FERRODING ROSENTHAL OPERATORS

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Abstract

In this paper we show that a Rosenthal operator factors through a Banach space containing no isomorphs of $I_1$.

All spaces are to be Banach spaces. If $X$ is a Banach space, $X''$ denotes the bidual of $X$, $J_X$ is the canonical isometry embedding $X$ in $X''$, $B_X$ is the closed unit ball of $X$ and $I_X$ denotes the identity operator on $X$. Let $L(X,Y)$ be the set of all continuous linear operators from $X$ into $Y$.

Recall that an operator $T \in L(X,Y)$ is said to be completely continuous if every weakly convergent sequence $(x_n)$ is mapped into a norm convergent sequence $(Tx_n)$. The class of all completely continuous operators from $X$ into $Y$ is denoted by $Cc(X,Y)$ and $Co(X,Y)$ will denote the space of all compact operators from $X$ into $Y$.

Let $T \in L(X,Y)$. The $T$ is called a Rosenthal operator if $ST \in Co(X,Y_0)$ for all $S \in Cc(Y,Y_0)$, where $Y_0$ is an arbitrary Banach space. Using a theorem which is due to Rosenthal [8] and Dor [2] in the real and complex case, respectively, one gets that $T$ is a Rosenthal operator if and only if it maps bounded sequences into sequences possessing weak Cauchy subsequences (see for example [6]), and hence $T$ is a Rosenthal operator if and only if $T(B_X)$ is $\sigma(Y'',Y')$-relatively sequentially compact in $Y''$.

The Rosenthal operators are called weak Cauchy operators by [5] and conditionally weakly convergent operators by [7].

An operator $T \in L(X,Y)$ is called a $\ell_1$-singular operator if for each $S \in L(\ell_1,X)$, the composition $TS$ is not an isomorphic embedding [5]. So, $T \in L(X,Y)$ is a Rosenthal operator if and only if it is a $\ell_1$-singular operator. $Ro(X,Y)$ will denote the class of all Rosenthal operators from $X$ into $Y$.

Let $N\ell_1$ be the space ideal of all Banach spaces containing no isomorphic copy of $\ell_1$ and let $Op(N\ell_1)$ be the operator ideal of all operators which factor through spaces in $N\ell_1$.

In this note we shall prove that $Op(N\ell_1)$ coincides with the class of all Rosenthal operators. For this end we shall use the construction of Davies-Figiel-Johnson-Pelczynski [1]. Let $W$ be a convex, symmetric and bounded
subset of a Banach space $X$. For $n = 1, 2, 3, \ldots$, the Minkowski functional $\| \cdot \|_n$ of the set $U_n = 2^n W + 2^{-n} B_X$ is a norm equivalent to $\| \cdot \|$. Define, for $x \in X$, $\| x \|_n = \left( \sum_{k=1}^{\infty} \| x \|_k^2 \right)^{\frac{1}{2}}$; let $Z = \{ x \in X : \| x \|_n < \infty \}$ and let $j$ denote the identity embedding of $Z$ into $X$; then $(Z, \| \cdot \|_n)$ is a Banach space and $j$ is continuous [1, lemma 1, (ii)].

**Theorem.** Let $X, Y$ be Banach spaces and let $T \in L(X,Y)$. Then the following properties are equivalent:

(i) $T \in Ro(X,Y)$

(ii) $T \in Op(N\ell_1)(X,Y)$.

**Proof:** (i) $\Rightarrow$ (ii). Suppose that $T \in Ro(X,Y)$. With reference to [1], put $W = T(B_X)$, $M = \{ y \in Y : \| y \|_n < \infty \}$ and let $j$ denote the identity embedding of $M$ into $Y$. Let $(y_n)$ be a sequence of elements of $W$, then $(y_n)$ has a subsequence $(y_{n_j})$ such that $(J_Y y_{n_j})$ is $(Y'', Y')$-convergent in $Y''$. Hence $J_Y W$ is $\sigma(Y'', Y')$-sequentially compact in $Y''$ and so by virtue of [1, lemma 1, (xii)], $J_M B_M$ is $\sigma(M'', M')$-sequentially compact in $M''$. This implies that $M \in N\ell_1$.

The operators $j^{-1}T : X \to M$ and $j : M \to Y$ provide the required factorization.

(ii) $\Rightarrow$ (i). It is trivial. ■

**Remark 1.** In a 1980 paper [4], S. Heinrich showed that if $T \in Ro(X,Y)$ then $T \in Op(N\ell_1)(X,Y)$ proving that the operator ideal of all Rosenthal operators is injective and surjective and satisfy the $\Sigma_p$-condition for $1 < p < \infty$. That is, for arbitrary Banach spaces $X_n, Y_n, n \in N$, the followings holds:

If $T \in T((\Sigma X_n)_p, (\Sigma Y_n)_p)$, and $Q_m T P_m \in Ro(X_m, Y_n), n, m \in N$ then $T \in Ro((\Sigma X_n), (\Sigma Y_n))$ where $P_m$ and $Q_n$ denote the projections of $(\Sigma X_n)_p, (\Sigma Y_n)_p$ onto the coordinates $X_m$ and $Y_n$, respectively.

**Remark 2.** Other characterizations of the Rosenthal operators were obtained by A. Fakhoury [3].

**References**


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