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A REPRESENTATION THEOREM FOR CERTAIN BOOLEAN LATTICES

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Abstract

Let R be an associative ring with 1 and R–tors the complete brouwerian lattice of all hereditary torsion theories on the category of left R–modules. A well known result asserts that R is a left semiartinian ring iff R–tors is a complete atomic boolean lattice. In this note, we prove that if $\mathfrak{L}$ is a complete atomic boolean lattice then, there exists a left semiartinian ring R such that $\mathfrak{L}$ is lattice–isomorphic to R–tors.

1. Background. Throughout the following, R will denote an associative ring with unit element and R–tors will denote the complete brouwerian lattice of all hereditary torsion theories on the category R–mod of unitary left R–modules. Notation and terminology concerning R–tors will follow [**1**].

If $M \in R$–mod, let us denote by $\xi(M)$ the smallest element of R–tors respect to which M is torsion; the unique minimal element of R–tors is then $\xi = \xi(0)$. We will denote by R–simp a complete set of representatives of isomorphism classes of simple left R–modules.

A lattice $\mathfrak{L}$ is said to be *complete* if arbitrary meets and joins are defined in $\mathfrak{L}$. In particular, in a complete lattice there are a minimal element and a maximal element, which will be denoted by ξ and χ respectively. An element a of a complete lattice $\mathfrak{L}$ is an *atom* if a$\neq \xi$ and $b < a$ implies $b = \xi$; we will denote by $\mathfrak{A}(\mathfrak{L})$, the set of atoms of $\mathfrak{L}$. $\mathfrak{L}$ is said to be *atomic* if for every $x \in \mathfrak{L}$, there exists $a \in \mathfrak{A}(\mathfrak{L})$ such that $a \leq x$. $\mathfrak{L}$ is *locally atomic* if every element of $\mathfrak{L}$ is a join of atoms. A complemented distributive lattice is called a *boolean lattice*; in a boolean lattice, complements are unique.

For any ring R, the atoms of R–tors are the torsion theories of the form $\xi(S)$ with $S \in R$–simp. As a result, R–tors is an atomic lattice and R is left semiartinian iff R–tors is a complete atomic boolean lattice or iff R–tors is locally atomic. See III of [**1**] for details.

2. Complete Atomic Boolean Lattices. In [**2**], professor Golan asks if for every boolean lattice $\mathfrak{L}$, there exists a left semiartinian ring R such that $\mathfrak{L}$ is lattice– isomorphic to R–tors. Inasmuch as for every ring R, R–tors is complete atomic lattice, in this note, we will consider complete atomic boolean lattices.

Lemma 1. *If $\mathfrak{L}$ is a complete atomic boolean lattice, then*

a) $\mathfrak{L}$ is locally atomic

b) The decomposition of every element $x \in \mathfrak{L}$, $x \neq \xi$ as join of atoms is unique.

Proof: The proof follows easily by using proposition 4.5 of Ch. III of [3]. ■

Corollary 2. *Let $\mathfrak{L}$ and $\mathfrak{L}'$ be complete atomic boolean lattices; if $\varphi : \mathfrak{A}(\mathfrak{L}) \to \mathfrak{A}(\mathfrak{L}')$ is a bijection, then φ can be extend to a lattice isomorphism $\overline{\varphi} : \mathfrak{L} \to \mathfrak{L}'$.*

Theorem 3. *Let $\mathfrak{L}$ be a complete atomic boolean lattice; then there exists a left semiartinian ring R such that $\mathfrak{L}$ is lattice-isomorphic to R-tors.*

Proof: Let $\mathfrak{L}$ be a complete atomic boolean lattice; by Corollary 2, it is enough to show a left semiartinian ring R and a bijective correspondence from $\mathfrak{A}(\mathfrak{L})$ to R-simp, then we will have that $\mathfrak{L}$ is isomorphic to R-tors.

Case 1: $\mathfrak{A}(\mathfrak{L})$ is a finite set.

Let us supose $|\mathfrak{A}(\mathfrak{L})| = n < \infty$ and let $R = \mathbb{Z}_2^n$ be the ring direct product of n copies of $\mathbb{Z}_2$, the integers modulo 2. Then $|R\text{-simp}| = n$ and inasmuch as R is semiartinian ring, we are done.

Case 2: $\mathfrak{A}(\mathfrak{L})$ is an infinite set.

Let us denote $|\mathfrak{A}(\mathfrak{L})| = \alpha$ and let $A = \mathbb{Z}_2^\alpha$ the ring direct product of α copies of $\mathbb{Z}_2$. Now, let R be the subring of A generated by the sum $I = \mathbb{Z}_2^{(\alpha)}$ and the unit element of A. In R, the projective simple modules are, exept by isomorphisms, the simple direct sumands in the decomposition $I = \mathbb{Z}_2^{(\alpha)}$. On the other hand, the singular simple modules are all isomorphic to R/I. Then $|R\text{-simp}| = \alpha$ and so, there exists a bijection between $\mathfrak{A}(\mathfrak{L})$ and R-simp.

Finally, it is not difficult to prove that R is a semiartinian ring, and so we complete the proof.

References

1. GOLAN J., "*Localization of noncommutative rings,*" Marcel Dekker, 1975.

2. GOLAN J., "*Thirty open problems concerning torsion theories,*" Secretariado de publicaciones y departamento de algebra y fundamentos, Universidad de Murcia,1986.

3. STENSTROM BO., "*Rings of Quotients*," New york– Berlin, Springer Verlag, 1975.

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