $n \longrightarrow \infty$ and thus $R_n f \longrightarrow 0$ a.e. if $f = g - Sg$.

Since the maximal operator M_S is bounded in $[L_\phi, \|\ \|_{(\phi)}]$ we obtain that, for any $f \in L_\phi$, $n^{-1}\sum_{k=0}^{n-1} e^{im\,k}T^k f$ converges a.e. and therefore for every trigonometric polynomial α and $f \in L_\phi$ we have that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \alpha(k)T^k f(x)$$

exists and is finite a.e.

Then, for every $f \in L_\phi \cap L_\infty$, $T_n f$ converges a.e. since for every $\varepsilon > 0$ there exists a trigonometric polynomial α_ε such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |b_k - \alpha_\varepsilon(k)| < \varepsilon$$

and consequently

$$\limsup_{n\to\infty} \left|T_n f(x) - \frac{1}{n}\sum_{k=0}^{n-1} \alpha_\varepsilon(k)T^k f(x)\right| < \varepsilon\|f\|_\infty \quad \text{a.e.}$$

In this way, since $L_\phi \cap L_\infty$ is dense in L_ϕ, we conclude that $T_n f$ converges almost everywhere for every $f \in L_\phi$.

Finally, let $f^*(x) = \lim_{n\to\infty} T_n f(x)$. It follows from a) that $f^* \in L_\phi$ and $\phi(|T_n f - f^*|)$ is dominated by $\phi(M_T f) \in L_1$; thus, taking into account the Lebesgue's dominated theorem, we get that $\lim_{n\to\infty} \int_X \phi(|T_n f - f^*|)d\mu = 0$ which proves that $\lim_{n\to\infty} \|T_n f - f^*\|_{(\phi)} = 0$.

Acknowledgments.- This paper contains some of the results of the author's Doctoral Thesis written under the direction of Professor Alberto de la Torre at the University of Málaga. The author is deeply indebted to Professor de la Torre for his generous help.

References

1. M.A. AKCOGLU AND R.V. CHACÓN, A convexity theorem for positive operators, *ZW* **3** (1965), 328–332.
2. B. BRU ET H. HEINICH, Isométries positives et propriétés ergodiques de quelques espaces de Banach, *Ann. Ins. Henri Poincaré* **17** (1981), 377–405.
3. F. CUNNINGHAM AND N. GROSSMAN, On Young's inequality, *Amer. Math. Monthly* **78** (1971), 781–788.
4. A. GARSIA, Topics in almost everywhere convergence, *Lectures in advances Math. Mark. Publ. Co.* (1970).
5. M.A. KRASNOSELSKY AND V.B. RUTITSKY, Convex functions and Orlicz spaces, *Noordhoff Groningen* (1961).
6. U. KRENGEL, Ergodic Theorems, *W. de Gruyter. Studies in Mathematics* (1985).
7. J. MUSIELAK, Orlicz spaces and modular spaces, "*Springer-Verlag,*" 1983.
8. J.H. OLSEN, The individual weighted ergodic theorem for bounded Besicovitch sequences, *Canad. Math. Bull.* **25** (1982), 468–471.
9. R. SATO, An extrapolation theorem for contractions with fixed points, *Canad. Math. Bull.* **24** (1981), 199–203.
10. A. DE LA TORRE, A dominated ergodic theorem for contractions with fixed points, *Canad. Math. Bull.* **20** (1977), 89–91.

Departamento de Matemáticas
Facultad de Ciencias
Universidad de Málaga
SPAIN

Rebut el 8 d'octubre de 1987