Publicacions Matemàtiques, Vol **32** (1988), 61–78.

EVALUATING A p-TH ORDER COHOMOLOGY OPERATION

J. HARPER AND A. ZABRODSKY

Abstract

A certain $p-th$ order cup product is detected by a $p-th$ order cohomology operation. The result is applied to finite H-spaces, to show that several properties of compact Lie groups do not hold for arbitrary torsion free finite H-spaces.

1. Summary of results

In this paper we give the detailed computations of results announced in [**3**]. As the computations involve many technical details we try to present the main line of proof separately in Chapter 2–filling in the missing details in Chapters 3–7. In Chapter 8 we bring some applications.

Let p be a prime. Throughout this paper we write $H^*(X) = H^*(X, \mathbf{Z}/p\mathbf{Z})$. Let ϕ_n denote an $n+1$ order mod p cohomology operation associated with the Toda brackets $\langle \overbrace{\mathcal{P}^a, \ldots, \mathcal{P}^{p-1}, \mathcal{P}^1}^{n+1 \text{ times}} \rangle$ ($a = 1$ if n is even $a = p-1$ if n is odd). By this we mean a stable cohomology operation universally defined on a spectra

Partially supported by grants of the U.S.-Israel Binational Science Foundation and the U.S. National Science Foundation.

The editor regrets very much to inform that Professor Zabrodsky died on November 86.
The first author dedicates this paper to his memory.

E_n with a Postnikov system given by:

$$
\text{(D1)} \qquad
\begin{array}{ccccc}
\Omega K_n & \xrightarrow{j_n} & E_n & \xrightarrow{k_n} & K_{n+1} \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
\Omega K_r & \xrightarrow{j_r} & E_r & \xrightarrow{k_r} & K_{r+1} \\
 & & \big\downarrow h_r & & \\
 & & E_{r-1} & & \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
\Omega K_1 & \xrightarrow{j_1} & E_1 & \xrightarrow{k_1} & K_2 \\
 & & \big\downarrow h_1 & & \\
 & & E_0 = K_0 & \xrightarrow{\mathcal{P}^1 = k_0} & K_1
\end{array}
$$

with the properties:

1.0.

(1) $K_0 = K(\mathbf{Z}/p\mathbf{Z}, 0)$ -the Eilenberg MacLane Spectrum.

(2) for $r \geq 1$, $K_r = \sum^{t_r} K_0$, $t_r = \begin{cases} spq + q - 2s & r = 2s+1 \\ spq - 2s + 1 & 0 < r = 2s \end{cases}$
$q = 2p - 2$.

(3) $\Omega K_r \xrightarrow{j_r} E_r \xrightarrow{h_r} E_{r-1}$ and for $0 \leq r < n$ $E_{r+1} \xrightarrow{h_{r+1}} E_r \xrightarrow{K_r} K_{r+1}$ are (co) fibrations.

(4) $[k_r \circ j_r] \in [\Omega K_r, K_{r+1}] = \mathcal{P}^{a_r} \in \underline{A}(p)$ where

$$a_r = \begin{cases} 1 & \text{for } r \text{ - even} \\ p-1 & \text{for } r \text{ - odd} \end{cases}$$

ϕ_n is then given by the universal example (in the sense of [**1**]) $\langle x = H^0(h_1 \circ \ldots \circ h_n)\imath$, E_n , $u = H^{t_{n+1}}(k_{n+1})(\Sigma^{t_{n+1}}\imath)\rangle$.

1.1. Remark. We shall use the notations of D1 to describe a non stable (say $m-1$ connected) representative of ϕ_n. In that case the maps in (3) are fibrations, $K_0 = K(\mathbf{Z}/p\mathbf{Z}, m)$, $K_r = K(\mathbf{Z}/p\mathbf{Z}, m + t_r)$.

The main purpose of this paper is to show:

The Main Theorem.

(1) ϕ_{p-1} exists and if $n \not\equiv (-1)$ mod p then $\mathcal{P}^n \imath_{2n+1} \in H^{2np+1}(K(\mathbf{Z}/p\mathbf{Z}, 2n+1))$ is in the domain of ϕ_{p-1} and $\imath_{2n+1} \cdot \mathcal{P}^1 \imath_{2n+1} \cdot \mathcal{P}^2 \imath_{2n+1} \cdot \ldots \cdot \mathcal{P}^{p-1} \imath_{2n+1} \in \phi_{p-1}(\mathcal{P}^n \imath_{2n+1})$.

(2) $0 \notin \phi_{p-1}(\mathcal{P}^n \imath_{2n+1})$.

The principal applications of the main theorem (which generalizes results in [**11**]) is the construction of counterexamples that the following properties shared by Lie groups, are not valid in general for finite H-spaces:

1.2. Examples.
1.2.1. Let X be a compact connected Lie group. If $H^*(X) \approx \wedge(x_{2m_1+1}, \ldots, x_{2m_r+1})$, $m_1 \leq m_2 \leq \cdots \leq m_r$ then there exists a map $f: X \to S^{2m_r+1}$ with $x_{2m_r+1} \in \operatorname{Im} H^*(f)$.
1.2.2. It was recently proved by H. Miller ([**4**]) that for $X = O(n), U(n), Sp(n)$ (and consequently X may belong to a larger family of Lie groups) X splits stably according to the (integral) algebraic weight of its generators, i.e. if $H^*(X)$ is as above $\Sigma^\infty X \approx Y_1 \vee Y_2 \vee \cdots \vee Y_r$ where $H^*(Y_j)$ contains all the monomials $x_{2m_{t_1}+1} \cdot x_{2m_{t_2}+1} \cdot \ldots \cdot x_{2m_{t_j}+1}$ in $H^*(X)$.
1.2.3. It is also well known (e.g: see [**2**, theorem 1.1] with $f = \Sigma T$, T - the λ-th power map) that for any H-space X with $H^*(X)$ as above $\Sigma X \approx Y_1 \vee \vee Y_2 \vee \cdots \vee Y_{p-1}$ where $H^*(Y_j)$ contains all the monomials in the x_{2m_i+1}'s of length j mod $p-1$ (these will be called elements of mod $p-1$ algebraic weight j).

Thus, the following question naturally arises:

1.3. (See e.g: [**9**, problem 14]) Which of these Lie groups properties are shared by finite H-spaces in general?
1.3.1. If X is a mod p H-space with

$$H^*(X) = \wedge(x_{2m_1+1}, x_{2m_2+1}, \ldots, x_{2m_r+1})$$

does there exist a map $f: X \longrightarrow S^{2m_r+1}$ with $x_{2m_r+1} \in \operatorname{Im} H^*(f)$?
1.3.2. Is there a mod p stable splitting of a mod p finite H- space according to the (integral) algebraic weight of its cohomology?
1.3.3. (The weakest common property): Let X be as in 1.3.2. Is there a stable map $\hat{f}: X \longrightarrow \Omega^\infty \Sigma^\infty S^{2m_r+1}$ with $x_{2m_r+1} \in \operatorname{Im} H^*(\hat{f})$?.

All these questions are negatively answered by the following:

1.4. Example. (See [**3**]): There exists a mod p H-space ($p \geq 5$) X with $H^*(X) = \wedge(x_{2p+1}, \mathcal{P}^1 x_{2p+1}, \ldots, \mathcal{P}^p x_{2p+1})$. It follows from the main theorem that there is no stable map $f: X \longrightarrow \Omega^\infty \Sigma^\infty S^{2p^2+1}$ with $\mathcal{P}^p x_{2p+1} \in \operatorname{Im} H^*(f)$. (See Corollary 8.6)

A consequence of the above example (and similar ∞ dimensional examples) is that for an arbitrary stable splitting of $K(\mathbf{Z}/p\mathbf{Z}, 2n+1)$ ($n \not\equiv -1(p)$) any summand containing $\mathcal{P}^n \imath_{2n+1}$ in its cohomology will have to include the product $\imath_{2n+1} \cdot \mathcal{P}^1 \imath_{2n+1} \cdot \ldots \cdot \mathcal{P}^{p-1} \imath_{2n+1}$.

Finally one can deduce a non realizability theorem:

1.5. Proposition. *(See Corollary 8.5) If $n \not\equiv -1 \mod p$ then there is no* $\bmod p$ *H-space (or even a λ-power space) X with*

$$H^*(X) = \wedge(x_{2n+1}, \mathcal{P}^1 x_{2n+1}, \dots, \mathcal{P}^{p-1} x_{2n+1}).$$

If n is large enough there is no space with the above cohomology. (Note that $\wedge(x_{2p-1}, \mathcal{P}^1 x_{2p-1}, \dots, \mathcal{P}^{p-1} x_{2p-1})$ is the $\bmod p$ *cohomology of one of Nishida's factors of $SU(p^2-p+1)$ ([7]). Hence $n \not\equiv -1$ (p) is essential).*

2. Outline of proof

Evaluating a p–th order cohomology operation cannot be expected to be an easy task. Generally it involves many technical details. Still, the core of the computations has a definite straight forward line of thought. In this section we shall try to outline the proof of our main results referring the reader to the auxiliary technical computations in the following sections. The first observation (3.1) is that ϕ_{p-1} exists. (This is obvious for $p=2$ thus we restrict ourselves to p–odd). The reason is quite simple: Assuming ϕ_{r-1} exists, the obstruction for the existence of ϕ_r is a class in the cohomology of E_{r-1}. It turns out that for small r the cohomology of E_{r-1} in the appropriate dimension is zero.

The fact that $\mathcal{P}^n \iota_{2n+1}$ is in the domain of ϕ_{p-1} is embedded in the inductive proof of the main theorem but essentially it is a consequence of 4.3.

The principal idea of the proof of the main theorem is as follows: Let $K = K(\mathbf{Z}/p\mathbf{Z}, 2n+1)$ and let

$$\Sigma K = B_1 \subset B_2 \subset B_3 \subset \cdots \subset B_\infty = BK = K(\mathbf{Z}/p\mathbf{Z}, 2n+2)$$

be the Milnor–Rothenberg–Steenrod–Milgram filtration ([5], [8]) of the classifying space of K. We shall use the following notations:

2.0.

(i) $i_{r,s}: B_r \longrightarrow B_s$ is the inclusion, with the abreviation $i_{r,r+1} = i_r$.

(ii) $B_r \cup CB_{r-1} = B_r, B_{r-1}$, $j_r: B_r \longrightarrow B_r, B_{r-1}$ denotes the inclusion.

(iii) $\delta_r: B_r, B_{r-1} \longrightarrow \Sigma B_{r-1}$ the third leg of the Puppe sequence.

(iv) $\iota = \iota_{2n+2}$ denotes all fundamental classes in $H^{2n+2}(B_r)$, $1 \le r \le \infty$.

We evaluate ϕ_{p-1} on $\mathcal{P}^n \iota_{2n+2} \in H^{2np+2}(\Sigma K) = H^{2np+2}(B_1)$ using a $2np+1$ connected non stable version of (D1). Thus one reads D1 with $K_r = K(\mathbf{Z}/p\mathbf{Z}, 2np+2+t_r)$ $(t_0 = 0)$, t_r as defined in 0.1 - (2).

We start an inductive process:

$$
(\mathrm{D2})_p \qquad
\begin{tikzcd}
B_p \arrow[r, "\tilde{u}_p"] \arrow[d, "j_p"] & E_0 = K(\mathbf{Z}/p\mathbf{Z}, 2np+2) \arrow[d, "k_0 = P^1"] \\
B_p, B_{p-1} \arrow[r, "u_p = \overbrace{\iota\otimes\iota\otimes\cdots\otimes\iota}^{p \text{ times}}"] \arrow[d, "\delta_p"] & K_1 \arrow[d, "P^{p-1}"] \\
\Sigma B_{p-1} \arrow[r, dashed, "\hat{u}_p"] & BK_2
\end{tikzcd}
$$

$$n \not\equiv -1 \pmod p$$

$[\tilde{u}_p] = \frac{1}{n+1} P^n \iota_{2n+2}$, so $P^1[\tilde{u}_p]\iota_{2n+2} = P^{n+1}\iota_{2n+2} = \iota^p_{2n+2} = [j_p](\iota\otimes\iota\otimes\cdots\otimes\iota) =$ $= [u_p \circ j_p]$. Note that $H^*(B_r, B_{r-1}) \approx \overbrace{\Sigma H^*(K)\otimes\Sigma H^*(K)\otimes\cdots\otimes\Sigma H^*(K)}^{r \text{ times}}$ thus the notation $u_p = \iota\otimes\cdots\otimes\iota$. Also note that the map induced by $\Sigma i_{r-1}\circ\delta_r$ from $H^*(B_{r-1}, B_{r-2})$ to $H^*(B_r, B_{r-1})$ corresponds to $d_{r-1}\colon \underline{B}_{r-1} \longrightarrow \underline{B}_r$ in the cobar construction of $A = H^*(K)$ (with the appropriate shift of dimension).

Any choice of homotopies $W\colon k_0\circ\tilde{u}_p \sim u_p\circ j_p$, $\ell\colon * \sim P^{p-1}\circ k_0$ induces a map $\hat{u}_p\colon \Sigma B_{p-1} \longrightarrow BK_2$ and its adjoint $\hat{u}_{p\#}\colon B_{p-1} \longrightarrow K_2$ factors up to homotopy as $B_{p-1} \xrightarrow{\tilde{u}_{p-1}} E_1 \xrightarrow{k_1} K_2$, hence $\hat{u}_{p\#}$ represents $\phi_1(P^n\iota_{2n+2})$. To continue the inductive step one has to show that $\hat{u}_{p\#}$ factors (up to homotopy) as

$$B_{p-1} \xrightarrow{j_{p-1}} B_{p-1}, B_{p-2} \xrightarrow{u_{p-1}} K_2.$$

Inductively, suppose one has a (homotopy) commutative diagram (for $3 \le r \le$ $\le p$)

$$
(\mathrm{D2})_r \qquad
\begin{tikzcd}
B_r \arrow[r, "\tilde{u}_r"] \arrow[d, "j_r"] & E_{p-r} \arrow[d, "k_{p-r}"] \\
B_r, B_{r-1} \arrow[r, "u_r"] \arrow[d, "\delta_r"] & K_{p-r+1} \arrow[d, "P^{a_{p-r+1}}"] \\
\Sigma B_{r-1} \arrow[r, dashed, "\hat{u}_r"] & BK_{p-r+2}
\end{tikzcd}
$$

where $[k_{p-r}\circ\tilde{u}_r]$ represents $\phi_{p-r}(P^n\iota_{2n+2})$. Any choice of homotopies $W\colon k_{p-r}\circ$ $\tilde{u}_r \sim u_r\circ j_r$ and $\ell\colon * \sim P^{a_{p-r+1}}\circ k_{p-r}$ (which exists by 3.1) induces a map $\hat{u}_r\colon \Sigma B_{r-1} \longrightarrow BK_{p-r+2}$ whose adjoint $(\hat{u}_r)_\#$ factors as $B_{r-1} \xrightarrow{\tilde{u}_{r-1}} E_{p-r+1}$ $\xrightarrow{k_{p-r+1}} K_{p-r+2}$, hence, $(\hat{u}_r)_\#$ represents $\phi_{p-r+1}(P^n\iota_{2n+2})$.

To proceed inductively one has to show that $\hat{u}_r$ factors as

$$\Sigma B_{r-1} \xrightarrow{\Sigma j_{r-1}} \Sigma B_{r-1}, \Sigma B_{r-2} \xrightarrow{(u_{r-1})_\#} BK_{p-r+2}$$

or equivalently, $(\hat{u}_r)_\#$ factors as

$$B_{r-1} \xrightarrow{j_{r-1}} B_{r-1}, B_{r-2} \xrightarrow{u_{r-1}} K_{p-r+2}.$$

This is proved in 7.1 using the following preparatory steps:

2.1. Fix λ –a primitive root of unity mod p. The λ–th power map on K is an ∞–loop map hence induces self maps $T_{B_r}, T_{B_r, B_{r-1}}, \Sigma T_{B_{r-1}} = T_{\Sigma B_{r-1}}$ on $B_r, (B_r, B_{r-1}), \Sigma B_{r-1}$ respectively and $i_{s,t}, j_r, \delta_r$ commute with the T's up to homotopy. Similarly the stable multiplication by λ induces self maps $T_{E_i}, T_{K_i}, T_{\Omega K_i} = \Omega T_{K_i}$ (which are ∞-loops maps) on $E_i, K_i, \Omega K_i$ in (D1). One can easily see that in $(\mathrm{D2})_p$ $\tilde{u}_p, u_p$ commute with these self maps T. Choosing $W: k_0 \circ \tilde{u}_p \sim u_p \circ j_p$, $\ell: * \sim \mathcal{P}^{p-1} \circ k_0$ appropriately (for ℓ any stable null homotopy will do) one may obtain $\hat{u}_p$ that commutes with the T's: $\hat{u}_p \circ T_{\Sigma B_{p-1}} \sim T_{BK_2} \circ \hat{u}_p$. (see 5.1, 5.2, and 5.3).

Now, using self maps theory (section 5) one can make sure that in all inductive steps $(D2)_r \Longrightarrow (D2)_{r-1}$ all maps obtained commute with the self maps T. This translates to the algebraic fact that the classes $[u_r] \in H^*(B_r, B_{r-1}) \approx$ $\approx \overbrace{\Sigma H^*(K) \otimes \cdots \otimes \Sigma H^*(K)}^{r \text{ times}}$ have algebraic weight 1 mod $p-1$. (i.e: $[u_r] = \Sigma a_i \cdot v_i$, $v_i = \Sigma z_1 \otimes \Sigma z_2 \otimes \cdots \otimes \Sigma z_r$, $z_i \in H^*(K)$ is a monomial of primitive $z_i = x_1^{(i)} \ldots x_{\ell_i}^{(i)}$, $x_j^{(i)} \in PH^*(K)$ and $\sum_{i=1}^r \ell_i \equiv 1 \pmod{p-1}$).

2.2. To prove 7.1 one has to show that $\hat{u}_r \circ \Sigma i_{r-2} \sim *$. The first step in the proof uses 4.3 to show that W could be chosen (without violating the restriction required to obtain the properties discussed in 2.1) so that $\hat{u}_r \circ \Sigma i_{s,r-1} \sim *$.

2.3. To show that $\hat{u}_r \circ \Sigma i_{s,r-1} \sim *$ implies $\hat{u}_r \circ \Sigma i_{s+1,r-1} \sim *$ for $s < r-2$ one observes: $\hat{u}_r \circ \Sigma i_{s,r-1} \sim *$ is equivalent to the existence of $\hat{u}'_s: \Sigma B_{s+1}, \Sigma B_s \longrightarrow BK_{p-r+2}$, $\hat{u}'_s \circ \Sigma j_{s+1} \sim \hat{u}_r \circ i_{s+1,r-1}$, $[\hat{u}'_s]$ has an algebraic weight 1 mod $p-1$ and $d_{s+1}[\hat{u}'_j] = 0$ in the cobar construction. Following computations in the cobar construction (6.2), $[\hat{u}'_s] = d_s(\hat{\hat{u}}'_s)$ and $\hat{u}_r \circ \Sigma i_{s+1,r-1} \sim *$.

2.4. Now one deduces the existence of $(\mathrm{D2})_r$ for $p \geq r \geq 2$, and $(\hat{u}_2)_\#: B_1 \longrightarrow K_p$ represents $\phi_{p-1}(\mathcal{P}^n \iota_{2n+2})$. As $\tilde{H}^*(B_1) \approx H^*(B_1, B_0)$ $(B_0 = *)$ one can write $(\hat{u}_2)_\# = u_1$ obtaining classes $u_r \in \underline{B}_r = H^*(B_r, B_{r-1})$, $1 \leq r \leq$ $\leq p$ subject to the conditions: u_r have $(\bmod\, p-1)$ algebraic weight 1, $u_p =$ $= \overbrace{\iota \otimes \iota \otimes \cdots \otimes \iota}^{p \text{ times}}$, $\mathcal{P}^{a_{p-r+1}} u_r = d_{r-1} u_{r-1}$. By 6.3 $\Sigma^{-1} u_1 \in H^*(K)$ has the form $\varepsilon_p \iota \cdot \mathcal{P}^1 \iota \cdot \mathcal{P}^2 \iota \cdot \ldots \cdot \mathcal{P}^{p-1} \iota \pmod{\mathrm{Im}\, \mathcal{P}^1}$ and the first part of the main theorem follows.

To show that $0 \notin \phi_{p-1}(\mathcal{P}^n \iota_{2n+1})$ $(n \not\equiv -1(p))$ suffices to show that for some space X and some class $x \in H^{2n+1}(X)$, $0 \notin \phi_{p-1}(\mathcal{P}^n x)$. ($\mathcal{P}^n x_{2n+1}$ is obviously in the domain of ϕ_{p-1} being the image of $\mathcal{P}^n \iota_{2n+1}$). This is done in Chapter 8 where some examples and applications are discussed.

3. The existence of ϕ_r

We use the notation of the stable version of (D1).

3.1. Proposition. *If $r\,(r < q)$ satisfies the inequality $[r/2] < \sum_{t=0}^{q-r-1} \sum_{i=0}^{t} p_i$ $(q = 2p-2)$ then ϕ_r exists. In particular ϕ_{p-1} exists.*

Proof: $\phi_0 = \mathcal{P}^1$ and ϕ_1 obviously exist so one may assume $p > 2$, $r \geq 1$. Now suppose ϕ_{n-1} exists, i.e. there exists a map $k_n : E_{n-1} \longrightarrow K_n = \Sigma^{t_n} K_0$. The obstruction for the existence of ϕ_n is the composition $E_{n-1} \xrightarrow{k_n} K_n \xrightarrow{\mathcal{P}^{a_n}} \Sigma K_{n+1}$ ($a_n = 1$ if n is even, $a_n = p-1$ if n is odd).

Now $\Omega K_{n-1} = \Sigma^{-1} K_{n-1} \xrightarrow{j_{n-1}} E_{n-1} \xrightarrow{k_{n-1}} K_n$ is given by $\mathcal{P}^{p-a_n}$, hence $* \sim \mathcal{P}^{a_n} \circ k_{n-1} \circ j_{n-1}$ and $\mathcal{P}^{a_n} \circ k_{n-1}$ factors as $E_{n-1} \xrightarrow{h_{n-1}} E_{n-2} \xrightarrow{\omega} \Sigma K_{n+1} = \Sigma^{t_{n+1}+1} K_0$. ■

If one can show that $H^{t_{n+1}+1}(E_{n-2}) = 0$, 3.1 will follow. Using the exact sequences $H^*(E_{j-1}) \longrightarrow H^*(E_j) \longrightarrow H^*(\Sigma^{-1}K_j)$ and $H^*(E_0) = H^*(K_0)$ suffices to prove that $H^{t_{n+1}+1}(\Omega K_j) = H^{t_{n+1}+1}(\Sigma^{t_j} K_0) = H^{t_{n+1}-t_j+2}(K_0) = 0$ for $j \geq 0$ $(t_0 = 0)$ provided that $[n/2]_p < \sum_{t=0}^{q-n-1} \sum_{i=0}^{t} p^i = \lambda_{q-n+1}$. Thus, putting $t_s = \nu_s q + q - s + 1$, $t_{n+1} - t_j + 2 = mq + k$ where $m \leq \nu_{n+1}$, $k \geq q - n + 1$, $\nu_{n+1} = \begin{cases} \frac{n}{2}p & \text{if } n \text{ is even} \\ [\frac{n}{2}]p - 1 & \text{if } n \text{ is odd.} \end{cases}$

3.1. Will follow from the following:

3.2. Lemma. *For $2 \leq k \leq q-1$ and $m < \sum_{t=0}^{k-2} \sum_{i=0}^{t} p^i = \lambda_k$, one has $H^{mq+k}(K_0) = 0$. In particular the inequality holds if $m \leq \nu_{n+1} \leq [\frac{n}{2}]p < \lambda_{q-n+1} \leq \lambda_k$.*

Proof: Every element in $H^{mq+k}(K_0)$ has the form $\alpha\imath$ where α is a sum of admissible monomials in $\underline{A}(p)$, $|\,\alpha\,| = mq + k$ (see [10, p. 77]). An admissible monomial of dim $\equiv k \mod q$ has to contain at least k Bocksteins. The lowest dimensional admissible monomial with k Bocksteins is $\beta\mathcal{P}^{p^{k-2}+\cdots+1} \beta\mathcal{P}^{p^{k-3}+\cdots+1} \ldots \beta\mathcal{P}^{p+1}\beta\mathcal{P}^1\beta$ and its dimension is $\lambda_k q + k$. ■

4. Some computations in the Steenrod algebra

Let $\omega \in \underline{A}(p)$ be a linear combination of admissible monomials of excess $\leq 2n+1$. (See [10]). If $\mathcal{P}^1\omega$ is a sum of admissible monomials of excess $> 2n+1$ then (modulo $\ker \mathcal{P}^1$) ω is a linear combination of elements of one of the two forms:

(i) $\omega = \mathcal{P}^m \omega_0$, $m \not\equiv -1(p)$, $[(|\omega_0| + 1)/2] = m - n$.

(ii) $\omega = ((m+1)\beta\mathcal{P}^m - \mathcal{P}^m\beta)\omega_0$, $m \not\equiv 0, -1(p)$, $[|\omega_0|/2] = m - n$

where ω_0 is an admissible monomial.

One has the following:

4.1. Proposition. *Let* $K = K(\mathbf{Z}/p\mathbf{Z}, 2n+1)$. *Given* $0 \neq u = \omega\iota_{2n+1} \in \in PH^t(k/\mathbf{Z}/p\mathbf{Z})(\omega \in \underline{A}(p))$ *where*

$$t = or \begin{cases} 2np + 1 + kpq - 2k + 1 & 2 \leq 2k \leq p-1 \\ 2np + 1 + kpq + sq - 2k & \begin{cases} 1 \leq s \leq p-1 \\ 2 \leq 2k \leq p-3 \end{cases} \end{cases}$$

(the latter is void for $p = 3$*).*

If $\mathcal{P}^1 u = 0$ *then* $\mathcal{P}^1\omega = 0$ *in* $\underline{A}(p)$.

Proof: We may assume that excess $\omega \leq 2n+1$ and $\mathcal{P}^1 u = 0$ implies excess $(\mathcal{P}^1\omega) > 2n+1$ hence ω has one of the two forms listed above:

(i) $u = \mathcal{P}^m\omega_0\iota_{2n+1}$, $|\omega_0\iota_{2n+1}| = 2m$ or $2m+1$.

(ii) $u = ((m+1)\beta\mathcal{P}^m - \mathcal{P}^m\beta)\omega_0\iota_{2n+1}$, $|\omega_0\iota_{2n+1}| = 2m+1$ or $2m+2$ hence

$|u| = 2mp$ or $2mp+1$ (case i).

$|u| = 2mp+2$ or $2mp+3$ (case ii).

The hypothesis on t excludes $|u| = 2mp+2$, $|u| = 2mp$ can occur only if $t = 2np + pq$ $(k = 1)$, $|\omega_0\iota_{2n+1}| = 2n+q$ and $|\omega_0| = q-1$. But there are no elements in $\underline{A}(p)$ of dimension $q-1$. $|u| = 2mp+1$ can occur only if $s + k = p > 3$ and then $|\omega_0| = kq + q - 2k$ and again, there are no elements in this dimension in $\underline{A}(p)$. Similarly, $|u| = 2mp+3$ occurs only if $s + k = p - 1 > 2$. Then $|\omega_0| = kq + q - 2k - 1$ and there are no elements in this dimension in $\underline{A}(p)$. ∎

4.2. Proposition. *If* $1 \leq a \leq p-1$ *then the following sequence is exact*

$$\ldots \to \underline{A}(P) \xrightarrow{\mathcal{P}^a\times} \underline{A}(p) \xrightarrow{\mathcal{P}^{p-a}\times} \underline{A}(p) \xrightarrow{\mathcal{P}^a\times} \underline{A}(p) \to \ldots$$

($\mathcal{P}^a\times$ *means the endomorphism of left multiplication by* $\mathcal{P}^a$*).*

Proof: We assume p-odd. The Milnor representation of $\underline{A}(p)$, [6], yields an isomorphism $\underline{A}(p) \approx \mathcal{P} \otimes Q$. This is an isomorphism of left $\mathcal{P}$ modules where $\mathcal{P}$ is the subalgebra of $\underline{A}(p)$ generated by $\mathcal{P}^{p^i}$. Thus, suffices to prove the exactness of the sequence where $\mathcal{P}$ replacing $\underline{A}(p)$. For $\mathcal{P}$ one can use again the basis of admissible monomials in the $\mathcal{P}^i$'s. Now, if $a < p$ and $\omega \in \mathcal{P}$ is an admissible monomial then $\mathcal{P}^a\omega$ is (a multiple of) an admissible monomial. If $\omega_1 \neq \omega_2$ are admissible monomials then $\mathcal{P}^a\omega_1 \neq \mathcal{P}^a\omega_2$ (unless both are zero).It follows that if $u = \Sigma_i k_i w_i \in \underline{A}(p)$, $k_i \in \mathbf{Z}/p\mathbf{Z}$, ω_i admissible monomials $\mathcal{P}^a u = 0$ if and only if $\mathcal{P}^a\omega_i = 0$ for all i and suffices to prove taht $\mathcal{P}^a\omega = 0$ if and only if $\omega = \mathcal{P}^{p-a}\tilde{\omega}$ for an admissible monomial ω. Now $\omega = \mathcal{P}^m\omega_0 = \hat{k}\mathcal{P}^b\mathcal{P}^{\hat{m}p}\omega_0$ for some admissible monomial ω_0, $0 \leq \hat{m}$, $0 \leq b < p$. $\mathcal{P}^a\omega = 0$ if and only if $a + b \geq p$ and then $\omega = \mathcal{P}^{p-a}(\hat{\hat{k}}\ \mathcal{P}^{a+b-p}\mathcal{P}^{\hat{m}p}\omega_0)$. ∎

4.3. Corollary. *Let $K = K(\mathbf{Z}/p\mathbf{Z}, 2n+1)$ and let $\omega\imath_{2n+1} = u \in$ $\in PH^{2np+1+t_r}(K)$, $\omega \in \underline{A}(p)$ (t_r as in 1.0-(2)). If $\mathcal{P}^{a_r}u = 0$ then $u = \mathcal{P}^{a_{r+1}}u_0$ (a_r as in 1.0-(4)).*

Proof: If r is even $t_r = t_{2k} = kpq - 2k + 1$, $|u| = t$ as in 4.1 and $0 =$ $\mathcal{P}^{a_r}u = \mathcal{P}^1 u$ implies $\mathcal{P}^1\omega = 0$. $\omega = \mathcal{P}^{p-1}\omega_0$ and $u = \mathcal{P}^{p-1}\omega_0\imath_{2n+1} = \mathcal{P}^{a_{r+1}}u_0$. If r is odd $t_r = t_{2k+1} = kpq + q - 2k$. If $0 = \mathcal{P}^{a_r}u = \mathcal{P}^{p-1}u$ then for some s, $1 \le s \le p-2$, $\mathcal{P}^{s-1}u \ne 0$ and $\mathcal{P}^1(\mathcal{P}^{s-1}u) = 0$. The proof $u = \mathcal{P}^1 u_0$ is by induction on s. $|\mathcal{P}^{s-1}u| = t$ as in 4.1 and $\mathcal{P}^1(\mathcal{P}^{s-1}u) = 0$ implies $\mathcal{P}^1(\mathcal{P}^{s-1}\omega) = 0$ (where $u = \omega\imath_{2n+1}$, $\omega \in \underline{A}(p)$). By 4.2 $\mathcal{P}^{s-1}\omega = \mathcal{P}^{p-1}\omega_0$. If $s = 1$, $\omega = -\mathcal{P}^1(\mathcal{P}^{p-2}\omega_0)$ and $u = \mathcal{P}^1(\mathcal{P}^{p-2}\omega_0)\imath_{2n+1}$. Otherwise one has for some $\varepsilon \ne 0$, $\mathcal{P}^{s-1}(\omega - \varepsilon\mathcal{P}^{p-s}\omega_0)\imath_{2n+1} = 0$, by induction $u - \varepsilon\mathcal{P}^{p-s}\omega_0\imath_{2n+1} =$ $= \mathcal{P}^1 u_0$ and $u = \mathcal{P}^1(u_0 + \varepsilon_1\mathcal{P}^{p-s-1}\omega_0\imath_{2n+1})$. ∎

5. Review of homotopy theory of self maps

In this section we extract some notations and statements from [12], needed in our computations. Assume all spaces and maps are pointed. Consider pairs X, T_X of spaces X with a self map $T_X : X \longrightarrow X$ and "morphisms" $f, V_f : X, T_X \longrightarrow Y, T_Y$ where $f : X \longrightarrow Y$ and $V_f : X \longrightarrow PY = \mathrm{map}(I, Y)$ is a homotopy $V_f : f \circ T_X \sim T_Y \circ f$. Our main object here is to show that some standard homotopy theoretic constructions extend to our "category".

Now, given pairs E, T_E , B, T_B , B_0, T_{B_0} , morphims $f_0, V_{f_0} : E, T_E \longrightarrow$ B, T_B , $f_1, V_{f_1} : B, T_B \longrightarrow B_0, T_{B_0}$ and a homotopy $\ell : * \sim f_1 \circ f_0$ ($\ell : E \longrightarrow$ $\underline{L}B_0 = \mathrm{map}_*(I, 0; B_0, *)$). We denote by $\alpha(\ell) = \alpha(\ell, V_{f_0}, V_{f_1})$ the class of $\ell \circ T_E + Pf_1 \circ V_{f_0} + V_{f_1} \circ f_0 - \underline{L}B_0 \circ \ell$ in $\pi_1(\mathrm{map}_*(E, B_0), *)$.

For convenience assume that $B_0 = K(G, m)$ where G is an elementary abelian p-group. Then one has:

5.1. Proposition ((3.9) of [12]). *One can choose ℓ so that after replacing T_E, T_B, T_{B_0} by their p^r-th iteration $\alpha(\ell) = 0$. If ℓ is altered by $\omega : E \longrightarrow \Omega B_0$ and $\omega \circ T_E \sim \Omega T_{B_0} \circ \omega$ then $\alpha(\omega + \ell) = \alpha(\ell)$.*

Similarly, suppose in addition that one is given pairs L, T_L, K, T_K and maps $\sigma, V_\sigma : L, T_L \longrightarrow E, T_E$, $\tau, V_\tau : K, T_K \longrightarrow B, T_B$, $h, V_h : L, T_L \longrightarrow K, T_K$ and a homotopy $W : f_0 \circ \sigma \sim \tau \circ h$, to obtain a diagram:

(D.3)
$$\begin{tikzcd}
L, T_L \arrow[r, "\sigma"] \arrow[d, "h"] \arrow[dr, phantom, "W"] & E, T_E \arrow[d, "f_0"] & \\
K, T_K \arrow[r, "\tau"] & B, T_B \arrow[d, "f_1"] & \ell : * \sim f_1 \circ f_0 \\
 & B_0, T_{B_0} &
\end{tikzcd}$$

One can define an element $\alpha(W) = \alpha(W, V_\sigma, V_{f_0}, V_h, V_\tau)in\pi_1(\text{map}_*(L,B), f_0 \circ \sigma \circ T_L)$ as the class of $Pf_0 \circ V_\sigma + V_{f_0} \circ \sigma + PT_B \circ W - V_\tau \circ h - P\tau \circ V_h - W \circ T_L$. If B is an H-space one can translate $\alpha(W)$ to $\pi_1(\text{map}_*(L,B),*)$ and then it coincides with α of the first type (replacing the square by maps $L \longrightarrow E \times K \longrightarrow B$). In this case 5.1 yields:

5.2. Proposition. *W can be chosen so that after replacing T_L, T_K, T_E, T_B by their p-th iteration $\alpha(W) = 0$. If W is altered by $w: L \longrightarrow \Omega B$ and $w \circ T_L \sim$ $\sim T_B \circ w$ then $\alpha(w * W) = \alpha(W)$ (where * denote the pointwise multiplication of paths in the function space).*

The classes $\alpha(\ell), \alpha(W)$ represent obstructions to the following problems:

Given $f_1, V_{f_1}: B, T_B \longrightarrow B_0, T_{B_0}$ one naturally obtains a self map $T_{F_{f_1}}$ of the homotopy fiber F_{f_1} of f_1 so that the "inclusion of the fiber map" $j_{f_1}: F_{f_1} \longrightarrow B$ strictly commutes with the self maps: $j_{f_1} \circ T_{F_{f_1}} = T_B \circ j_{f_1}$. If $\ell: * \sim f_1 \circ$ $\circ f_0$, $f_0, V_{f_0}: E, T_E \to B, T_B$, then ℓ induces a lifting $f_\ell: E \longrightarrow V_{f_1}$ of $f_0: j_{f_1} \circ$ $\circ f_\ell = f_0 \cdot \alpha(\ell) = \alpha(\ell, V_{f_0}, V_{f_1})$ is the obstruction to lift V_{f_0} to a homotopy $V_{f_\ell}: f_\ell \circ T_E \sim T_{V_{f_1}} \circ f_\ell$.

Similarly, f_0, V_{f_0} induces a self map $T_{C_{f_0}}$ on the mapping cone C_{f_0} of f_0 so that $i_{f_0}: B \longrightarrow C_{f_0}$ strictly commutes with the self maps: $i_{f_0} \circ T_B = T_{C_{f_0}} \circ$ $\circ i_{f_0}$. $\ell: * \sim f_1 \circ f_0$ induces a map $\hat{f}_\ell: C_{f_0} \longrightarrow B_0$ extending f_1: $\hat{f}_\ell \circ i_{f_0} =$ $= f_1 \cdot \alpha(\ell) = \alpha(\ell, V_{f_0}, V_{f_1})$ is also the obstruction to extend V_{f_1} to a homotopy $V_{\hat{f}_\ell}: \hat{f}_\ell \circ T_{C_f} \sim T_{B_0} \circ \hat{f}_\ell$. Analogously, $T_K, T_E, T_B, V_{f_0}, V_\tau$ in D3 induce a self map $T_{U(f_0,\tau)}$ on the the homotopy pull back $U(f_0,\tau)$ of f_0 and τ. W induces a lifting $h_W: L \longrightarrow U(f_0,\tau)$ of

$$(\sigma \times h) \circ \Delta: L \longrightarrow E \times K.$$

$\alpha(W) = \alpha(W, V_\sigma, V_{f_0}, V_h, V_\tau)$ is the obstruction to lift $(V_\sigma \times V_h) \circ \Delta$ to a homotopy $V_{h_W}: h_W \circ T_L \sim T_{U(f_0,\tau)} \circ h_W$. Now, if in D3 σ, τ, W, ℓ are given, they induce a map $\rho: C_h \longrightarrow B_0$.

5.3. Proposition. *Suppose in (D3) B, B_0 are $K(G,n)$ and $K(G_0,m)$ respectively where G, G_0 are elementary abelian p-groups. If $W: f_0 \circ \sigma \sim \tau \circ$ $\circ h$, $\ell: * \sim f_1 \circ f_0$ are chosen so that $\alpha(W) = 0$, $\alpha(\ell) = 0$ then the natural map ρ admits a homotopy $V_\rho: \rho \circ T_{C_h} \sim T_{B_0} \circ \rho$.*

Proof: Writing down the formulas for the various maps involved one gets $(x, t \in L \times I,\ y \in K)$:

$$\rho(x,t) = \begin{cases} \ell\sigma(x)\ [2t] \\ Pf_1 \circ W(x)\ [2t-1] \end{cases} \qquad T_{C_h}(x,t) = \begin{cases} T_L(x),\ 2t \\ V_h(x)\ [2t-1] \end{cases}$$

$$\rho(y) = f_1 \circ \tau(y) \qquad\qquad T_{C_h}(y) = T_K(y)$$

$$\rho \circ T_{C_h}(x,t) = \begin{cases} \ell\sigma T_L(x)\ [4t] \\ Pf_1 \circ W \circ T_L(x)\ [4t-1] \\ P(f_1 \circ \tau) \circ V_h(x)\ [2t-1] \end{cases}$$

$$\rho T_{C_h}(y) = f_1 \circ \tau \circ T_k(y)$$

$$T_{B_0} \circ \rho(x,t) = \begin{cases} \underline{L}T_{B_0} \circ \ell \circ \sigma(x)\ [2t] \\ PT_{B_0} \circ Pf_1 \circ W(x)\ [2t-1] \end{cases}$$

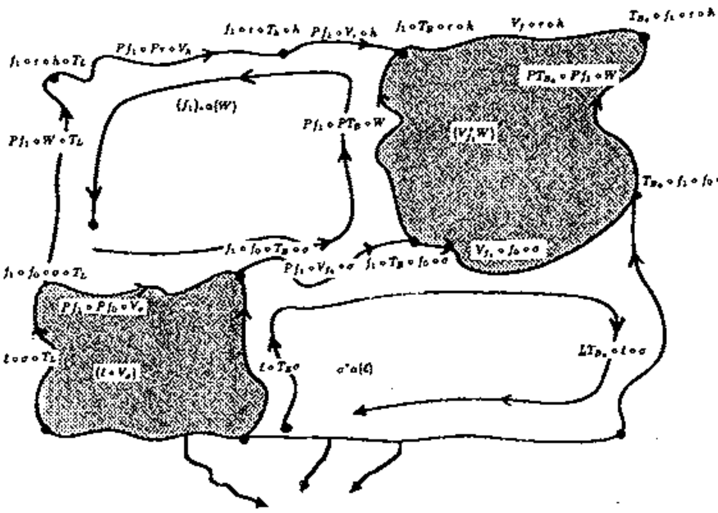
$$T_{B_0} \circ \rho(y) = T_{B_0} \circ f_1 \circ \tau(y)$$

The obstruction to extend the homotopy

$Pf_1 \circ V_\tau + V_{f_1} \circ \tau \colon f_1 \circ \tau \circ T_k \sim T_{B_0} \circ f_1 \circ \tau$ to a homotopy $\rho \circ T_{C_h} \sim T_{B_0} \circ \rho$ is the class in $\pi_1(\mathrm{map}_*(L, B_0), *)$ of the loop

$$\gamma = \ell \circ \sigma \circ T_L + Pf_1 \circ W \circ T_L + Pf_1 \circ P\tau \circ V_h + \\ + Pf_1 \circ V_\tau \circ h + V_{f_1} \circ \tau \circ h - PT_{B_0} \circ Pf_1 \circ W - \underline{L}T_{B_0} \circ \ell \circ \sigma$$

which can be illustrated as follows as a loop in $\mathrm{map}_*(L, B_0)$:



Thus $[\gamma] = -(\ell \circ \sigma \circ T_L)_\# \pi_1(f_1 *)\alpha(W) + \pi_1(\sigma^*)\alpha(\sigma)$ where

$$f_{1*} \colon \mathrm{map}_*(L, B) \longrightarrow \mathrm{map}_*(L, B_0)$$
$$\sigma^* \colon \mathrm{map}(E, B_0) \longrightarrow \mathrm{map}_*(L, B_0)$$

are the left and right compositions with f_1 and σ respectively.

$$(\ell \circ \sigma \circ T_L)_\# \colon \pi_1(\mathrm{map}_*(L, B_0), f_1 \circ f_0 \circ \sigma \circ T_L) \longrightarrow \pi_1(\mathrm{map}_*(L, B_0), *)$$

is the isomorphism induced by the path $\ell \circ \sigma \circ T_L$. Hence, $\alpha(W) = 0, \quad \alpha(\ell) = 0$ implies $[\gamma] = 0$. ∎

6. Derivations in the co-bar construction

Let A be a graded connected primitively generated Hopf algebra of finite type over $\mathbf{Z}/p\mathbf{Z}$ p-odd. Then the elements of A are sums of monomials in the primitives. One has a mod $p-1$ algebraic weight function w defined on homogeneous polynomials: $w(a) \in \mathbf{Z}/(p-1)\mathbf{Z}$, satisfying: $w(1) = 0$, $W(PA) = 1$, $w(\mathrm{Im}(\overbrace{PA \otimes \cdots \otimes PA}^{k \text{ times}} \xrightarrow{\Delta^k} A)) = k$ (Δ - the multiplication in A). Thus, together with the grading this gives a bigrading of A, $A_{s,t}$, $s \in \mathbf{Z}/(p-1)\mathbf{Z}$, $t \in Z^+$.

Let $\underline{B} = \bigoplus_{r=0}^{\infty} \underline{B}_r$ be the reduced cobar construction on A: $\underline{B}_r = \overbrace{\overline{A} \otimes \cdots \otimes \overline{A}}^{r \text{ times}}$ with a bigrading $\underline{B}_{r,t} = \bigoplus_{\substack{\Sigma m_i = t \\ m_i > 0}} A_{m_1} \otimes \cdots \otimes A_{m_r}$. Then w extends to $\underline{B}_r$ by $\omega(a_1 \otimes \cdots \otimes a_r) = \sum \omega(a_i)$ for monomials a_i in A. Let $d_r : \underline{B}_r \longrightarrow \underline{B}_{r+1}$ be the derivation $d_r(a_1 \otimes \cdots \otimes a_r) = \sum_{i=1}^{r} (-1)^{i+1} a_1 \otimes \cdots \otimes \overline{\mu} a_i \otimes \cdots \otimes a_r$ ($\overline{\mu}$ -the reduced coproduct in A). Then d_r respects the weight function thus extending it to the homology of d_r, namely to $\mathrm{Ext}_A^{*,*}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$. The latter is a free algebra generated by $\mathrm{Ext}_A^{*}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$ and by some elements in $\bigoplus_k \mathrm{Ext}_A^{2,2kp}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$. All generators in $\mathrm{Ext}_A^{*,*}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$ are of weight one. ($\mathrm{Ext}_A^{1,m}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z}) \approx PA_m$ and the generators in $\mathrm{Ext}_A^{2,2kp}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$ are represented by $\sum_{a=1}^{p-1} \frac{1}{p}\binom{p}{a} v^a \otimes v^{p-a}$, $v \in PA_{2k}$). Thus we have:

6.1. Lemma. $\mathrm{Ext}_A^{s,r}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$ *contains no elements of weight 1 for* $2 < s < p$ *and for* $s = 2$, $r \not\equiv 0 \pmod{2p}$.

6.2. Corollary. *If* $u \in \ker d_r \subseteq \underline{B}_r$, $w(u) = 1$, $2 < r < p$ *or if* $r = 2$ *and* $|u| \not\equiv 2 \pmod{2p}$ *then* $u \in \mathrm{Im}\, d_{r-1}$ *(* $|\ |$ *- the total degree* $s + r$ *in* $\mathrm{Ext}^{s,r}$*).*

If A above is a non stable module over $\mathbf{Z}/p\mathbf{Z}[\mathcal{P}^1]/(\mathcal{P}^1)^p \subset \underline{A}(p)$ (i.e., $\mathcal{P}^1 u = u^p$ for $u \in A_2$, $\mathcal{P}^1 A_1 = 0$ and $\mathcal{P}^1$ acts as a derivation on $a \cdot b$ and $a \otimes \otimes b$. $\mathcal{P}^1(PA) \subset PA$). Then $\mathcal{P}^i = \frac{1}{i!}(\mathcal{P}^1)^i$ $(i < p)$ respects the mod $p-1$ algebraic weight and $\mathcal{P}^i$ commuted with d_r in $\overbrace{\overline{A} \otimes \cdots \otimes \overline{A}}^{r \text{ times}} \approx \underline{B}_r$.

6.3. Proposition. *Let A be a primitively generated graded connected Hopf algebra and a module over* $\mathbf{Z}/p\mathbf{Z}[\mathcal{P}^1]/(\mathcal{P}^1)^p$. *Suppose* $PA_m = 0$ *for* $m < 2n+1$ *and let* $x \in PA_{2n+1}$. *Then there exist elements* $u_s \in \overbrace{\overline{A} \otimes \cdots \otimes \overline{A}}^{s \text{ times}}$ *of* mod $p-1$ *algebraic weight* 1, $1 \le s \le p$, *so that:*

$$(i) \quad u_p = \overbrace{x \otimes x \otimes \cdots \otimes x}^{p \text{ times}} \qquad (ii) \quad \mathcal{P}^{a_{s+1}} u_s = d_{s-1} u_{s-1} \quad \text{for } s > 1.$$

$(u_{s-1} \in \overbrace{\overline{A} \otimes \cdots \otimes \overline{A}}^{s-1 \text{ times}}$ *considered as an element in* $\underline{B}_{s-1})$ a_{s+1} *as in 1.0 (4). For any choice of* u_s *of weight 1* $(\operatorname{mod} p-1)$ *satisfying (i) and (ii) one has* $u_1 = \varepsilon_p x \cdot \mathcal{P}^1 x \cdot \mathcal{P}^2 x \dots \mathcal{P}^{p-1} x \ (\operatorname{mod}\ \operatorname{Im} \mathcal{P}^1) \quad \varepsilon_p = (-2)^{p-\frac{1}{2}}$.

Proof: Suppose by induction u_{p-r} of weight one were found for $0 \leq r \leq r_0$ so that, $u_p = x \otimes \cdots \otimes x$ and $\mathcal{P}^{a_r} u_{p-r+1} = d_{p-r} u_{p-r}, \quad |u_s| =$ odd. Further assume:

$$(\text{for} \quad r = 2s) \quad u_{p-2s} = [\mathcal{P}^{p-1}(\mathcal{P}^1 x \cdot x)]^s \cdot x \overbrace{\otimes x \otimes \cdots \otimes x}^{p-2s-1 \text{ times}} + \textstyle\sum_i w_1^i \otimes w_2^i$$

where $w_2^i \in \overbrace{A \otimes \cdots \otimes A}^{p-2s-1 \text{ times}}, \quad |w_2^i| > (2n+1)(p-2s-1)$

$$(\text{for } r = 2s+1) \quad u_{p-2s-1} = [\mathcal{P}^{p-1}(\mathcal{P}^1 x \cdot x)]^s (\mathcal{P}^1 x \cdot x) \otimes \overbrace{x \otimes \cdots \otimes x}^{p-2s-2 \text{ times}} +$$

$$+ \sum_i v_1^i \otimes v_2^i$$

where $v_2^i \in \overbrace{A \otimes \cdots \otimes A}^{p-2s-2 \text{ times}}, \quad |v_2^i| > (2n+1)(p-2s-2)$.

If $r_0 = 0, \quad \mathcal{P}^1 u_p = \mathcal{P}^1(x \otimes \cdots \otimes x) = \sum_{i=1}^{p} x \otimes \cdots \otimes \overset{|\text{ith-place}}{\mathcal{P}^1 x} \otimes \cdots \otimes x =$ $= d_{p-1} \sum_{i=1}^{p-1}((-1)^{i+1} \cdot i) x \otimes \cdots \otimes \overset{|\text{ith-place}}{\mathcal{P}^1 x} \cdot x \otimes \cdots \otimes x$ and u_{p-1} exists having the desired form. If $r_0 = 2s_0 + 1, \quad s_0 \geq 0, \quad \mathcal{P}^1 u_{p-2s_0} = d_{p-2s_0-1} u_{p-2s_0-1}$ imply $d_{p-2s_0-1} \mathcal{P}^{p-1} u_{p-2s_0-1} = 0$. As $w(\mathcal{P}^{p-1} u_{p-2s_0-1}) = 1 \quad p > p - 2s_0 - 1 >$ $> 1, |\mathcal{P}^{p-1} u_{p-2s_0-1}| =$ odd, hence $\not\equiv 2 \ (\operatorname{mod} 2p)$, by 6.2 $\mathcal{P}^{p-1} u_{p-2s_0-1} \in$ $\in \operatorname{Im} d_{p-2s_0-2}$.

Now $\mathcal{P}^{p-1} u_{p-2s_0-1} = [\mathcal{P}^{p-1}(\mathcal{P}^1 x \cdot x)]^{s_0+1} \otimes \overbrace{x \otimes x \cdots \otimes x}^{p-2s-2 \text{ times}} +$ $+ \sum_i \tilde{v}_1^i \otimes \tilde{v}_2^i, \ |\tilde{v}_2^i| > (2n+1)(p-2s-2)$. One can easily argue that if $\mathcal{P}^{p-1} u_{p-2s_0-1} = d_{p-2s_0-2} u_{p-2s_0-2}$, then u_{p-2s_0-2} has the desired leading term $(\operatorname{mod} \ker d_{p-2s_0-2})$. If $r_0 = 2s_0, 0 < s_0 < \frac{p-1}{2}, \quad \mathcal{P}^{p-1} u_{p-2s_0+1} =$ $= d_{p-2s_0} u_{p-2s_0}$ and $d_{p-2s_0} \mathcal{P}^1 u_{p-2s_0} = 0$. As $p > p - 2s_0 > 1, \quad |\mathcal{P}^1 u_{p-2s_0}| =$ $=$ odd $\not\equiv 2 \ (\operatorname{mod} 2p)$, again $\mathcal{P}^1 u_{p-2s_0} \in \operatorname{Im} d_{p-2s_0-1}$. $\mathcal{P}^1 u_{p-2s_0} =$ $= [\mathcal{P}^{p-1}(\mathcal{P}^1 x \cdot x)]^s \mathcal{P}^1 x \otimes \overbrace{x \otimes \cdots \otimes x}^{p-2s-1 \text{ times}} + \sum_i \tilde{w}_1^i \otimes \tilde{w}_2^i \ |\tilde{w}_2^i| > (p-2s-1)(2n+1)$. Again one can see that $\mathcal{P}^1 u_{p-2s_0} = d_{p-2s_0-1} u_{p-2s_0-1}, u_{p-2s_0-1}$ has the desired leading term $(\operatorname{mod}\ \operatorname{Im}\ d_{p-2s_0-1})$.

Now, one can see by induction that u_{p-r} are uniquely determined mod $(\ker d_{p-r} = \operatorname{Im}\ d_{p-r-1}) + \operatorname{Im}\ \mathcal{P}^{a_r+1}$. Thus u_1 is determined up to image $\mathcal{P}^{a_2} = \mathcal{P}^1$ and $u_1 = [\mathcal{P}^{p-1}(\mathcal{P}^1 x \cdot x)]^{p-\frac{1}{2}} \cdot x = \varepsilon_p x \mathcal{P}^1 x \dots \mathcal{P}^{p-1} x$ mod $\operatorname{Im}\ \mathcal{P}^1$. ∎

7. Proof of the main theorem, part (1)

Fix p odd and $n \not\equiv -1 \pmod p$. We follow the notations of 2.0 (i) -(iv) and recall the observation that $H^*(B_r, B_{r-1}) \approx \underline{B}_r A$ where $A = H^*(K)$, $\underline{B}_r A \approx$ $\approx \overbrace{\Sigma A \otimes \cdots \otimes \Sigma A}^{r \text{ times}}(\ \Sigma A = \tilde{H}^*(\Sigma K)\ )$. The composition $B_{r+1}, B_r \xrightarrow{\delta_r} \Sigma B_r \xrightarrow{\Sigma j_r}$ $\Sigma B_r, B_{r-1}$ induces a morphism $H^*(B_r, B_{r-1}) \longrightarrow H^*(B_{r+1}, B_r)$ corresponding to the derivation in the cobar construction

$$d_r: \overbrace{\bar{A} \otimes \bar{A} \otimes \cdots \otimes \bar{A}}^{r \text{ times}} \longrightarrow \overbrace{\bar{A} \otimes \cdots \otimes \bar{A}}^{r+1}, \quad d_r(a_1 \otimes \cdots \otimes a_r) =$$

$$= \sum_{j=1}^{r} (-1)^j a_1 \otimes \cdots \otimes \bar{\mu}^* a_j \otimes \cdots \otimes a_r$$

where $\bar{\mu}^*: \bar{A} \longrightarrow \bar{A} \otimes \bar{A}$ is the reduced coproduct (note that $\bar{A} \approx \Sigma A$ with a shift of dimensions).

Now, starting of with $(D2)_p$ one observes that all maps (except possibly $\hat{u}_p$) commute with the self maps T induced by the λ-th power maps in K (and consequently in $\Sigma K, B_i, (B_i, B_{i-1})$) and by those in E_i, K_i. One can see that if $\ell: * \sim p^{p-1} \circ k_0, V_{k_0}: k_0 \circ T_{E_0} \sim T_{K_1} \circ k_0, V_{p^{p-1}}: p^{p-1} \circ T_{K_1} \sim T_{BK_2} \circ p^{p-1}$ are chosen to be the stable homotopies then $\alpha(\ell, V_{k_0}, V_{p^{p-1}}) = 0$.

By 5.2, 5.3 one can choose $W: k_0 \circ \tilde{u}_p \sim u_p \circ j_p$ so that $\hat{u}_p: \Sigma B_{p-1} \longrightarrow BK_2$ will satisfy $\hat{u}_p \circ T_{\Sigma B_{p-1}} \sim T_{BK_2} \circ \hat{u}_p$ (where the T's may have to be replaced by their p^t-th iteration).

The same observation will hold in all the following inductive steps: If $(D2)_r$ is given so that all maps (with the possible exception of $\hat{u}_r$) commute up to homotopy with the self maps T then one chooses the stable homotopies $\ell: * \sim$ $\sim p^{a_{p-r+1}} \circ k_{p-r}$, $V_{k_{p-r}}: k_{p-r} \circ T_{E_{p-r}} \sim T_{K_{p-r+1}} \circ k_{p-r}$, $V_{p^{a_{p-r+1}}}: p^{a_{p-r+1}} \circ$ $\circ T_{K_{p-r+1}} \sim T_{BK_{p-r+2}} \circ p^{a_{p-r+1}}$ and a suitable $W: k_{p-r} \circ \tilde{u}_r \sim u_r \circ j_r$ so that $\hat{u}_r$ induced by ℓ and W will satisfy $\hat{u}_r \circ T_{B_{r-1}} \sim T_{BK_{p-r+2}} \circ u_r$. Thus if one assumes inductively (induction on $p-r$) that $(D2)_r$ exists for all $p \geq r \geq r_0 \geq 3$ so that all maps commute up to homotopy with the self maps T one has to prove the following proposition in order to continue the inductive process:

7.1. Proposition. $\hat{u}_{r_0} \circ \Sigma i_{r_0-2} \sim *$.

If 7.1 is proved then $\hat{u}_{r_0}$ may be factored as

$$\Sigma B_{r_0-1} \xrightarrow{\Sigma j_{r_0-1}} \Sigma B_{r_0-1}, \Sigma B_{r_0-2} \xrightarrow{(u_{r_0-1})_\#} BK_{p-r_0+2}$$

and $(u_{r_0-1})_\#$ commutes with the T's (5.1). The same holds for the adjoint map $u_{r_0-1}: B_{r_0-1}, B_{r_0-2} \longrightarrow K_{p-r_0+2}$.

By induction $[k_{p-r_0} \circ \tilde{u}_{r_0}]$ represents $\Phi_{p-r_0}(\mathcal{P}^n \imath_{2n+2})$ and consequently $\Phi_{p-r_0+1}(\mathcal{P}^n \imath_{2n+2})$ is represented by $[(\hat{u}_{r_0})_\#] \in [B_{r_0-1}, K_{p-r_0+2}]$, in particular $(\hat{u}_{r_0})_\#$ factors as

$$B_{r_0-1} \xrightarrow{\tilde{u}_{r_0-1}} E_{p-r_0+1} \xrightarrow{k_{p-r_0+1}} K_{p-r_0+2} : \tilde{u}_{r_0-1}$$

is the lifting of $\tilde{u}_{r_0} \circ i_{r_0-1}$ induced by $\underline{L}u_{r_0} \circ \ell_0 + W \circ i_{r_0-1} : * \sim k_{p-r_0} \circ \tilde{u}_{r_0} \circ i_{r_0-1}$ where ℓ_0 is the standard homotopy $\ell_0 : * \sim j_{r_0} \circ r_{r_0}$. One can easily see that $\alpha(\ell_0) = 0$ and W was chosen to have $\alpha(W) = 0$, hence, by a dual of 5.3 $\tilde{u}_{r_0-1}$ commutes with the self maps. Applying 7.1 one obtains $(D2)_{r_0-1}$ as required. This inductive process yields diagrams $(D2)_r$ for $2 \le r \le p$. If one denotes the adjoint of $\hat{u}_2 : \Sigma B_1 \longrightarrow BK_p$ by $u_1 : B_1 \longrightarrow K_p$ (and as all maps are pointed $u_1 : B_1, B_0 \longrightarrow K_p$ where $B_0 = *$) then $[u_1]$ represents $\Phi_{p-1}(\mathcal{P}^n \imath_{2n+2})$. In this way one obtains a sequence of maps $[u_r] \in [B_r, B_{r-1};\ K_{p-r+1}] = = H^{2np+2+t_{p-r+1}}(B_r, B_{r-1})$ satisfying:

(i) $u_p = \imath_{2n+2} \otimes \cdots \otimes \imath_{2n+2}$,

(ii) $\mathcal{P}^{a_{p-r+1}}[u_r] = [u_{r-1} \circ \Sigma i_{r-1} \circ \delta_r]$.

One shifts dimensions and considers $\Sigma^{-r} u_r \in \overbrace{\bar{A} \otimes \cdots \otimes \bar{A}}^{r \text{ times}} = \underline{B}_r (\bar{A} = \overline{H}^*(K))$ then (i), (ii) are shifted to satisfy the hypothesis of 6.3 (u_r of mod p-1 algebraic weight 1 is equivalent to the property that u_r commutes with the self maps). Hence, $\Sigma^{-1} u_1 = \varepsilon_p \imath_{2n+1} \cdot \mathcal{P}^1 \imath_{2n+1} \ldots \mathcal{P}^{p-1} \imath_{2n+1}$ (mod Im $\mathcal{P}^1$). As Im $\mathcal{P}^1$ is part of the indeterminancy of Φ_{p-1} part (1) of the main theorem follows.

Proof of 7.1: First consider the composition:

$$\Sigma B_1 \xrightarrow{\Sigma i_{1,r_0-1}} \Sigma B_{r_0-1} \xrightarrow{\hat{u}_{r_0}} BK_{p-r_0+2} \quad (r_0 \ge 3).$$

Now

$$\tilde{H}^*(\Sigma B_1) \approx H^*(\Sigma B_1, \Sigma B_0) \approx \Sigma^2 \tilde{H}(K) \approx \Sigma^2 \bar{A}$$

and the image of $H^*(\Sigma i_{1,r_0-1})$ is $\Sigma^2 PA$. Now, $(\hat{u}_{r_0})_\# : B_{r_0-1} \longrightarrow K_{p-r_0+2}$ represents $\Phi_{p-r_0+1}(\mathcal{P}^n \imath_{2n+2})$, hence, $* \sim \mathcal{P}^{a_{p-r_0}} \circ \hat{u}_{r_0}$ and $\mathcal{P}^{a_{p-r_0}}[\hat{u}_{r_0} \circ \Sigma i_{1,r_0-1}] = = 0$, $[\hat{u}_{r_0} \circ \Sigma i_{1,r_0-1}] \in \Sigma^2 PA$. Let $u = \Sigma^{-2}[\hat{u}_{r_0} \circ \Sigma i_{1,r_0-1}] \in PA$ then u has weight 1 $(\bmod\, p-1)$ and its dimension is $2np + 1 + t_{p-r_0}$. By 4.3 $u = \mathcal{P}^{a_{p-r_0+1}} v$, $v \in PA$. Now the map $H^*(BK) = H^*(B_\infty) \longrightarrow H^*(B_1) \xrightarrow{\Sigma^{-1}} H^*(K)$ induces a surjection $PH^*(BK) \longrightarrow PH^*(K)$ and $\Sigma^2 v$ belongs to Im $H^*(\Sigma B_{r_0}) \longrightarrow H^*(\Sigma B_1)$, say $\Sigma^2 v = H^*(i_{1,r_0})\tilde{v}$. Thus one can alter $W : k_{p-r_0} \circ \tilde{u}_{r_0} \sim u_{r_0} \circ j_{r_0}$ by $\tilde{v} \in [\Sigma B_{r_0}, K_{p-r_0+1}]$ (as $\tilde{v}$ is of weight 1 it commutes with the self maps, by 5.2 $\alpha((-v) * W) = 0$) and $\tilde{u}_{r_0}$ is thus altered to obtain $\tilde{u}_{r_0} \circ \Sigma i_{1,r_0-1} \sim *$.

Suppose inductively that $\hat{u}_{r_0} \circ \Sigma i_{r,r_0-1} \sim *$ for some r, $1 \le r < r_0 - 2$. Then the composition $\Sigma B_r \to \Sigma B_{r+1} \to \Sigma B_{r_0-1} \xrightarrow{\hat{u}_{r_0}} Bk_{p-r_0+2}$ is null homotopic. We also assume that all maps commute with the self maps T.

By 5.1 (and by replacing the self maps by their p^t iterations, if necessary) $u_{r_0}\circ$ $\circ\Sigma i_{r+1,r_0-1}$ extends to a map $\Sigma B_{r+1},\Sigma B_r \xrightarrow{u'_r} BK_{p-r_0+2}$ which homotopically commutes with the self maps. The λ power map induces a multiplication by λ on the fundamental class of BK_{p-r_0+2}, the fact that u'_r commutes with the self maps is equivalent to the fact that $[u'_r]\in H^*(\Sigma B_{r+1},\Sigma B_r)$ is a λ characteristic vector of $H^*(T_{\Sigma B_{r+1},\Sigma B_r})$, hence it is a class of mod $p-1$ algebraic weight 1. $r<r_0-2$ implies that the composition $B_{r+1},B_r \xrightarrow{\delta_{r+1}} \Sigma B_{r+1}\longrightarrow B_{r_0-1}$ is null homotopic, hence, $\hat{u}'_r\circ\Sigma j_{r+1}\circ\delta_{r+1}\sim\hat{u}_{r_0}\circ\Sigma i_{r+1,r_0-1}\circ\delta_{r+1}\sim *$, $d_{r+1}[\Sigma^{-1}\hat{u}'_r]=0$ in the cobar construction. As $p-1\geq r+1\geq 2$ and $|\Sigma^{-1}\hat{u}'_2|=2np+2+t_{p-r_0+2}\not\equiv 2 \mod 2p$ by 6.2 $\Sigma^{-1}\hat{u}'_r\in \operatorname{Im} d_r$, $\hat{u}'_r\sim$ $\sim\hat{\hat{u}}'_r\circ\Sigma^2 j_r\circ\Sigma\delta_{r+1}$,

$$\begin{tikzcd}
\Sigma B_{r+1},\Sigma B_r \arrow[r, "\Sigma\delta_{r+1}"] \arrow[rrr, bend right, "\hat{u}'_r"'] & \Sigma^2 B_r \arrow[r, "\Sigma j_r"] & \Sigma^2 B_r,\Sigma^2 B_{r-1} \arrow[r, "\hat{\hat{u}}'_r"] & BK_{p-r_0+2}
\end{tikzcd}$$

hence, $\hat{u}_{r_0}\circ\Sigma i_{r+1,r_0-1}\sim\hat{u}'_r\circ\Sigma j_{r+1}\sim\hat{\hat{u}}'_r\circ\Sigma^2 j_r\circ\Sigma\delta_{r+1}\circ\Sigma j_{r+1}\sim *$. By induction $\hat{u}_{r_0}\circ\Sigma i_{r,r_0-1}\sim *$ for all $r\leq r_0-2$, $\hat{u}_{r_0}\circ\Sigma i_{r_0-2,r_0-1}=\hat{u}_{r_0}\circ\Sigma i_{r_0-2}\sim$ $\sim *$ and 7.1 follows. ■

8. The main theorem (part (2)) and applications

The proof of the main theorem (2) (see 8.3) is a consequence of some applications of the main theorem part (1).

8.1. Proposition. *Let λ be a primitive root of unity* mod p. *Let X,T be a $Q-\lambda$ power space, i.e. $T\colon X\longrightarrow X$ induces multiplication by λ on the quotient module $QH^*(X)$ of algebra generators. If for some n, $n\not\equiv -1$* mod p, *$QH^i(X)=0$ for all $i\not\equiv 2n+1$* mod $p-1$ *and if $x\in H^{2n+1}(X)$ is an indecomposable λ characteristic vector of $H^*(T)$ then $\Phi_{p-1}(\mathcal{P}^n x_{2n+1})=$ $=\varepsilon_p x\cdot\mathcal{P}^1 x\ldots\mathcal{P}^{p-1}x+\operatorname{Im}\ \mathcal{P}^1+\bigoplus_{i\neq 1}W_i$ where $W_i\subset H^*(X)$ is the λ^i eigenspace, $1\leq i\leq p-1$. (One may assume that $H^m(T)$ has a diagonal form for every m).*

Proof: ΣT induces a mod p splitting of ΣX, $\Sigma X\approx_p\bigvee_{i=1}^{p-1}Y_i$, $H^*(Y_i)=$ $=\Sigma W_i$. (Y_{i_0} is the mapping telescope of $\prod_{i\neq i'_0}(\Sigma T-\lambda^i)$). $\mathcal{P}^n x$ is obviously in the domain of Φ_{p-1}. A representative of $\Phi_{p-1}(\mathcal{P}^n x)$ could be obtained by $X\xrightarrow{x}K(\mathbf{Z}/p\mathbf{Z},2n+1)\xrightarrow{(u_1)_\#}E_{p-1}\xrightarrow{k_p}K_p$, hence, by the main theorem (1) $\varepsilon_p x\cdot\mathcal{P}^1 x\ldots\mathcal{P}^{p-1}x\in\Phi_{p-1}(\mathcal{P}^n x)$. Let $:X\xrightarrow{f}E_{p-1}\xrightarrow{k_{p-1}}$ $\xrightarrow{k_{p-1}}K_p$ be any other representative of $\Phi_{p-1}(\mathcal{P}^n x)$, hence the composition $X\xrightarrow{f}E_{p-1}\xrightarrow{h_1\circ\cdots\circ h_{p-1}}K(\mathbf{Z}/p\mathbf{Z},2np+1)$ is $\mathcal{P}^n x$. Denote $f=(u_1)_\#\circ x$ then

$\hat f_\# - f_\# : \Sigma X \longrightarrow BE_{p-1}$ satisfies $B(h_1 \circ \cdots \circ h_{p-1}) \circ (\hat f_\# - f_\#) \sim *$ and $\hat f_\# - f_\#$ lifts to $\Sigma X \longrightarrow B\hat E_{p-1}$. ($\hat E_{p-1}$–the homotopy fiber of $h_1 \circ \ldots h_{p-1}$). Now one can easily see that $\pi_{2np+1+i}(\hat E_{p-1}) = 0$ for $i \equiv 0\,(p-1), 0 < i < t_{p-1} - 1$. On the other hand $W_1 \subset \bigoplus_s H^{2n+1+(p-1)s}(X)$ (as W_1 is spanned by monomials in the indecomposable λ-eigenvectors of length $\equiv 1 \mod p-1$). It follows that the map $\hat j_{p-1} : \Omega K_{p-1} \longrightarrow \hat E_{p-1}$ (the lifting of j_{p-1} in (D1)) induces a surjection $[Y_1, K_{p-1}] \longrightarrow [Y_1, B\hat E_{p-1}]$. Hence, $Y_1 \xrightarrow{i_1} \Sigma X \xrightarrow{\hat f_\# - f_\#} BE_{p-1} \xrightarrow{Bk_{p-1}} BK_p$ factors as $Y_1 \xrightarrow{w} K_{p-1} \xrightarrow{\mathcal{P}^1} BK_p$, and $[k_p \circ \hat f] - [k_p \circ f] = \mathcal{P}^1(\Sigma^{-1}\omega) + \bigoplus_{i\neq 1} W_i$. Hence $[k_p \circ \hat f] = \epsilon_p x \cdot \mathcal{P}^1 x \ldots \mathcal{P}^{p-1}x + \mathrm{Im}\ \mathcal{P}^1 + \bigoplus_{i \neq 1} W_i$. ■

8.2. Example. Let $X = B(2n+1)$ be the $2n$ connected Wilson's irreducible factor of the $2n$ connected element of the Ω-spectrum of BP. If $n \not\equiv -1 \mod p$ then for the fundamental class $x \in H^{2n+1}(X)$ one has $\epsilon_p x \cdot \mathcal{P}^1 x \ldots \mathcal{P}^{p-1} x \notin$ $\notin \mathrm{Im}\ \mathcal{P}^1 + \bigoplus_{i\neq 1} W_i$ hence, by 8.1 $0 \notin \Phi_{p-1}(\mathcal{P}^n x)$.

8.3. Proof of the main theorem (part(2)): $0 \in \Phi_{p-1}\mathcal{P}^n \iota_{2n+1}$ will contradict 8.2. ■

8.4. Corollary. *Let X, T, n, x be as in 8.1. If $0 \in \Phi_{p-1}(\mathcal{P}^n x_{2n+1})$ (in particular if $\mathcal{P}^n x_{2n+1} = 0$) then $x \cdot \mathcal{P}^1 x \ldots \mathcal{P}^{p-1}x \in \mathrm{Im}\ \mathcal{P}^1$.*

Proof: By 8.1, $0 \in \Phi_{p-1}(\mathcal{P}^n x) = \epsilon_p x \mathcal{P}^1 x \ldots \mathcal{P}^{p-1}x + \mathrm{Im}\ \mathcal{P}^1 + \bigoplus_{i\neq 1} W_i =$ $= \epsilon_p x \mathcal{P}^1 x \ldots \mathcal{P}^{p-1} x + \mathcal{P}^1 W_1 + \bigoplus_{i \neq 1} W_i$. As $x\mathcal{P}^1 x \ldots \mathcal{P}^{p-1}x \in W_1$, $\mathcal{P}^1 W_1 \subset W_1$ and $W_1 \cap \bigoplus_{i\neq 1} W_i = 0$ one has $x\mathcal{P}^1 x \ldots \mathcal{P}^{p-1} x \in \mathcal{P}^1 W_1$. ■

8.5. Corollary. *There is no Q-$\overline{\lambda}$ power space X having $H^*(X) =$ $= \wedge(x, \mathcal{P}^1 x, \ldots, \mathcal{P}^{p-1}x)$, $\dim X = 2n+1$, $n \not\equiv -1 \pmod p$.*

Proof: $\mathcal{P}^n x = 0$ by hypothesis but there are no elements w with $\mathcal{P}^1 w =$ $= x \cdot \mathcal{P}^1 x \ldots \mathcal{P}^{p-1}x$ to satisfy the conclusions of 8.4 ■

8.5.1. Remark. If n is large enough ($n > \frac{1}{2}(p^2 - p - 1)(p-1)$) then there is no space (not necessarily a power space) with $H^*(X) = \wedge(x, \mathcal{P}^1 x, \mathcal{P}^2 x, \ldots, \mathcal{P}^{p-1}x)$. This follows from the fact that $H^m(X) = 0$ for $m \not\equiv 2n+1 \pmod{p-1}$ in the relevant range (as in the proof of 8.1), hence, Im $\mathcal{P}^1$ is the total indeterminancy. In [3] a mod p ($p \geq 5$) H–space is constructed with $H^*(X) = \wedge(x, \mathcal{P}^1 x, \ldots$ $\ldots, \mathcal{P}^p x)$, $|x| = 2p+1$.

8.6. Corollary. *There is no stable map $f : X \longrightarrow \Omega^\infty \Sigma^\infty S^{2p^2+1}$ with $\mathcal{P}^p x \in$ $\in \mathrm{Im}\ H^*(f)$.*

Proof: As Φ_{p-1} is stable $0 \in \Phi_{p-1}(u)$ where $0 \neq u \in H^{2p^2+1}(\Omega^\infty \Sigma^\infty S^{2p^2+1})$. A map f with $H^*(f)u = \mathcal{P}^p x$ will yield (by 8.4) $x \cdot \mathcal{P}^1 x \ldots \mathcal{P}^{p-1} x \in \mathrm{Im}\ \mathcal{P}^1$ which is impossible. ■

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J. Harper: Department of Mathematics
The University of Rochester, U.S.A.

A. Zabrodsky: Institute of Mathematics
Hebrew University of Jerusalem, ISRAEL.

Rebut el 20 d'Octubre de 1986