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ON THE SMOOTHNESS OF LEVI–FOLIATIONS

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Abstract

We study the regularity of the induced foliation of a Levi–flat hypersurface in $\mathbb{C}^n$, showing that the foliation is as many times continuously differentiable as the hypersurface itself. The key step in the proof given here is the construction of a certain family of approximate plurisubharmonic defining functions for the hypersurface in question.

1. Introduction

Let S be a real hypersurface in $\mathbb{C}^n$ of class $C^k (k \geq 2)$ with vanishing Levi–form. The maximal complex subspace $TS \cap JTS$ is an integrable distribution of class C^{k-1} and codimension one on S, so the Frobenius theorem [**5**] guarantees that S admits a (unique) foliation of class C^{k-1} by complex hypersurfaces. The goal of this paper is to establish the following result.

Theorem. *The induced foliation of a C^k Levi–flat hypersurface is actually of class C^k.*

For $k = 1$ the Levi–form is not defined, but our proof will nevertheless show that if a real C^1 hypersurface admits a continuous foliation by complex hypersurfaces then that foliation is of class C^1. In fact, the existence of the Levi–foliation in the C^1 case was proved by Shcherbina for $n = 2$[**6**] and by Airepetian for general n [**1**]; moreover, Airepetian's paper also establishes the C^1–smoothness of the foliation. It appears that his technique (which uses both the Bishop disc method and the Frobenius theorem) could also be used to prove our result.

Our proof is based on the construction of a family of approximate pluriharmonic defining functions for S. (See the Proposition below). The existence of such functions is also useful in the study of holonomy of Levi–foliations (see [**2**]).

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1. Proof of Theorem

We work locally near some point in S, which we may take to be the origin, and we choose coordinates $z = (z', z_n)$, $z_n = x_n + iy_n$ so that $T_0S = \{(z', z_n) : y_n = 0\}$. Then we may write $y_n = r(z', x_n)$ on S, where r is of class C^k. Let $\psi(t, z')$ by defined for (t, z') near $(0, 0')$ by the condition that $\Psi(t, z')= = (z', \psi(t, z'))$ lies on the leaf L_t passing through $(0', t + ir(0', t))$. Then ψ is of class C^{k-1} in (t, z') and holomorphic in z'. Our goal is to show that ψ is actually of class C^k in (t, z'); it is easy to see that all partial derivatives involving fewer than k differentiations with respect to t exist and are continuous, so it will suffice to establish the existence and continuity of $(\partial/\partial t)^k \psi$.

The major step in proving the Theorem is the approximation of S along each leaf L_t by the zero set of a pluriharmonic function, as specified in the following Proposition.

Proposition. *There is a function $h(t, z)$ defined in a neighborhood U of $(0, 0)$ such that the following conditions hold for $(t, z) \in U$:*

(i) h is continuous in (t, z) and holomorphic in z,

(ii) $h(t, z) = 0$ for $z \in L_t$,

(iii) $\partial h/\partial z_n \neq 0$, and

(iv) Im $h = o(|h|^k)$ for $z \in S$ as $z \to L_t$, uniformly in (t, z).

Proof of Proposition:

We will construct a sequence of functions $h_0, h_1, \ldots, h_k = h$ defined in a neighborhood U_j of $(0, 0)$ such that the following conditions hold for $(t, z) \in U_j$:

$(i)_j$ h_j is continuous in (t, z) and holomorphic in z,

$(ii)_j$ $h_j(t, z) = 0$ for $z \in L_t$,

$(iii)_j$ $\partial h_j/\partial z_n \neq 0$, and

$(iv)_j$ Im $h_j = o(|h_j|^j)$ for $z \in S$ as $z \to L_t$, uniformly in (t, z).

We may take $h_0(t, z) = z_n - \psi(t, z')$.

Let $T = (\partial/\partial x_n) + (\partial r/\partial x_n)(\partial/\partial y_n)$; T is a vector field of class C^{k-1} tangent to S and transverse to the Levi-foliation near 0. Let

$$\begin{aligned}\Theta_1(t, z') &= \arg\, (Th_0)(\Psi(t, z')) \\ &= \arg\, (1 + i(\partial r/\partial x_n)(\Psi(t, z')).\end{aligned}$$

Then Θ_1 is continuous in (t, z'), and the function Im $e^{-i\Theta_1(t,z')}h_0(t, z)$ vanishes along L_t as does its derivative with respect to the vector field T. Thus

$$\text{Im } e^{-i\Theta_1(t,z')}h_0(t, z) = o(|h_0(t, z)|)$$

for $z \in S$ as $z \to L_t$, uniformly on a neighborhood of $(0, 0)$. We have $\Theta_1(0, 0') = = 0$ so that working on a smaller neighborhood we may assume that $|\Theta_1| < < \pi/4$. ■

Claim. $\Theta_1(t, z')$ *is pluriharmonic in* z'.

Proof of Claim: It will simplify notation to suppress the parameter t temporarily. Also, it will be useful to perform the change of coordinates

$$\zeta' = z'$$
$$\zeta_n = \tilde{x}_n + i\tilde{y}_n = h_0(z).$$

In the ζ–coordinates S is defined by an equation of the form

$$\tilde{y}_n = \tilde{r}(\zeta', \tilde{x}_n) = \tan \Theta_1(\zeta') \cdot \tilde{x}_n + o(|\tilde{x}_n|).$$

To show that $\Theta_1(\zeta')$ is pluriharmonic it suffices to show that for every complex-linear disc Δ near $0'$ in $\mathbf{C}^{n-1}$ and for every f continuous on $\bar{\Delta}$ and holomorphic on Δ with $\operatorname{Re} f = \Theta_1$ on $\partial\Delta$ we have $\operatorname{Re} f(\zeta_0') = \Theta_1(\zeta_0')$, where ζ_0' is the center of Δ.

Consider the two–parameter family of discs

$$\Gamma_{\varepsilon,\lambda}(\zeta') = (\zeta', \varepsilon\{e^{if(\zeta')} + i\lambda\}),\ \zeta' \in \Delta.$$

We have

$$\tilde{y}_n - \tilde{r}(\zeta', \tilde{x}_n) = \varepsilon\{\lambda + e^{-\operatorname{Im} f(\zeta')} \cdot \cos \operatorname{Re} f(\zeta') \cdot (\tan \operatorname{Re} f(\zeta') - \\ - \tan \Theta_1(\zeta'))\} + o(|\varepsilon|)$$

for $\zeta = \Gamma_{\varepsilon,\lambda}(\zeta')$, $\zeta' \in \Delta$.

Suppose that $\Theta_1(\zeta_0') > \operatorname{Re} f(\zeta_0')$. Pick a and b so that

$$0 < a < e^{-\operatorname{Im} f(\zeta_0')} \cdot \cos \operatorname{Re} f(\zeta_0') \cdot (\tan \Theta_1(\zeta_0') - \tan \operatorname{Re} f(\zeta_0'))$$

and

$$b > 2 \max_{\zeta' \in \bar{\Delta}} e^{-\operatorname{Im} f(\zeta')}.$$

Then for sufficiently small $\varepsilon > 0$ we have

$$\tilde{y}_n < \tilde{r}(\zeta', \tilde{x}_n) \text{ when } \zeta = \Gamma_{\varepsilon,a}(\zeta_0'),$$
$$\tilde{y}_n > \tilde{r}(\zeta', \tilde{x}_n) \text{ when } \zeta = \Gamma_{\varepsilon,\lambda}(\zeta'),\ \zeta' \in \partial\Delta,\ a \le \lambda \le b,$$
$$\tilde{y}_n > \tilde{r}(\zeta', \tilde{x}_n) \text{ when } \zeta = \Gamma_{\varepsilon,b}(\zeta'),\ \zeta' \in \bar{\Delta}.$$

But this violates the disc theorem [4, p. 53], since S is clearly pseudoconvex from both sides.

The case $\Theta_1(\zeta_0') > \operatorname{Re} f(\zeta_0')$ is similar. ■

Now choose $f_1(t, z')$ continuous in (t, z') and holomorphic in z' with $\Theta_1 = \operatorname{Re} f_1$. Let $h_1(t, z) = e^{if_1(t,z')}h_0(t, z)$. Then h_1 satisfies $(i)_1$, $(ii)_1$, $(iii)_1$, and $(iv)_1$ on a suitable neighborhood U_1 of $(0, 0)$.

Let us assume that h_{j-1} has been constructed and proceed to construct h_j. Let

$$\Theta_j(t,z') = \frac{(T^j \text{ Im } h_{j-1})(t,z)}{j!((T \text{ Re } h_{j-1})(t,z))^j} \Big|_{z=\Psi(t,z')}.$$

(Condition $(iii)_{j-1}$ shows that the denominator doesn't vanish at $(0,0')$). Then Θ_j is continuous in (t,z') (recall that z– derivatives of h_{j-1} come for free), and the function

$$\text{Im } h_{j-1}(t,z) - \Theta_j(t,z') \cdot (\text{ Re } h_{j-1}(t,z))^j$$

vanishes along L_t along with its derivatives of order $\leq j$ with respect to the vector field T. Hence

$$\text{Im } h_{j-1}(t,z) - \Theta_j(t,z') \cdot (\text{ Re } h_{j-1}(t,z))^j = o(|h_{j-1}(t,z)|^j)$$

for $z \in S$ as $z \to L_t$, uniformly on a neighborhood of $(0,0)$.

Claim. *$\Theta_j(t,z')$ is pluriharmonic in z'.*

Proof of Claim: Again we suppress t temporarily and perform a change of coordinates

$$\zeta' = z'$$
$$\zeta_n = \tilde{x}_n + i\tilde{y}_n = h_{j-1}(z)$$

Thus S is defined by an equation of the form

$$\tilde{y}_n = \Theta_j(\zeta') \cdot \tilde{x}_n^j + o(|\tilde{x}_n|^j)$$

As before, it suffices to show that for every complex–linear disc Δ near $0'$ in $\mathbf{C}^{n-1}$ and for every f continuous on $\bar{\Delta}$ and holomorphic on Δ with Re $f = \Theta_j$ on $\partial\Delta$ we have Re $f(\zeta_0') = \Theta_j(\zeta_0')$, where ζ_0' is the center of Δ.

Again we consider a two–parameter family of discs

$$\Gamma_{\varepsilon,\lambda}(\zeta') = (\zeta', \varepsilon + i\varepsilon^j\{\lambda + f(\zeta')\}),\ \zeta' \in \Delta.$$

We have

$$\tilde{y}_n - \tilde{r}(\zeta', \tilde{x}_n) = \varepsilon^j\{\lambda + \text{ Re } f(\zeta') - \Theta_j(\zeta')\} + o(|\varepsilon|^j)$$

for $\zeta = \Gamma_{\varepsilon,\lambda}(\zeta')$, $\zeta' \in \Delta$.

Suppose that $\Theta_j(\zeta_0') >$ Re $f(\zeta_0')$. Pick a and b so that

$$0 < a < \Theta_j(\zeta_0') - \text{ Re } f(\zeta_0')$$

and

$$b > 2 \max_{\zeta' \in \bar{\Delta}}\{\Theta_j(\zeta') - \text{ Re } f(\zeta')\}$$

Then for sufficiently small $\varepsilon > 0$ we have

$$\tilde{y}_n < \tilde{r}(\zeta', \tilde{x}_n) \text{ when } \zeta = \Gamma_{\varepsilon,a}(\zeta_0'),$$
$$\tilde{y}_n > \tilde{r}(\zeta', \tilde{x}_n) \text{ when } \zeta = \Gamma_{\varepsilon,\lambda}(\zeta'),\ \zeta' \in \partial\Delta,\ a \le \lambda \le b,$$
$$\tilde{y}_n > \tilde{r}(\zeta', \tilde{x}_n) \text{ when } \zeta = \Gamma_{\varepsilon,b}(\zeta'),\ \zeta' \in \bar{\Delta}.$$

But this violates the disc theorem as before.

The case $\Theta_1(\zeta_0') <$ Re $f(\zeta_0')$ is again similar. ■

Again choose $f_j(t, z')$ continuous in (t, z') and holomorphic in z' with $\Theta_j =$ Re f_j. Let $h_j(t,z) = h_{j-1}(t,z) - i f_j(t,z')(h_{j-1}(t,z))^j$. Then h_j satisfies $(i)_j$, $(ii)_j$, $(iii)_j$, and $(iv)_j$ on a suitable neighborhood U_j of $(0,0)$.

The proposition is proved, by induction. ■

Remarks.

1) In the case $k = \infty$ it need not be the case that S can be approximated to infinite order along a given leaf by the zero set of a pluriharmonic function. For $n = 1$, for example, a C^∞ curve need not be approximable to infinite order at a given point by a real–analytic curve.

2) For $j \le k - 2$ the claims in the above proof can be proved by a straightforward Levi–form computation.

3) One can avoid explicit mention of pseudoconvexity in the above proof by observing that the winding number of the boundary of a holomorphic disc around a given leaf cannot jump under small perturbations.

4) The functions h_j can actually be chosen to be of class C^{k-j} in (t, z).

5) If $\tilde{h}(z)$ is a holomorphic function vanishing on L_0 with Im $\tilde{h}(z) = o(|\tilde{h}(z)|^k)$ on some neighborhood of 0 in S then

$$\tilde{h}(z) = P(h(0,z)) + \beta(z) \cdot (h(0,z))^{k+1},$$

where P is a polynomial of degree k with real coefficients and β is holomorphic. Indeed, we may write

$$\tilde{h}(z) = \sum_{j=0}^{k} \alpha_j(z') \cdot (h(0,z))^j + \beta(z) \cdot (h(0,z))^{k+1},$$

where β and the α_j are holomorphic. Thus

$$\sum_{j=0}^{k} (\text{ Im } \alpha_j(z'))(\text{ Re } h(0,z))^j = \text{ Im } \tilde{h}(z) + o(|h(0,z)|^k = o(|\text{ Re } h(0,z)|^k)$$

on S, forcing Im $\alpha_j \equiv 0$ for $0 \le j \le k$, so that each α_j is a real constant.

6) If S is a real hypersurface of class C^k which is pseudoconvex from one side and which contains a complex hypersurface then the functions h_j can be

constructed for $j <$ some even integer j_0; the corresponding function Θ_{j_0} will be sub–or superharmonic. (The pluriharmonicity of Θ_1 has been used in several papers, for example in [**3**, p. 290].)

To prove the Theorem we first note that Re h has constant sign on each leaf so that by Harnack's inequality we have

$$\text{Re } h(t_0, \Psi(t, z')) = 0(|\text{ Re } h(t_0, \Psi(t, 0'))|)$$

for t_0, t, z' close enough to zero. But (iv) implies that Re h and h are comparable so it follows that

$$h(t_0, \Psi(t, z')) = 0(|h(t_0, \Psi(t, 0'))|)$$

and so

$$\text{Im } h(t_0, \Psi(t, z')) = o(|h(t_0, \Psi(t, 0'))|^k).$$

Thus from bounds for pluriharmonic conjugates we have

$$h(t_0, \Psi(t, z')) - h(t_0, \Psi(t, 0')) = o(|h(t_0, \Psi(t, 0'))|^k) = o(|t - t_0|^k)$$

after shrinking the domain of z'.

By (iii) and the inverse function theorem we may write

$$z_n = \Phi(t, z', h(t, z)),$$

where Φ is continuous in (t, w) and holomorphic in w. Thus

$$\psi(t, z') = \Phi(t_0, z', h(t_0, \Psi(t, z'))) = \Phi(t_0, z', h(t_0, \Psi(t, z'))) + o(|t - t_0|^k).$$

Now the main term of this last expression is C^k with respect to t, so that the following Lemma will establish the existence and continuity of $(\partial/\partial t)^k \psi(t, z')$ by showing that

$$(\partial/\partial t)^k \psi(t, z') = (\partial/\partial t)^k \Phi(t_0, z', h(t_0, \psi(t, 0')))\,|_{t_0=t}.$$

Lemma. *Let f be a C^{k-1} function on an interval $I \subset \mathbf{R}$. Suppose that there is a function g on $I \times I$ such that*

(i) $(\partial/\partial t)^j g(s, t)$ exists and is continuous on $I \times I$ for $0 \le j \le k$, and

(ii) $f(t) = g(s, t) + o(|t - s|^k)$ uniformly on $I \times I$.

Then

$$f^{(k)}(t) = (\partial/\partial t)^k g(t, s)\,|_{s=t}.$$

Proof of Lemma: It is clear from the hypotheses that

$$f^{(j)}(t) = (\partial/\partial t)^j g(s, t)\,|_{s=t}$$

for $0 \le j \le k - 1$. Let $P(s, t)$ denote the $(k-1)^{st}$ Taylor polynomial for f at s, and let

$$\alpha(t) = (\partial/\partial t)^k g(s, t)\,|_{s=t}.$$

Then applying Taylor's theorem to g in (ii) we have

$$(*) \qquad f(t) = P(s,t) + \alpha(t)(t-s)^k/k! + o(|t-s|^k)$$

uniformly on compact subsets of $I \times I$. Let ∇_h denote the difference operator

$$\nabla_h \varphi(t) = \{\varphi(t+h) - \varphi(t)\}/h.$$

Applying $(\nabla_h)k - 1$ to both sides of (*) and taking $s = t$ we get

$$(\nabla_h)^{k-1} f(t) = f^{(k-1)}(t) + c_k \alpha(t) h + o(h),$$

uniformly on compact subsets of I, where

$$c_k = h^{-1}(\nabla_h)^{k-1}(t-s)^k \,|_{s=t} = (k-1)(k!)/2.$$

Thus

$$
\begin{aligned}
(1) \qquad f^{(k)}(t) &= \lim_{h \to 0} \nabla_h f^{(k-1)}(t) \\
&= \lim_{h \to 0} (\nabla_h)^k f(t) - c_k\{\alpha(t+h) - \alpha(t)\} + o \\
&= \lim_{h \to 0} (\nabla_h)^k f(t) \\
&= \lim_{h \to 0} \alpha(t) + o(1) \qquad \text{(by (*) again)} \\
&= \alpha(t). \blacksquare
\end{aligned}
$$

This completes the proof of the Theorem.

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