The Infinite as Method in Set Theory and Mathematics

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Resumen. *El infinito como método en la teoría de conjuntos y la matemática*

Este artículo da cuenta de la aparición histórica de lo infinito en la teoría de conjuntos, y de cómo lo tratamos dentro y fuera de las matemáticas. La primera sección analiza el surgimiento de lo infinito como una cuestión de método en la teoría de conjuntos. La segunda sección analiza el infinito dentro y fuera de las matemáticas, y cómo deben adoptarse.

Palabras clave: infinito, la teoría de conjuntos, matemáticas.

Abstract

This article address the historical emergence of the infinite in set theory, and how we are to take the infinite in and out of mathematics.The first section discusses the emergence of the infinite as a matter of method in set theory. The second section discusses the infinite in and out of mathematics, and how it is to be taken.

Key words: infinite, set theory, mathematics.

The infinite, of course, is a large topic. At the outset, one can historically discern two overlapping clusters of concepts: (1) wholeness, completeness, universality, absoluteness. (2) endlessness, boundlessness, indivisibility, continuousness. The first, *the metaphysical infinite*, I will set aside. It is the second, *the mathematical infinite*, that I will address. Furthermore, I will address this by considering the historical emergence of the infinite in set theory, and how we are to take the infinite in and out of mathematics. Insofar as physics and, more broadly, science deals with the mathematical infinite through mathematical language and techniques, my remarks should be subsuming and consequent.

The main underlying point will be that how the mathematical infinite is approached, assimilated, and applied in mathematics is not a matter of "ontological commitment", of coming to terms with whatever that might mean, but rather of epistemological articulation, of coming to terms through knowledge. *The mathematical infinite in mathematics is a matter of method.* How we deal with the specific individual issues involving the infinite turns on the narrative we present about how it fits into mathematical frameworks established and being established.

The first section discusses the emergence of the infinite as a matter of method in set theory. The second section discusses the infinite in and out of mathematics, and how it is to be taken.

1. The Emergence of the Infinite in Mathematics

What role does the infinite play in modern mathematics? In modern mathematics, infinite sets abound both in the workings of proofs and as subject matter in statements, and so do universal statements, often of $\forall\exists$ form, which are indicative of direct engagement with the infinite. In many ways the role of the infinite is importantly "second-order" in the sense that Frege regarded number generally, in that the concepts of modern mathematics are understood as having infinite instances over a broad range.

But all this has been the case for only about a century. Infinite totalities and operations on them only emerged in mathematics in a recent period of algebraic expansion and rigorization of proof. It becomes germane, even crucial, to see how the infinite emerged through interaction with proof to come to see that the infinite in mathematics is a matter of method. If one puts the history of mathematics through the sieve of proof, one sees the emergence of methods drawing in the mathematical infinite, and the mathematical infinite came in at three levels: the countably infinite, the infinite of the natural numbers; the continuum, the infinite of analysis; and the empyrean infinite of higher set theory.

As a thematic entrée into the matter of proof and the countably infinite, we can consider the Pigeonhole Principle:

> If n pigeons fly into fewer than n pigeonholes, then one hole has more than one pigeon.

Taken primordially, this may be considered immediate as part of the meaning of the natural numbers and requires no proof. On the other hand, after its first explicit uses in algebraic number theory in mid-19th Century, it has increased in prominence in modern combinatorics, its consequences considered substantive and at times quite surprising given its immediacy. Rendered as a $\forall\exists$ statement about natural numbers, it however does not have the feel of a basic law of arithmetic but of a "non-constructive" existence assertion, and today it is at the heart of combinatorics, and indeed is the beginning of Ramsey Theory, a field full of non-constructive existence assertions.

So how does one prove the Pigeonhole Principle? For 1729 pigeons and 137 pigeonholes one can systematically generate all assignments from {1,...,1729} to {1,...,137} and check that there are always at least two pigeons assigned to the same pigeonhole. But we "see" nothing here, nor from any other particular brute force analysis. With the Pigeonhole Principle seen afresh as being at the heart of the articulation of finite cardinality and requiring proof based on prior principles, Richard Dedekind in his celebrated 1888 essay Was sind und was sollen die Zahlen? (120) first gave a proof applying mathematical induction on n. Today, the Pigeonhole Principle is regarded as a theorem of Peano Arithmetic (PA). In fact, in the presence of the elementary axioms of PA, the Pigeonhole Principle is equivalent to the central Principle of Induction. Moreover, the proof complexity of weak forms of the Pigeonhole Principle have been investigated in weak systems of arithmetic.1

This raises a notable historical point drawing in the infinite. The Pigeonhole Principle seems to have been first applied in mathematics by Gustav Lejeune Dirichlet in papers of 1842, one to the study of Pell's equation and another to establish a crucial approximation lemma for his well-known Unit Theorem describing the group of units of an algebraic number field.[2] The principle occurred in Dirichlet's *Vorlesungen über Zahlentheorie*, edited and published after his death by Dedekind in 1863. The occurrence is in the second, 1871 edition, in a short Supplement VIII by Dedekind on Pell's equation, and it was in the famous Supplement X that Dedekind laid out his theory of ideals in algebraic number theory, in which he worked directly with infinite totalities. In 1872 Dedekind was putting together *Was sind*, and he would be the first to define "infinite set", with the definition being a set for which there is a one-to-one correspondence with a proper subset. This is just the negation of the Pigeonhole Principle. Dedekind in effect had inverted a negative aspect of finite cardinality into a positive existence definition of the infinite.

The Pigeonhole Principle example brings out a crucial point about method. The proof by induction is an example of what Hilbert called formal induction. Insofar as the natural numbers do have an antecedent sense, a universal statement $\forall\nu\phi(\nu)$ about them should be correlated with all the informal counterparts $\phi(0),\phi(1),\phi(2),\ldots$ taken together. We have seen that, at least for existence statements $\phi(\nu)$, this may become intensionally problematic for particular instantiations. Contra Poincaré, Hilbert in his 1928 distinguished between contentual [inhaltlich] induction, the intuitive construction of each integer as numeral,3 and formal induction, by which $\forall n f(n)$ follows immediately from the two statements f(0) and $\forall\nu(\phi(\nu) \rightarrow \phi(\nu+1))$ and "through which alone the mathematical variable can begin to play its role in the formal system." In the case of the Pigeonhole Principle, we see the proof by formal induction, but it bears little constructive relation to any particular

1 See for example Maciel-Pitassi-Woods (2002).

2 See Dirichlet 1889/97, pp.579,636. The principle in the early days was called the Schubfachprinzip (``drawer principle''), though not however by Dirichlet. The second, 1899 edition of Heinrich Weber's Lehrbuch der Algebra used the words ``in Faecher verteilen'' (``to distribute into boxes'') and Edmund Landau's 1927 Vorlesungen über Zahlentheorie had ``Schubfachschluss''.

3 For Hilbert, the numeral for the integer n consists of n short vertical strokes concatenated together.

instance. Be that as it may, the schematic sense of the countably infinite, the infinite of the natural numbers, is carried in modern mathematics by formal induction, a principle used everywhere in combinatorics and computer science to secure statements about the countably infinite. There is no larger mathematical sense to the Axiom of Infinity in set theory other than to provide an extensional counterpart to formal induction, a method of proof. The Cantorian move against the traditional conception of the natural numbers as having no end in the "after" sense is neatly rendered by extensionalizing induction itself in modern set theory with the ordinal ω, with "after" recast as membership.

The next level of the mathematical infinite would be the continuum, the infinite of mathematical analysis. Bringing together the two traditional Aristotlean infinities of infinite divisibility and of infinite progression, one can ask: How many points are there on the line? This would seem to be a fundamental, even primordial, question. However, to cast it as a mathematical question, underlying concepts would have to be invested with mathematical sense and a way of mathematical thinking provided that makes an answer possible, if not informative. First, the real numbers as representing points on the linear continuum would have to be precisely described. A coherent concept of cardinality and cardinal number would have to be developed for infinite mathematical totalities. Finally, the real numbers would have to be enumerated in such a way so as to accommodate this concept of cardinality. Georg Cantor made all of these moves as part of the seminal advances that have led to modern set theory. His Continuum Hypothesis would propose a specific, structured resolution about the size of the continuum in terms of his transfinite numbers, a resolution that would become pivotal where set-theoretic approaches to the continuum became prominent in mathematical investigations.

Set theory had its beginnings in the great 19th Century transformation of mathematics, a transformation beginning in analysis. With the function concept having been steadily extended beyond analytic expressions to infinite series, sense for the new functions could only be developed through carefully specific deductive procedures. Proof reemerged in mathematics as an extension of algebraic calculation and became the basis of mathematical knowledge, promoting new abstractions and generalizations. The new articulations to be secured by proof and proof in turn to be based on prior principles, the regress lead in the early 1870s to the appearance of several definitions of the real numbers, of which Cantor's and Dedekind's are the best known. It is at first quite striking that the real numbers as a totality came to be developed so late, but this can be viewed against the background of the larger conceptual shift from intensional to extensional mathematics, that is, from rules to objects. Infinite series outstripping sense, it became necessary to adopt an arithmetical view of the continuum given extensionally as a totality of points.

Cantor's definition of the real numbers appeared in his seminal paper *1872* on trigonometric series; proceeding in terms of fundamental sequences, he laid the basis for his theorems on sequential convergence. Dedekind in his *1872* formulated the real numbers in terms of his cuts to express the completeness of the continuum; deriving the least upper bound principle as a simple consequence, he thereby secured the basic properties of continuous functions. In the use of arbitrary sequences and infinite totalities, both Cantor's and Dedekind's objectifications of the continuum helped set the stage for the subsequent development of that extensional mathematics par excellence, set theory. Cantor's formulation was no idle conceptualization, but to the service of specific mathematics, the articulation of his results on uniqueness of trigonometric series involving his derived sets, the first instance of topological closure. Dedekind describes how he came to his formulation much earlier, but also acknowledges Cantor's work. In modern terms, Cantor's reals are equivalence classes according to an equivalence relation which importantly is a congruence relation, a relation that respects the arithmetical structure of the reals. It is through Cantor's definition that completeness would be articulated for general metric spaces, thereby providing the guidelines for proof in new contexts involving infinite sets.

Set theory was born on that day in December 1873 when Cantor established that *the continuum is not countable:* There is no one-to-one correspondence between the natural numbers N = {0,1,2,3,...} and the real numbers R. Like the irrationality of the square root of 2, the uncountability of the continuum was an impossibility result established via *reductio ad absurdum* that opened up new possibilities. Cantor addressed a specific problem, embedded in the mathematics of the time, in his seminal paper *1874* entitled "On a property of the totality of all real algebraic numbers". Dirichlet's algebraic numbers, it will be remembered, are the roots of polynomials with integer coefficients; Cantor established the countability of the algebraic numbers. This was the first substantive correlation of an infinite totality with the natural numbers, and it was the first application of what now goes without saying, that finite words based on a countable alphabet are countable. Cantor then established: *For any (countable) sequence of reals, every interval contains a real not in the sequence.* The following is Cantor's argument, in brief:

> Suppose that s is a sequence of reals and I an interval. Let $a < b$ be the first two reals of s, if any, in I. Then let $a' < b'$ be the first two reals of s, if any, in the open interval (a,b); $a'' < b''$∃ the first two reals of s, if any, in (a',b'); and so forth. Then however long this process continues, the intersection of the nested intervals must contain a real not in the sequence s.

Cantor went on, of course, to develop his concept of cardinality based on one-to-one correspondence. Having made the initial breach with a negative result about the lack of a one-to-one correspondence, he turned infinite cardinality into a positive concept and investigated the possibilities for one-to-one correspondences. Just as the discovery of the irrational numbers had led to one of the great achievements of Greek mathematics, Eudoxus's theory of geometric proportions, Cantor began his move toward a full-blown mathematical theory of the infinite. Cantor notably established that there is a one-to-one correspondence between the line and the plane, and generally R^n, and this stimulated the study of dimension and the possibilities of continuous correspondences eventually leading to the Brouwer Fixed Point Theorem. By his 1878 *Beitrag* Cantor had come to the focal Continuum Hypothesis. In his 1883 *Grundlagen* Cantor developed the transfinite numbers and the key concept of well-ordering, in significant part to take a structured approach to infinite cardinality and the Continuum Hypothesis. He developed the number classes by bundling the transfinite numbers together according to cardinality, and then propounded this basic well-ordering principle: "It is always possible to bring any well-defined set into the form of a well-ordered set." Sets are to be well-ordered, and they and their cardinalities are to be gauged via the transfinite numbers of his structured conception of the infinite.

Almost two decades after his *1874* Cantor in a short note *1891* gave his now celebrated diagonal argument, establishing Cantor's Theorem: For any set X the totality of functions from X into a fixed two-element set has a larger cardinality than X, i.e. there is no one-to-one correspondence between the two. This result generalized his 1874 result that the continuum is not countable, since the totality of functions from N into a fixed two-element set has the same cardinality as R. In retrospect the diagonal argument can be drawn out from the *1874* proof:

> Starting with a sequence s of reals and a half-open interval I_0, instead of successively choosing delimiting pairs of reals in the sequence, avoid the members of s one at a time: Let I_1 be the left or right half-open subinterval of I_0 demarcated by its midpoint, whichever does not contain the first element of s. Then let I_2 be the left or right half-open subinterval of I_1 demarcated by its midpoint, whichever does not contain the second element of s; and so forth. Again, the nested intersection contains a real not in the sequence s. Abstracting the process in terms of reals in binary expansion, one is just generating the binary digits of the diagonalizing real.[4]

Cantor had been shifting his notion of set to a level of abstraction beyond sets of reals and the like, and the casualness of his 1891 may reflect an underlying cohesion with his 1874. Whether the new proof is really "different" from the earlier one, through this abstraction Cantor could now dispense with the recursively defined nested sets and limit construction, and he could apply his argument to any set. He had proved for the first time that there is a cardinality larger than that of R, and moreover affirmed "the general theorem, that the powers of well-defined sets have no maximum." Thus, Cantor for the first time entertained

4. Cantor first gave a proof of the uncountability of the reals in a letter to Dedekind of 7 December 1873 (Ewald 1996 pp.845ff), professing that ". . . only today do I believe myself to have finished with the thing . . .". It is remarkable that in this letter already appears a doubly indexed array of real numbers and a procedure for traversing the array downward and to the right, as in a now common picturing of the diagonal proof.

the third level of the mathematical infinite, the empyrean level, beyond the continuum, of higher set theory.

Nowadays it goes without saying that each function from a set X into a two-element set corresponds to a subset of X, so Cantor's Theorem is usually stated as: For any set X its power set, the totaity of all of its subsets, has a larger cardinality than X. However, it would be an exaggeration to assert that Cantor at this point was working on power sets; rather, he was expanding the 19th Century concept of function by ushering in arbitrary functions. At the end of his 1891 Cantor had dealt explicitly with "all" functions with a specified domain X and range {0,1}; regarded these as being enumerated by one super-function $\Phi(x,z)$ with enumerating variable z; and formulated the diagonalizing function $g(x) = 1 - \Phi(x,x)\exists$. This argument, even to its notation, would become method, flowing into descriptive set theory, the Gödel Incompleteness Theorem, and recursion theory, the paradigmatic means of transcendence over an established context.

The first decade of the new century saw Ernst Zermelo make his major advances in the development of set theory. In his 1904 Zermelo analyzed Cantor's well-ordering principle by reducing it to the Axiom of Choice (AC), the abstract existence assertion that every set X has a choice function, i.e. a function f such that for every non-empty $Y \in X$, $f(Y) \in Y$. Zermelo thereby shifted the notion of set away from Cantor's principle that every well-defined set is well-orderable and replaced that principle by an explicit axiom. His Well-Ordering Theorem showed specifically that a set is well-orderable exactly when its power set has a choice function. How AC brought to the fore issues about the non-constructive existence of functions is well-known, and how AC became increasingly accepted in mathematics has been well-documented. The expansion of mathematics into large abstract contexts was navigated with axioms and proofs, and this led to more and more appeals to AC.

In his *1908* Zermelo published the first full-fledged axiomatization of set theory, partly to establish set theory as a discipline free of the emerging paradoxes and particularly to put his Well-Ordering Theorem on a firm footing. In addition to codifying generative set-theoretic principles, a substantial motive for Zermelo's axiomatizing set theory was to buttress his Well-Ordering Theorem by making explicit its underlying set existence assumptions.[5] Initiating the first major transmutation of the notion of set after Cantor, Zermelo thereby ushered in a new abstract, prescriptive view of sets as structured solely by membership and governed and generated by axioms, a view that would soon come to dominate. Thus, proof played a crucial role by stimulating an axiomatization of a field of study and a corresponding transmutation of its underlying notions.

In the tradition of Hilbert's axiomatization of geometry, Zermelo's axiomatization was in the manner of an implicit definition, with axioms providing rules for procedure and

5. Moore 1982 pp.155ff supports this contention using items from Zermelo's Nachlass.

generating sets and thereby laying the basis for proofs. Fully two decades earlier Dedekind in his essay *Was sind* had provided certain rules for dealing with his sets [Systeme], and several overlapping aspects can serve as points of departure for Zermelo's axiomatization. Both Dedekind and Zermelo set down rules for sets in large part to articulate arguments involving simple set operations like "set of", union, and intersection. In particular, both had to argue for the equality of sets resulting after involved manipulations, and extensionality became operationally necessary. However vague the initial descriptions of sets, sets are to be determined solely by their elements, and the membership question is to be determinate.[6] The looseness of Dedekind's description of sets allowed him in *Was sind* (66) the latitude as it were to "prove" the existence of infinite sets, but Zermelo just stated the Axiom of Infinity as a set existence principle.

The main point of departure has to do with the larger issue of the role of proof for articulating sets. Like algebraic constructs, sets were new to mathematics and would be incorporated by setting down the rules for their proofs. Just as calculations are part of the sense of numbers, so proofs would be part of the sense of sets, as their "calculations". Just as Euclid's axioms for geometry had set out the permissible geometric constructions, the axioms of set theory would set out the specific rules for set generation and manipulation. But unlike the emergence of mathematics from marketplace arithmetic and Greek geometry, sets and transfinite numbers were neither laden nor buttressed with substantial antecedents. Like strangers in a strange land stalwarts developed a familiarity with them guided hand in hand by their axiomatic framework. For Dedekind in *Was sind* it had sufficed to work with sets by merely giving a few definitions and properties, those foreshadowing Extensionality, Union, and Infinity. Zermelo provided more rules: Separation, Power Set, and Choice.

The standard axiomatization ZFC was completed by 1930, with the Axiom of Replacement brought in through the work of von Neumann and the Axiom of Foundation, through axiomatizations of Zermelo and Bernays. The Cantorian transfinite is contextualized by von Neumann's incorporation of his ordinals and the Axiom of Replacement, which underpins transfinite induction and recursion as methods of proof and definition. And Foundation provides the basis for applying transfinite recursion and induction procedures to get results about all sets, they all appearing in the cumulative hierarchy. One is simply extending methods through limit points, and the seeming exacerbation of the breach into the actual infinite amounts to new rules and methods for proofs. From this point of view, the Axiom of Choice can be regarded as a similarly necessary principle for infusing the contextualized transfinite with the order already inherent in the finite.

6. Dedekind in Was sind begins a footnote to his statement about extensional determination with: "In what manner this determination is brought about, and whether we know a way of deciding upon it, is a matter of indifference for all that follows; the general laws to be developed in no way depend upon it; they hold under all circumstances."

2. The Infinite In and Out of Mathematics

It is evident that mathematics has been much inspired by problems and conjectures and has progressed autonomously through the communication of proofs and the assimilation of methods. Being socially and historically contingent, mathematics has advanced when individuals could collectively make mathematics out of concepts, whether they involve infinite totalities or not. The commitment to the infinite is to what is communicable about it, to the procedures and methods in articulated contexts, to language and argument. Infinite sets are what they do, and their sense is carried in the methods we collectively employ on their behalf.

When considering the infinite as method in modern mathematics and its relation to the primordial mathematical infinite, there is a deep irony about mathematical objects and their existence. Through the rigor and precision of modern mathematics, mathematical objects achieve a sharp delineation in mathematical practice as founded on proof. The contextual objectification then promotes, perhaps even urges, some larger sense of reification. Or, there is confrontation with some prior held belief or sense about existence that then promotes a skeptical attitude about what mathematicians do and prove, especially about the infinite. Whether mathematics inadvertently promotes realist attitudes or not, the applicability of mathematics to science should not extend to philosophy if the issues have to do with existence itself, for again, existence in mathematics is contextual and governed by rules and procedures, and metaphysical existence, especially concerning the infinite, does not inform and is not informed by mathematical work.

Mathematicians themselves are prone to move in and out of mathematics in their existential assessments, stimulated by their work and the urge to put a larger stamp of significance to it. We quoted Hilbert above, and he famously expressed larger metaphysical views about finitism and generated a program to establish the overall consistency of mathematics. In set theory, Cantor staked out the Absolute, which he associated with the transcendence of God, and applied it in the guise of the class of all ordinal numbers for delineative purposes. Gödel attributed to his conceptual realism about sets and his philosophical standpoint generally his relative consistency results with the constructible universe. What are we to make of what mathematicians say and their motivations? Despite the directions in which mathematics has been led by individuals' motivations it is crucial to point out, again, that what has been retained and has grown in mathematics is communicable as proofs and results. We should assess the role of the individual, but keep in mind the larger autonomy of mathematics. A delicate but critical point here is that, as with writers and musicians, we should be dispassionate, sometimes even skeptical, about what mathematicians say about their craft. As with the surface Platonism often espoused by mathematicians, we must distinguish what is said from what is done.

The interplay between philosophical views of individual mathematicians, historically speaking, and the space of philosophical possibility, both in their times and now, is what needs to be explored.

There is one basic standpoint about the infinite which seems to underly others and leads to prolonged debates about the "epistemic". This is the (Kantian) standpoint of human finitude. We are cast into a world which as a whole must be infinite, yet we are evidently finite, even to the number of particles that makes us up. So how can we come to know the infinite in any substantive way? This long-standing attitude is part of a venerable tradition, and to the extent that we move against it our approach may be viewed as bold and iconoclastic. Even phenomenologically, we see before us mathematicians working coherently and substantially with infinite sets and concepts. The infinite is embedded in mathematics as method, we can assimilate methods, and we use the infinite through method in proofs. Even those mathematicians who would take some sort of metaphysical stance against the actuality of the infinite in mathematics can nevertheless follow and absorb a proof by mathematical induction.

There is a final, large point in this direction. As mathematics has expanded with the incorporation of the infinite, several voices have advocated the restriction of proof procedures and methods. Brouwer and Weyl were early figures and Bishop, a recent one. How are we to take all this? We now have a good grasp of intuitionistic and constructivist approaches to mathematics as various explicit, worked-out systems. We also have a good understanding of hierarchies of infinitistic methods through quantifier complexity, proof theory, reverse mathematics and the like. Commitments to the infinite can be viewed as the assimilation of methods along hierarchies. Be that as it may, an ecumenical approach to the infinite is what seems to be called for: There is no metamathematics, in that how we are able to argue about resources and methods is itself mathematical. As restrictive approaches were advocated, they themselves have been brought into the fold of mathematics, the process itself having mathematical content. Proofs about various provabilities are themselves significant proofs. It is interesting to carry out a program to see how far a strictly finitist or predicativist approach to mathematics can go, not to emasculate mathematics or to tout the one true way, but to find new, informative proofs and to gain an insight into the resources at play, particularly with regard to commitments to infinity.

Stepping back, the study of the infinite in mathematics urges us to develop a larger ecumenicism about the role of the infinite. Like the modern ecumenical approach to proofs in all their variety and complexity, proofs about resources provide new mathematical insights about the workings of method. Even then, in relation to later "elementary" proofs or formalized proofs in an elementary system of a statement, prior proofs may well retain an irreducible semantic content. In this content reside robustly aspects of the infinite, displaying the autonomy of mathematics as an evolving practice.

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