EULERIAN NUMBERS AND OPERATORS

by

L. Carlitz

1. INTRODUCTION. The Eulerian numbers $A_{n,k}$ are usually introduced by means of [1], [6, Ch. 8]

\[
1 - \frac{\lambda}{1 - \lambda e^{i - k})x} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=1}^{n} A_{n,k} x^k.
\]

It follows from (1.1) that

\[
A_{n+1,k} = (n - k + 2) A_{n,k-1} + k A_{n,k}
\]

and

\[
A_{n,k} = A_{n,n-k+1} \quad (1 \leq k \leq n).
\]

It is evident from (1.2) and $A_{1,1} = 1$ that the $A_{n,k}$ are positive integers for $n \geq k \geq 1$.

The symmetry relation (1.3) is by no means obvious from the generating function (1.1). This has motivated the introduction of the symmetric notation [3]

\[
A(r, s) = A_{r+s+1, r+1} = A_{r+s+1, s+1} = A(s, r),
\]

where now $r \geq 0$, $s \geq 0$. It then follows from (1.1) that

\[
\sum_{r,s=0}^{\infty} A(r, s) \frac{x^r y^s}{(r + s + 1)!} = F(x, y),
\]

where

\[
F(x, y) = \frac{e^x - e^y}{xe^y - ye^x}
\]
The recurrence (1.2) becomes

(1.7) \[ A(r, s) = (r + 1)A(r, s - 1) + (s + 1)A(r - 1, s). \]

Moreover in addition to (1.5) there is a second generating function

(1.8) \[ \sum_{r,s=0}^{\infty} A(r, s) \frac{x^r y^s}{(r+s)!} = (1 + xF(x, y))(1 + yF(x, y)) \]

with \( F(x, y) \) defined by (1.6).

If we put

(1.9) \[ A_n = A_n(x, y) = \sum_{r+s=n} A(r, s) x^r y^s, \]

it follows from (1.7) that

(1.10) \[ A_n(x, y) = (x + y + xy(D_x + D_y))A_{n-1}(x, y), \]

where \( D_x = \partial/\partial x, \ D_y = \partial/\partial y \). Iteration of (1.10) gives

(1.11) \[ A_n(x, y) = (x + y + xy(D_x + D_y))^n \cdot 1. \]

It is accordingly of interest to consider the expansion of the operator

(1.12) \[ \Omega^n = (x + y + xy(D_x + D_y))^n. \]

We shall show that

(1.13) \[ \Omega^n = \sum_{k=0}^{n} C_{n,k}(x, y) (xy)^k (D_x + D_y)^k, \]

where

(1.14) \[ C_{n,k}(x, y) = \frac{1}{k!(k+1)!} (D_x + D_y)^k A_n(x, y) \quad (0 \leq k \leq n). \]

The generating function (1.8) suggests the generalization [3]

(1.15) \[ \sum_{r,s=0}^{\infty} A(r, s|\alpha, \beta) \frac{x^r y^s}{(r+s)!} = (1 + xF(x, y))^\alpha(1 + yF(x, y))^\beta \]
where again \( F(x, y) \) is defined by (1.6). Thus
\[
A(r, s) = A(r, s|1,1). 
\]

It follows from (1.15) that
\[
(1.16) \quad A(r, s|\alpha, \beta) = (r + \beta) A(r, s - 1|\alpha, \beta) + (s + \alpha) A(r - 1, s|\alpha, \beta). 
\]

which evidently reduces to (1.7) when \( \alpha = \beta = 1 \); also
\[
(1.17) \quad A(r, s|\alpha, \beta) = A(s, r|\beta, \alpha). 
\]

By (1.16), \( A(r, s|\alpha, \beta) \) is a polynomial in \( \alpha, \beta \) with positive integral coefficients. Combinatorial properties of \( A(r, s|\alpha, \beta) \) are discussed in [3].

Put
\[
(1.18) \quad A_n(x, y|\alpha, \beta) = \sum_{r+s=n} A(r, s|\alpha, \beta) x^r y^s. 
\]

Then by (1.16)
\[
(1.19) \quad A_n(x, y|\alpha, \beta) = (\alpha x + \beta y + xy(D_x + D_y)) A_{n-1}(x, y|\alpha, \beta), 
\]

so that
\[
(1.20) \quad A_n(x, y|\alpha, \beta) = (\alpha x + \beta y + xy(D_x + D_y))^n. 
\]

It is therefore of interest to consider the expansion of the operator
\[
(1.21) \quad \Omega_{\alpha, \beta}^n = (\alpha x + \beta y + xy(D_x + D_y))^n. 
\]

We shall show that
\[
(1.22) \quad \Omega_{\alpha, \beta}^n = \sum_{k=0}^{n} C_{\alpha, \beta}^{(\alpha, \beta)}(\alpha, \beta)(xy)^k (D_x + D_y)^k, 
\]

where
\[
(1.23) \quad C_{\alpha, \beta}^{(\alpha, \beta)}(\alpha, \beta) = \frac{1}{k!} \left(\frac{1}{(\alpha + \beta)^k}\right) (D_x + D_y)^k A_n(x, y|\alpha, \beta) \quad (0 \leq k \leq n), 
\]

where
\[
(\alpha + \beta)_k = (\alpha + \beta)(\alpha + \beta + 1) \ldots (\alpha + \beta + k - 1). 
\]
The case \( \alpha + \beta \) equal to zero or a negative integer requires special treatment.

We consider also the inverse of (1.22), that is,

\[(1.24) \quad (xy)^n (D_x + D_y)^n = \sum_{k=0}^n B_{n,k}^{(\alpha,\beta)} D_x^k.
\]

We show that

\[(1.25) \quad (D_x + D_y) B_{n,k}^{(\alpha,\beta)} (x, y) = n (\alpha + \beta + n - 1) B_{n-1,k}^{(\alpha,\beta)} (x, y)
\]

and

\[(1.26) \quad \sum_{n=0}^\infty \frac{t^n}{n!} \sum_{k=0}^n B_{n,k}^{(\alpha,\beta)} (x, y) (x-y)^k y^k = \left( \frac{1-xu}{1-yu} \right)^{-\alpha} (1-xu)^{-\beta}.
\]

Additional properties of \( B_{n,k}^{(\alpha,\beta)} (x, y) \) are given in §§ 8-10.

In recent years the Eulerian numbers and certain generalizations have been encountered in a number of combinatorial problems [2], [3], [4], [5], [6], [7]. The study of Eulerian operators is of intrinsic interest and may be useful for applications.

2. It is convenient to first discuss (1.13), that is,

\[(2.1) \quad (x + y + xy (D_x + D_y))^n = \sum_{k=0}^n C_{n,k} (x, y) (xy)^k (D_x + D_y)^k.
\]

We shall require the following operational formulas:

\[(2.2) \quad (D_x + D_y)^k (x + y) = 2k (D_x + D_y)^{k-1} + (x + y) (D_x + D_y)^k,
\]

\[(2.3) \quad (D_x + D_y)^k xy = k (k - 1) (D_x + D_y)^{k-2} +
\]

\[+ \quad k (x + y) (D_x + D_y)^{k-1} + xy (D_x + D_y)^k.
\]

The proof is by induction on \( k \). For (2.2) we have

\[(D_x + D_y)^k (x + y) = 2k (D_x + D_y)^{k-1} + (D_x + D_y) (x + y) (D_x + D_y)^k
\]

\[= 2k (D_x + D_y)^{k-1} + [2 + (x + y) (D_x + D_y)] (D_x + D_y)^k
\]

\[= 2 (k + 1) (D_x + D_y)^{k-1} + (x + y) (D_x + D_y)^{k+1}.
\]
As for (2.3), we have
\[
(D_x + D_y)^{h+1} xy = k (k - 1) (D_x + D_y)^{h-1} + \\
+ k (D_x + D_y) (x + y) (D_x + D_y)^{h-1} + (D_x + D_y) xy (D_x + D_y)^h \\
= k (k - 1) (D_x + D_y)^{h-1} + k [2 + (x + y) (D_x + D_y)] (D_x + D_y)^{h-1} \\
+ [x + y + xy (D_x + D_y)] (D_x + D_y)^h \\
= k (k + 1) (D_x + D_y)^{h-1} + (k + 1) xy (D_x + D_y)^h + xy (D_x + D_y)^{h+1}.
\]

Incidentally, the special case $k = 1$ of (2.3) may be noted:

(2.4) \[ (D_x + D_y) xy = x + y + xy (D_x + D_y) \equiv \Omega \]

Thus

(2.5) \[ \Omega^n = [(D_x + D_y) xy]^n. \]

We now apply $\Omega$ to both sides of (2.1). Then

\[
\Omega^{n+1} = \sum_{h=0}^{n} \Omega \left( C_{n,h} (x, y) (xy)^h \right) (D_x + D_y)^h.
\]

Since
\[
(D_x + D_y) \left( C_{n,h} (x, y) (xy)^h \right) \\
= k (xy)^{h-1} (x + y) C_{n,h} (x, y) \\
+ (xy)^h (D_x + D_y) C_{n,h} (x, y) + (xy)^h C_{n,h} (x, y) (D_x + D_y),
\]

it follows that

\[
\Omega^{n+1} = (x + y) \sum_{h=0}^{n} C_{n,h} (x, y) (xy)^h (D_x + D_y)^h \\
+ xy \sum_{h=0}^{n} (k (xy)^{h-1} (x + y) C_{n,h} (x, y) + (xy)^h (D_x + D_y) C_{n,h} (x, y) \\
+ (xy)^h C_{n,h} (x, y) (D_x + D_y)) (D_x + D_y)^h \\
= \sum_{h=0}^{n} (xy)^h \left( [(k + 1) (x + y) + xy (D_x + D_y)] C_{n,h} (x, y) \\
+ C_{n,h-1} (x, y) \right) (D_x + D_y)^h.
\]

We therefore have the recurrence

(2.6) \[ C_{n+1,h} (x, y) = [(k + 1) (x + y) + xy (D_x + D_y)] C_{n,h} (x, y) + C_{n,h-1} (x, y). \]
This establishes the existence of the expansion (2.1) and indeed shows that \( C_{n,k}(x, y) \) is a homogeneous polynomial in \( x, y \) of degree \( n - k \).

In the next place we apply \( \Omega \) to both sides of (6.1) but now on the right. Then

\[
\Omega^{m+1} = \sum_{h=0}^{n} C_{n,h}(x, y) (xy)^h (D_x + D_y)^h [x + y + xy (D_x + D_y)].
\]

Applying (2.2) and (2.3), we get

\[
\Omega^{m+1} = \sum_{h=0}^{n} C_{n,h}(x, y) (xy)^h (2k (D_x + D_y)^{k-1} + (x + y) (D_x + D_y)^h)
\]

\[
+ \sum_{h=0}^{n} C_{n,h}(x, y) (xy)^h (k (k - 1) (D_x + D_y)^{h-2} + k (x + y) (D_x + D_y)^{k-1}
\]

\[+ xy (D_x + D_y)^h) (D_x + D_y).\]

It follows that

\[
\begin{align*}
C_{n+1,k}(x, y) &= (k + 1) (x + y) C_{n,k}(x, y) \\
&+ (k + 1) (k + 2) xy C_{n,k+1}(x, y) + C_{n,k-1}(x, y).
\end{align*}
\]

Comparing (2.7) with (2.6), we get

\[
(D_x + D_y) C_{n,k}(x, y) = (k + 1) (k + 2) C_{n,k+1}(x, y).
\]

It is clear from (2.8) that

\[
C_{n,k}(x, y) = \frac{1}{k!(k + 1)!} (D_x + D_y)^k C_{n,0}(x, y).
\]

Since, by (1.1),

\[
C_{n,0}(x, y) = A_n(x, y),
\]

(2.9) becomes

\[
C_{n,k}(x, y) = \frac{1}{k!(k + 1)!} (D_x + D_y)^k A_n(x, y) \quad (0 \leq k \leq n).
\]

so that we have proved (1.14).

3. Put

\[
f_n(x, y, z) = \sum_{k=0}^{n} (k + 1)! C_{n,k}(x, y) z^k.
\]
Then $f_n(x, y, z)$ is homogeneous in $x, y, z$ of degree $n$. We also define

$$g_n(x, y) = \sum_{n-k} \frac{1}{(n+1)!} C_{n,k}(x, y).$$

Since, by (1.5) and (1.9),

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)!} A_n(x, y) = F(x, y),$$

it follows that

$$g_n(x, y) = \frac{1}{k!(k+1)!} (D_x + D_y)^k F(x, y).$$

(3.3)

It is easily verified that

$$(D_x + D_y) F = F^2$$

and therefore

$$g_n(x, y) = \frac{1}{k!(k+1)!} F^{k+1}(x, y).$$

(3.4)

Thus (3.3) becomes

$$g_n(x, y) = \frac{1}{(k+1)!} F^{k+1}(x, y).$$

(3.5)

Therefore

$$G(x, y, z) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} g_n(x, y) z = \frac{F(x, y)}{1 - zF(x, y)}.$$  

(3.6)

Also, since

$$G(x, y, z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} f_n(x, y, z),$$

we get

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)!} f_n(x, y, z) = \frac{F(x, y)}{1 - zF(x, y)}.$$  

(3.7)

By (1.6),

$$\frac{F(x, y)}{1 - zF(x, y)} = \frac{e^x - e^y}{(x e^y - y e^x) - z (e^x - e^y)} = \frac{e^x - e^y}{(x + z) e^x - (y + z) e^y} = \frac{e^{x+z} - e^{y+z}}{(x + z) e^{x+z} - (y + z) e^{x+z}} = F(x + z, y + z).$$

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Thus (3.7) becomes

\[(3.8) \quad \sum_{n=0}^{\infty} \frac{1}{(n+1)!} f_n(x, y, z) = F(x + z, y + z).\]

Since

\[F(x + z, y + z) = \sum_{n=0}^{\infty} A_n(x + z, y + z),\]

it follows that

\[(3.9) \quad f_n(x, y, z) = A_n(x + z, y + z).\]

This formula can also be proved without the use of generating functions.

4. We now consider the general case:

\[(4.1) \quad \Omega_{a,b}^n = \sum_{h=0}^{n} C^{(a,b)}_{n,k} (x, y) (xy)^h (D_x + D_y)^h,\]

where

\[(4.2) \quad \Omega_{a,b} = ax + by + xy (D_x + D_y).\]

We apply \(\Omega_{a,b}\) on the left of each side of (4.1). Since

\[\Omega_{a,b} \left[ C^{(a,b)}_{n,k} (x, y) (xy)^h (D_x + D_y)^h \right]
\[= (ax + by) C^{(a,b)}_{n,k} (x, y) (xy)^h + h (x + y) C^{(a,b)}_{n,k} (x, y) (xy)^h + (xy)^{h+1} (D_x + D_y) C^{(a,b)}_{n,k} (x, y) (xy)^h + (xy)^{k+1} C^{(a,b)}_{n,k} (x, y) (D_x + D_y),\]

we get the recurrence

\[(4.3) \quad C^{(a,b)}_{n+1,k} (x, y) = (ax + by) C^{(a,b)}_{n,k} (x, y) + (h (x + y) + xy (D_x + D_y)) C^{(a,b)}_{n,k} (x, y) + C^{(a,b)}_{n,k-1} (x, y).\]

Next, apply \(\Omega_{a,b}\) on the right. Since

\[(D_x + D_y)^h (ax + by) = \sum_{i=0}^{h} \binom{h}{i} D_x^i D_y^{h-i} (ax + by)\]

\[= \sum_{i=0}^{h} \binom{h}{i} (x D_x^i + j D_x^{i-1}) D_y^{h-i} + \beta D_x^i (y D_y^{h-i} + (k - j) D_y^{h-i-1}).\]
\[= \alpha x \sum_{i=0}^{k} \binom{k}{i} D_x^i D_y^{k-i} + k \alpha \sum_{i=1}^{k-1} \binom{k-1}{i-1} D_x^{i-1} D_y^{k-i} + \beta y \sum_{i=0}^{k} \binom{k}{i} D_x^i D_y^{k-i} + k \beta \sum_{i=0}^{k-1} \binom{k-1}{i} D_x^i D_y^{k-i}\]
\[= (\alpha x + \beta y) (D_x + D_y)^k + k (\alpha + \beta) (D_x + D_y)^{k-1}\]

and, by (2.3),

\[(D_x + D_y)^k xy + k (k - 1) (D_x + D_y)^{k-2}\]
\[+ k (x + y) (D_x + D_y)^{k-1} + xy (D_x + D_y)^k,\]

we get

\[Q_{n+1}^{x+1} = \sum_{k=0}^{n} C_{n,k}^{(a,b)} (x, y) (xy)^k ((\alpha x + \beta y) (D_x + D_y)^k + k (\alpha + \beta) + (D_x + D_y)^{k-1}\]
\[+ k (k - 1) (D_x + D_y)^{k-1} + k (x + y) (D_x + D_y)^k + xy (D_x + D_y)^{k+1}).\]

It follows that

\[C_{n+1,k}^{(a,b)} (x, y) = ((k + \alpha) x + (k + \beta) y) C_{n,k}^{(a,b)} (x, y)\]
\[+ (k + 1) (k + \alpha + \beta) xy C_{n,k+1}^{(a,b)} (x, y) + C_{n,k}^{(a,b)} (x, y).\]

Comparison of (4.4) with (4.3) gives

\[(D_x + D_y) C_{n,k}^{(a,b)} (x, y) = (k + 1) (k + \alpha - \beta) C_{n,k-1}^{(a,b)} (x, y).\]

It follows that

\[C_{n,k}^{(a,b)} (x, y) = \frac{1}{k! (\alpha + \beta)_k} (D_x + D_y)^k C_{n,0}^{(a,b)} (x, y),\]

provided \(\alpha + \beta\) is not equal to zero or a negative integer. Moreover, by (1.20),

\[C_{n,0}^{(a,b)} (x, y) = A_n (x, y | \alpha, \beta),\]

so that (4.6) becomes

\[C_{n,k}^{(a,b)} (x, y) = \frac{1}{k! (\alpha + \beta)_k} (D_x + D_y)^k A_n (x, y | \alpha, \beta).\]
It follows from (1.19) and (4.7) that

\[ A_{m+n} (x, y \mid a, b) = \sum_{k=0}^{\min(m,n)} \frac{1}{k! (a + b)^k} (xy)^k (D_x + D_y)^k A_m (x, y \mid a, b) (D_x + D_y)^k A_n (x, y \mid a, b). \]

5. Put

\[ f_a (x, y, z \mid a, b) = \sum_{k=0}^{n} (a + b)^k C_{n,k}^{(a,b)} (x, y) z^k, \]

\[ g_k (x; y \mid a, b) = \sum_{n=0}^{k} \frac{1}{n!} C_{n,k}^{(a,b)} (x, y), \]

\[ \Phi_{a,b} (x, y) = (1 + xF (x, y))^a (1 + yF (x, y))^b. \]

Since

\[ \sum_{n=0}^{\infty} \frac{1}{n!} A_n (x, y \mid a, b) = \Phi_{a,b} (x, y), \]

it follows from (4.7) and (5.2) that

\[ g_k (x, y) = \frac{1}{k! (a + b)^k} (D_x + D_y)^k \Phi_{a,b} (x, y). \]

But

\[ (D_x + D_y)^k \Phi_{a,b} (x, y) = (a + b)^k F^k (x, y) \Phi_{a,b} (x, y), \]

so that (5.4) becomes

\[ g_k (x, y \mid a, b) = \frac{1}{k!} F^k (x, y) \Phi_{a,b} (x, y). \]

Now put

\[ G (x, y, z \mid a, b) = \sum_{k=0}^{\infty} (a + b)^k g_k (x, y \mid a, b) z^k. \]

Then, by (5.6),

\[ G (x, y, z \mid a, b) = \frac{\Phi_{a,b} (x, y)}{(1 - zF (x, y))^a + b}. \]

Since

\[ \Phi_{a,b} (x, y) = \frac{(x - y)^a \rho^a + y}{(x e^\rho - y e^\rho)^a + b}, \]
and
\[ \frac{\Phi_{\alpha, \beta}(x, y)}{(1 - z F(x, y))^{x+\beta}} = \frac{(x - y)^{x+\beta} e^{x+\beta y}}{[(xe^y - ye^x) - z (e^x - e^y)]} \]
\[ = \frac{(x - y)^{x+\beta} e^{x+\beta y}}{[(x + z) e^y - (y + z) e^x]} \]
\[ = \frac{(x - y)^{x+\beta} e^{(x+z)+\beta(x+z)}}{[(x + z) e^{x+z} - (y + z) e^{x+z} e^{x-\beta}]} \]

(5.7) becomes

(5.8) \[ G(x, y, z | \alpha, \beta) = \Phi_{\alpha, \beta}(x + z, y + z). \]

On the other hand, by (5.1) and (5.2),
\[ G(x, y, z | \alpha, \beta) = \sum_{k=0}^{\infty} (\alpha + \beta)_k z^k \sum_{n=0}^{\infty} \frac{1}{n!} C^{(\alpha, \beta)}_{\alpha, \beta}(x, y) z^n \]
\[ = \sum_{k=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{\infty} (\alpha + \beta)_k C^{(\alpha, \beta)}_{\alpha, \beta}(x, y) z^k \]
\[ = \sum_{n=0}^{\infty} \frac{1}{n!} f_n(x, y, z | \alpha, \beta), \]
so that, by (5.8),

(5.9) \[ \sum_{n=0}^{\infty} \frac{1}{n!} f_n(x, y, z | \alpha, \beta) = \Phi_{\alpha, \beta}(x + z, y + z). \]

Therefore, by (1.15) and (1.18), we get

(5.10) \[ f_n(x, y, z | \alpha, \beta) = A_n(x + z, y + z | \alpha, \beta). \]

This identity implies

(5.11) \[ A_n(x + z, y + z | \alpha, \beta) = \frac{z^k}{k!} (D_x + D_y)^k A_n(x, y | \alpha, \beta), \]

which can also be obtained by applying Taylor's theorem to \( A_n(x, y | \alpha, \beta). \)

6. As noted above, (4.7) is not valid when \( x + \beta \) is zero or a negative integer. We shall now consider the excluded values. It is
convenient to begin with the special case \( z = \beta = 0 \) In place of (4.1) we now have

\[
(6.1) \quad (xy(D_x + D_y))^n = \sum_{k=1}^{n} C_{n,k}^{(0,0)}(x,y)(xy)^k (D_x + D_y) \quad (n \geq 1)
\]

The recurrence (4.3) reduces to

\[
(6.2) \quad C_{n+1,k}^{(0,0)}(x,y) = [k(x+y) + xy(D_x + D_y)]C_{n,k}^{(0,0)}(x,y) + C_{n,k-1}^{(0,0)}(x,y),
\]

while (4.4) becomes

\[
(6.3) \quad C_{n+1,k}^{(0,0)}(x,y) = k(x+y)C_{n,k}^{(0,0)}(x,y) + k(k+1)xyC_{n,k+1}^{(0,0)}(x,y) + C_{n,k-1}^{(0,0)}(x,y)
\]

Hence

\[
(6.4) \quad (D_x + D_y)C_{n,k}^{(0,0)}(x,y) = k(k+1)C_{n,k+1}^{(0,0)}(x,y),
\]

so that

\[
(6.5) \quad C_{n,k}^{(0,0)}(x,y) = \frac{1}{k!(k-1)!} (D_x + D_y)^{k-1} C_{n+1,k}^{(0,0)}(x,y) \quad (k \geq 1)
\]

For \( k = 1 \), (6.2) reduces to

\[
C_{n+1,1}^{(0,0)}(x,y) = [x + y + xy(D_x + D_y)]C_{n,1}^{(0,0)}(x,y),
\]

which yields

\[
C_{n,1}^{(0,0)}(x,y) = [x + y + xy(D_x + D_y)]^{n-1} C_{1,1}^{(0,0)}(x,y)
\]

It is clear from (6.1) that \( C_{1,1}^{(0,0)}(x,y) = 1 \) and therefore

\[
(6.6) \quad C_{n,1}^{(0,0)}(x,y) = A_{n-1}(x,y) \equiv A_{n-1}(x,y | 1,1)
\]

Thus (6.5) becomes

\[
(6.7) \quad C_{n,k}^{(0,0)}(x,y) = \frac{1}{k!(k-1)!} (D_x + D_y)^{k-1} A_{n-1}(x,y)
\]
Before discussing the general case \( \alpha + \beta \) equal to zero or a negative integer, we consider the expansion

\[
Q_{n,\alpha,\beta}^n = \sum_{k=0}^{n} Q_{n,k}^{(n,\alpha,\beta)}(x, y) (xy(D_x + D_y))^k,
\]

where the \( Q_{n,k}^{(n,\alpha,\beta)}(x, y) \) are to be determined. Clearly

\[
Q_{n,\alpha,\beta}^{n+1} = (\alpha x + \beta y) \sum_{k=0}^{n} Q_{n,k}^{(n,\alpha,\beta)}(x, y) (xy(D_x + D_y))^k
\]
\[+\ x y \sum_{k=0}^{n} (D_x + D_y) Q_{n,k}^{(n,\alpha,\beta)}(x, y) \cdot (xy(D_x + D_y))^k
\]
\[+\ \sum_{k=0}^{n} Q_{n,k}^{(n,\alpha,\beta)}(x, y) (xy(D_x + D_y))^k,
\]

so that

\[
Q_{n+1,\alpha,\beta}^{(n+1,\alpha,\beta)}(x, y) = \Omega_{n,\alpha,\beta} Q_{n+1,k}^{(n,\alpha,\beta)}(x, y) + Q_{n,k+1}^{(n,\alpha,\beta)}(x, y)
\]

For \( k = 0 \), (6.9) reduces to

\[
Q_{n+1,0}^{(n,\alpha,\beta)}(x, y) = \Omega_{n,\alpha,\beta} Q_{n,0}^{(n,\alpha,\beta)}(x, y)
\]

Since, by (6.8),

\[
Q_{1,0}^{(n,\alpha,\beta)}(x, y) = \alpha x + \beta y = A_1(x, y | x, y),
\]

it is clear that

\[
Q_{n,0}^{(n,\alpha,\beta)}(x, y) = A_n(x, y | \alpha, \beta)
\]

We shall now show that

\[
Q_{n,0}^{(n,\alpha,\beta)}(x, y) = \binom{n}{k} A_{n-k}(x, y | \alpha, \beta) \quad (0 \leq k \leq n)
\]

Clearly (6.11) holds for \( n = 0 \) assuming that it holds up to and including the value \( n \), we have by (6.9),
\[ Q^{(n, \beta)}_{n+1, \lambda} (x, y) = \binom{n}{k} Q^n_{n-k \beta} A_{n-k} (x, y \mid x, \beta) + \binom{n}{k-1} A_{n-k+1} (x, y \mid x, \beta) \]

\[ = \binom{n}{k} A_{n-k+1} (x, y \mid x, \beta) + \binom{n}{k-1} A_{n-k+1} (x, y \mid x, \beta) \]

\[ = \binom{n}{k} A_{n-k+1} (x, y \mid x, \beta) \]

We have therefore proved

\[ \Omega^n_{x, \beta} = \sum_{k=0}^{n} \binom{n}{k} A_{n-k} (x, y \mid x, \beta) (xy (D_x + D_y))^k \]

This suggests the following more general result:

\[ \Omega^n_{x+y, \beta+\delta} = \sum_{k=0}^{n} \binom{n}{k} A_{n-k} (x, y \mid x, \beta) \Omega^k_{y, \delta} \]

To prove (6.13), consider

\[ \Omega^n_{x+y, \beta+\delta} = \sum_{k=0}^{n} R_{n,k} \Omega^k_{y, \delta}, \]

where the \( R_{n,k} \) are functions of \( x, y, \alpha, \beta, \gamma, \delta \). Then

\[ \Omega^{n+1}_{x+y, \beta+\delta} = [(x+y) x + (\beta+\delta) y + xy (D_x + D_y)] \sum_{k=0}^{n} R_{n,k} \Omega^k_{y, \delta} \]

\[ = [(x+y) x + (\beta+\delta)] \sum_{k=0}^{n} R_{n,k} \Omega^k_{y, \delta} \]

\[ + xy \sum_{k=0}^{n} (D_x + D_y) R_{n,k} \cdot \Omega^k_{y, \delta} + \sum_{k=0}^{n} R_{n,k} \cdot xy (D_x + D_y) \Omega^k_{y, \delta} \]

\[ = \sum_{k=0}^{n} \Omega^k_{x, \beta} R_{n,k} \cdot \Omega^k_{y, \delta} + \sum_{k=0}^{n} R_{n,k} \Omega^k_{y, \delta} \]

This evidently implies

\[ R_{n+1, \lambda} = \Omega^n_{x, \beta} R_{n, \lambda} + R_{n, \lambda-1} \]

Then, exactly as above, we show first that

\[ R_{n,0} = A_n (x, y \mid x, \beta) \]
and generally
\[ R_{n,k} = \binom{n}{k} A_{n-k}(x, y | \alpha, \beta) \quad (0 \leq k \leq n) \]

This completes the proof of (6.13)

As a special case of (6.13), we note

(6.15) \((x y (D_x + D_y))^n = \sum_{k=0}^{n} \binom{n}{k} A_{n-k}(x, y | -\alpha, -\beta) \Omega_{n,k}^x\)

7 We now treat the general excluded case in (4.7), \(\alpha + \beta\) equal to zero or a negative integer. We shall require the following formulas:

(7.1) \[ \Phi_{n,\beta}(x, y) = \sum_{\ell=0}^{n} \frac{1}{\ell!} A_{\ell}(x, y | \alpha, \beta) \]
\[ = (1 + x F(x, y))^\alpha (1 + y F(x, y))^\beta, \]

(7.2) \[ (D_x + D_y)^k F(x, y) = k! F^{k+1}(x, y), \]

(7.3) \[ (D_x + D_y)^k \Phi_{n,\beta}(x, y) = (\alpha + \beta)_k F^k(x, y) \Phi_{n,\beta}(x, y). \]

If \(\alpha + \beta\) is not equal to zero or a negative integer, we have seen that
\[ C_{n,k}^{(n,\beta)}(x, y) = \frac{1}{k! (\alpha + \beta)_k} (D_x + D_y)^k A_n(x, y | \alpha, \beta). \]

It follows that
\[ \sum_{n=k}^{\infty} \frac{1}{n!} C_{n,k}^{(n,\beta)}(x, y) = \frac{1}{k! (\alpha + \beta)_k} (D_x + D_y)^k \Phi_{n,\beta}(x, y) \]
and therefore, by (7.3),

(7.4) \[ \sum_{n=k}^{\infty} \frac{1}{n!} C_{n,k}^{(n,\beta)}(x, y) = \frac{1}{k!} F^k(x, y) \Phi_{n,\beta}(x, y). \]

Put

(7.5) \[ F^k(x, y) = \sum_{n=0}^{\infty} \frac{1}{(n+k)!} A_n^{(k)}(x, y) \quad (k = 1, 2, 3, ...), \]
where \( A_n^{(k)}(x, y) \) is homogeneous of degree \( n \) in \( x, y \). For \( k = 1 \) we have

\[
A_n^{(1)}(x, y) = A_n(x, y) = A_n(x, y | 1, 1).
\]

(7.6)

It follows from (7.5) and (7.6) that

\[
A_n^{(k+1)}(x, y) = \sum_{r=0}^{n} \binom{n + k + 1}{r + 1} A_r(x, y) A_{n-r}^{(k)}(x, y)
\]

(7.7)

and therefore the coefficients in \( A_n^{(k)}(x, y) \) are positive integers

By (7.4) and (7.5) we get, since \( C_{n,k}^{(a,b)}(x, y) \) is of degree \( n - k \),

\[
C_{n,k}^{(a,b)}(x, y) = \frac{1}{k!} \sum_{r=0}^{n} \binom{n}{r} A_r^{(b)}(x, y) A_{n-r}^{(a)}(x, y | a, b).
\]

(7.8)

Since both \( C_{n,k}^{(a,b)}(x, y) \) and \( A_n(x, y | a, b) \) are polynomials in \( a, b \) (as well as in \( x, y \)), it follows that (7.8) is valid for all \( a, b \). The numerical coefficients in \( C_{n,k}^{(a,b)}(x, y) \) are integers; however that is not obvious from (7.8)

Since

\[
A_n(x, y | 0, 0) = 0 \quad (n > 0),
\]

it is evident that (7.8) implies

\[
C_{n,k}^{(0,0)}(x, y) = \frac{1}{k!} A_n^{(0)}(x, y).
\]

(7.9)

By (7.2) and (7.5) we have

\[
(k - 1)! \sum_{n=0}^{\infty} \frac{1}{(n + k)!} A_n^{(k)}(x, y) = (D_x + D_y)^{k-1} \frac{1}{(n + 1)!} A_n(x, y),
\]

\[
(k - 1)! \sum_{n=0}^{\infty} \frac{1}{(n + k)!} A_n^{(k)}(x, y) = (D_x + D_y)^{k-1} \frac{1}{(n + 1)!} A_n(x, y),
\]

so that

\[
A_n^{(k)}(x, y) = \frac{1}{(k - 1)!} (D_x + D_y)^{k-1} A_n (x, y) \quad (1 \leq k \leq n).
\]

(7.10)

Thus (7.8) becomes

\[
C_{n,k}^{(a,b)}(x, y) = \frac{1}{k! (k - 1)!} \sum_{r=0}^{n} \binom{n}{r} (D_x + D_y)^{k-1} A_{n-r}^{(a,b)}(x, y)
\]

(7.11)

\[
\cdot A_{n-r}^{(a,b)}(x, y | a, b) \quad (1 \leq k \leq n).
\]
For \( \alpha = \beta = 0 \), (7.11) reduces to

\[
(7.12) \quad C_{n,k}^{(0,0)}(x, y) = \frac{1}{k!(k-1)!} (D_x + D_y)^{k-1} A_{n-1}(x, y)
\]

in agreement with (6.7)
Both (7.8) and (7.11) are valid for all \( \alpha, \beta \).

8. We now consider the inverse of (4.1), that is,

\[
(8.1) \quad (xy)^n (D_x + D_y)^n = \sum_{k=0}^{n} (-1)^{n-k} B_{n,k}^{(\alpha, \beta)}(x, y) \Omega_{n, \beta}^k,
\]

where, as will appear presently, \( B_{n,k}^{(\alpha, \beta)}(x, y) \) is a homogeneous polynomial in \( x, y \) of degree \( n - k \). The existence of a formula of this kind is evidently implied by (4.1). For example

\[
x y (D_x + D_y) = -(\alpha x + \beta y) + \Omega_{n, \beta},
\]

\[
(xy)^2 (D_x + D_y) = (\alpha x + \beta y)^2 + \alpha x^2 + \beta y^2
\]

\[
- [(2 \alpha + 1) x + (2 \beta + 1) y] \Omega_{n, \beta} + \Omega_{n, \beta}^2.
\]

To get a recurrence for the coefficients \( B_{n,k}^{(\alpha, \beta)} \) we apply the operator \( xy (D_x + D_y) \) to both sides of (8.1) — on the left. This gives

\[
n (x + y) (xy)^n (D_x + D_y) + (xy)^{n+1} (D_x + D_y)^{n+1}
\]

\[
= xy \sum_{k=0}^{n} (-1)^{n-k} \left\{ (D_x + D_y) B_{n,k}^{(\alpha, \beta)}(x, y) + B_{n,k}^{(\alpha, \beta)}(x, y) (D_x + D_y) \right\} \Omega_{n, \beta}^k
\]

\[
= \sum_{k=0}^{n} (-1)^{n-k} \left\{ xy (D_x + D_y) B_{n,k}^{(\alpha, \beta)}(x, y) - (\alpha x + \beta y) B_{n,k}^{(\alpha, \beta)}(x, y),
\text{ } + B_{n,k}^{(\alpha, \beta)}(x, y) \Omega_{n, \beta} \right\} \Omega_{n, \beta}^k.
\]

It follows that

\[
(xy)^{n+1} (D_x + D_y)^{n+1} = -n (x + y) \sum_{k=0}^{n} (-1)^{n-k} B_{n,k}^{(\alpha, \beta)}(x, y) \Omega_{n, \beta}^k
\]

\[
+ \sum_{k=0}^{n} (-1)^{n-k} \left\{ xy (D_x + D_y) B_{n,k}^{(\alpha, \beta)}(x, y)
\text{ } - (\alpha x + \beta y) B_{n,k}^{(\alpha, \beta)}(x, y) \right\} \Omega_{n, \beta}^k
\]

\[
+ \sum_{k=1}^{n+1} (-1)^{n-k+1} B_{n,k-1}^{(\alpha, \beta)}(x, y) \Omega_{n, \beta}^k.
\]
Therefore

\[(8.2) \quad B^{(\alpha, \beta)}_{n+1, k}(x, y) = [(x + n) x + (\beta + n) y - xy (D_x + D_y)]
\cdot B^{(\alpha, \beta)}_{n, k}(x, y) + B^{(\alpha, \beta)}_{n, k-1}(x, y).\]

On the other hand, if we multiply both sides of (8.1) on the right by \(Q_{\alpha, \beta}\) we get

\[\sum_{k=0}^{n} (-1)^{n-k} B^{(\alpha, \beta)}_{0, k}(x, y) Q_{\alpha, \beta}^{k+1}\]

\[= (xy)^n (D_x + D_y)^n [(x + \beta) y + xy (D_x + D_y)]\]

\[= (xy)^n (n (x + \beta) (D_x + D_y)^{n-1} + (x + \beta y) (D_x + D_y)^n)
+ (xy)^n (n (n - 1) (D_x + D_y)^{n-1} + n (x + y) (D_x + D_y)^n
+ xy (D_x + D_y)^{n+1})\]

\[= n (x + \beta + n - 1) (x y)^n (D_x + D_y)^{n-1}
+ [(x + n) x + (\beta + n) y] (xy)^n (D_x + D_y)^n + (xy)^{n+1} (D_x + D_y)^{n+1}.\]

This implies

\[(8.3) \quad B^{(\alpha, \beta)}_{n+1, k}(x, y) = [(x + n) x + (\beta + n) y] B^{(\alpha, \beta)}_{n, k}(x, y)
- n (x + \beta + n - 1) xy B^{(\alpha, \beta)}_{n-1, k} + B^{(\alpha, \beta)}_{n, k-1}(x, y).\]

Comparing (8.3) with (8.2), we get

\[(8.4) \quad (D_x + D_y) B^{(\alpha, \beta)}_{n, k}(x, y) = n (x + \beta + n - 1) B^{(\alpha, \beta)}_{n-1, k}(x, y).\]

In the next place, it follows at once from (4.1) and (8.1) that

\[(8.5) \quad \sum_{k=0}^{n} (-1)^{k-i} C^{(\alpha, \beta)}_{n, i}(x, y) B^{(\alpha, \beta)}_{n, i}(x, y) = \delta_{n, i}\]

and

\[(8.6) \quad \sum_{k=0}^{n} (-1)^{n-k} B^{(\alpha, \beta)}_{n, k}(x, y) C^{(\alpha, \beta)}_{n, i}(x, y) = \delta_{n, i}.\]

A formula of a different kind can be obtained by applying each side of (8.1) to \(A_{\alpha}(x, y | x, \beta)\). Since

\[(D_x + D_y)^n A_{\alpha}(x, y | x, \beta) = 0 \quad (0 \leq r < n),\]

\[Q^{n}_{\alpha, \beta} A_{\alpha}(x, y | x, \beta) = A_{r+1, \alpha}(x, y | x, \beta),\]
we get
\[(8.7) \quad \sum_{k=0}^{n} (-1)^{n-k} B_{n,k}^{(a,b)}(x, y) A_{r+k}(x, y | \alpha, \beta) = 0 \quad (0 \leq r < n).\]

In either (8.5) or (8.6) take \(j = n\). Since \(C_{n,n}^{(a,b)}(x, y) = 1\), we have
\[(8.8) \quad B_{n,n}^{(a,b)}(x, y) = 1.\]

Also it is easily verified that
\[\Omega_{a,\beta}(x^{-\beta} y^{-\alpha}) = 0,\]
so that
\[\Omega_{a,\beta}(x^{-\beta} y^{-\alpha}) = 0 \quad (k = 1, 2, 3, \ldots).\]
Thus (8.1) implies
\[(-1)^{n} B_{n,0}^{(a,b)}(x, y) = (xy)^{n}(D_{x} + D_{y})^{n} x^{-\beta} y^{-\alpha}.\]

A little manipulation leads to
\[(8.9) \quad B_{n,0}^{(a,b)}(x, y) = \sum_{k=0}^{n} \binom{n}{k} (\alpha x + (\beta - k)y) x^{k} y^{n-k}.\]

This is equivalent to
\[(8.10) \quad \sum_{n=0}^{\infty} B_{n,0}^{(a,b)}(x, y) \frac{n!}{x^{n}} = (1 - xz)^{-\alpha} (1 - yz)^{-\beta}.\]

We remark that \(B_{n,0}^{(a,b)}(x, y)\) satisfies the following recurrence:
\[(8.11) \quad B_{n+1,0}^{(a,b)}(x, y) = (ax + \beta y + x^{2}D_{x} + y^{2}D_{y}) B_{n,0}^{(a,b)}(x, y),\]
Indeed, by (8.9),
\[\begin{align*}
(ax + \beta y + x^{2}D_{x} + y^{2}D_{y}) B_{n,0}^{(a,b)}(x, y) \\
= (ax + \beta y) \sum_{k=0}^{n} \binom{n}{k} (\alpha x + (\beta - k)y) x^{k} y^{n-k} \\
+ \sum_{k=0}^{n} \binom{n}{k} (\alpha x + (\beta - k)y) x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} (\alpha x + (\beta - k)y) x^{k} y^{n-k+1}.
\end{align*}\]
The coefficient of $x^k y^{n-k+1}$ on the right is equal to
\[
\alpha \binom{n}{k-1} (\alpha)_{n-k+1} + \beta \binom{n}{k} (\beta)_{n-k} \\
+ (k-1) \binom{n}{k-1} (\alpha)_{k-1} (\beta)_{n-k+1} + (n-k) \binom{n}{k} (\alpha)_{k} (\beta)_{n-k} \\
= \binom{n}{k-1} (\alpha)_{k} (\beta)_{n-k+1} + \binom{n}{k} (\alpha)_{k} (\beta)_{n-k+1} = \binom{n+1}{k} (\alpha)_{k} (\beta)_{n-k+1}.
\]

It follows from (8.11) that

\[(8.12) \quad B^{(\alpha, \beta)}_{n,0} (x, y) = (\alpha x + \beta y + x^2 D_x + y^2 D_y)^n \cdot 1.\]

9. When $\alpha = \beta = 0$, (8.1) reduces to

\[(9.1) \quad (xy)^n (D_x + D_y)^n = \sum_{k=0}^{n} B^{(0,0)}_{n,k} (x, y) (xy (D_x + D_y))^k,\]

while (8.3) becomes

\[(9.2) \quad B^{(0,0)}_{n,1,k} (x, y) = n (x + y) B^{(0,0)}_{n,k} (x, y) - n (n - 1) xy B^{(0,0)}_{n-1,k} (x, y) + B^{(0,0)}_{n,k-1} (x, y).\]

It is evident from (9.1) that

\[(9.3) \quad B^{(0,0)}_{n,0} (x, y) = 0 \quad (n > 0).\]

For brevity, put

\[b_{n,k} = \frac{1}{(n-1)!} B^{(0,0)}_{n,k} (x, y) \quad (n \geq 1).\]

Then (9.2) becomes

\[(9.4) \quad b_{n+1,k} - (x + y) b_{n,k} + xy b_{n-1,k} = \frac{1}{n} b_{n,k-1} \quad (k \geq 1).\]

For $k = 1$, (9.4) reduces to

\[(9.5) \quad b_{n+1,1} - (x + y) b_{n,1} + xy b_{n-1,1} = 0 \quad (n > 1).\]
The recurrence (9.5) implies
\[ b_{n,1} = c_1 x^n + c_2 y^n, \]
where \( c_1, c_2 \) are constant. Since
\[ b_{1,1} = 1, \quad b_{2,1} = x + y, \]
we get
\[ (9.6) \quad b_{n,1} = \frac{x^n - y^n}{x - y} = \sigma_n. \]

Next, for \( k = 2 \), we have
\[ (9.7) \quad b_{n+1,2} - (x + y) b_{n,2} + xy b_{n-1,2} = \frac{n}{1} \sigma_n. \]
Since
\[ b_{1,2} = b_{2,0} = 1, \]
(9.7) holds for \( n \geq 1 \). It follows that
\[ (1 - (x + y) z + xyz^2) \sum_{n=1}^{\infty} b_{n+1,2} z^n = \sum_{n=1}^{\infty} \frac{n}{1} \sigma_n z^n. \]
Since
\[ \frac{1}{1 - (x + y) z + xyz^2} = \sum_{n=0}^{\infty} \sigma_{n+1} z^n, \]
we get
\[ b_{n+1,2} = \sum_{j=1}^{n} \frac{1}{j} \sigma_j \sigma_{n-j+1}. \]

Generally (9.4) implies
\[ \sum_{n=1}^{\infty} (b_{n+1,k} - (x + y) b_{n,k} + x y b_{n-1,k}) z^n = \sum_{n=k-1}^{\infty} \frac{n}{1} b_{n,k-1} z^n, \]
that is,
\[ (9.8) \quad \sum_{n=k-1}^{\infty} b_{n+1,k} z^n = \frac{1}{1 - (x + y) z + xyz^2} \sum_{n=k-1}^{\infty} \frac{n}{1} b_{n,k-1} z^n. \]
Therefore, as above, we get
\[ (9.9) \quad b_{n+1,k} = \sum_{j=k-1}^{n} \frac{1}{j} b_{j,k-1} \sigma_{n-j+1}. \]
We may rewrite (9.9) in the form

\[(9.10) \quad b_{n+k, h} = \sum_{j=0}^{n} \frac{1}{j + k - 1} b_{j+k-1, h-1} \sigma_{n-f+1}.\]

Using this formula we get

\[b_{n+3, 3} = \sum_{0 \leq i \leq j \leq n} \frac{1}{(i + 1)(j + 2)} \sigma_{i+1} \sigma_{j-i+1} \sigma_{n-f+1},\]

\[b_{n+4, 4} = \sum_{0 \leq i \leq j \leq k \leq n} \frac{1}{(i + 1)(j + 2)(k + 3)} \sigma_{i+1} \sigma_{j-i+1} \sigma_{k-i+1} \sigma_{n-k+1}.\]

and so on.

We may also mention an operational formula for \(b_{n,k}\). Define the operator \(D_x^{-1}\) by means of

\[D_x^{-1} f(x) = \int_{0}^{x} f(t) \, dt.\]

Thus

\[\sum_{n=1}^{\infty} \frac{1}{n} \sigma_n z^n = D_x^{-1} \sum_{n=0}^{\infty} \sigma_{n+1} z^n = D_x^{-1} \frac{1}{(1-xz)(1-zy)}.\]

By (9.7)

\[\sum_{n=1}^{\infty} b_{n+1,2} z^n = \frac{1}{(1-xz)(1-zy)} D_x^{-1} \frac{1}{(1-xz)(1-zy)} ,\]

so that

\[\sum_{n=1}^{\infty} \frac{1}{n+1} b_{n+1,2} z^n = [D_x^{-1} (1-xz)^{-1} (1-zy)^{-1}]^2 \cdot 1.\]

At the next stage we get

\[\sum_{n=2}^{\infty} b_{n+1,3} z^n = [(1-xz)^{-1} (1-zy)^{-1} D_x^{-1}]^3 (1-xz)^{-1} (1-zy)^{-1}\]

The general formula is

\[(9.11) \quad \sum_{n=k-1}^{\infty} b_{n+1,k} z^n = [(1-xz)^{-1} (1-zy)^{-1} D_x^{-1}]^{k-1} (1-xz)^{-1} (1-zy)^{-1} (k \geq 1).\]
10. A generating function for $B_{n,k}^{(n,\beta)}(x,y)$ in the general case can be found in the following way. It follows from (8.5) that

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} (-1)^{k-i} C_{n,k}^{(n,\beta)}(x,y) B_{k,i}^{(n,\beta)}(x,y) = \frac{z^i}{i!}. \tag{10.1}$$

By (7.4) we have

$$\sum_{n=k}^{\infty} \frac{1}{n!} C_{n,k}^{(n,\beta)}(x,y) = \frac{1}{k!} F^k(x,y) \Phi_{n,\beta}(x,y).$$

Since $C_{n,k}^{(n,\beta)}(x,y)$ is homogeneous of degree $n - k$, this implies

$$\sum_{n=k}^{\infty} \frac{z^n}{n!} C_{n,k}^{(n,\beta)}(x,y) = \frac{z^k}{k!} F^k(xz,yz) \Phi_{n,\gamma}(xz,yz).$$

Thus (10.1) becomes

$$\frac{z^i}{i!} = \sum_{k=i}^{\infty} (-1)^{k-i} B_{k,i}^{(n,\beta)}(x,y) \sum_{n=k}^{\infty} \frac{z^n}{n!} C_{n,k}^{(n,\beta)}(x,y)$$

$$= \Phi_{n,\beta}(xz,yz) \sum_{k=i}^{\infty} (-1)^{k-i} B_{k,i}^{(n,\beta)}(x,y) \frac{z^k F^k(xz,yz)}{k!}.$$

Multiplying by $\psi^i$ and summing over $j$, we get

$$e^{\psi} = \Phi_{n,\beta}(xz,yz) \sum_{k=0}^{\infty} \sum_{j=0}^{n} (-1)^{k-j} B_{k,j}^{(n,\beta)}(x,y) \frac{z^k F^k(xz,yz)}{k!} \psi^j. \tag{10.2}$$

Consider the equation

$$z F(xz,yz) = u, \tag{10.3}$$

that is,

$$e^{\psi} - e^{\psi x} \frac{e^{\psi x} - y e^{\psi y}}{xe^{\psi x} - ye^{\psi y}} = u.$$ 

This reduces to

$$e^{(x-y)z} = \frac{1 - xu}{1 - yu},$$

so that

$$z = \frac{1}{x - y} \log \frac{1 + xu}{1 + yu} = \sum_{i=1}^{\infty} (-1)^{n-1} \frac{1}{n} \sigma_n u^n. \tag{10.4}$$
where as above

$$\sigma_n = (x^n - y^n)/(x - y).$$

Since

$$\Phi_{n,\beta}(xz, yz) = (1 + xzF(xz, yz)\beta(1 + yzF(xz, yz))\beta$$

$$= (1 + xu)^{\beta}(1 + yu)^{\beta},$$

(10.2) becomes

(10.5) $$e^{\nu} = (1 + xu)^{\beta}(1 + yu)^{\beta} \sum_{k=0}^{\infty} \frac{\nu^k}{k!} \sum_{i=0}^{k} (-1)^{k-i} B_{k,i}^{(\alpha,\beta)}(x, y) \nu^i.$$

But, by (10.4),

$$e^{\nu} = \left( \frac{1 + xu}{1 + yu} \right)^{\nu(y - y)} ;$$

so that (10.5) may be replaced by

(10.6) $$\sum_{k=0}^{\infty} \frac{\nu^k}{k!} \sum_{i=0}^{k} B_{k,i}^{(\alpha,\beta)}(x, y) (x - y)^i \nu^i = \left( \frac{1 - xu}{1 - yu} \right)^{-\nu} (1 - xu)^{-\alpha}(1 - yu)^{-\beta}.$$

In particular, for \( \nu = 0 \), (10.6) reduces to

(10.7) $$\sum_{k=0}^{\infty} \frac{k!}{\nu^k} B_{k,0}^{(\alpha,\beta)}(x, y) = (1 - xu)^{-\alpha}(1 - yu)^{-\beta},$$

which is evidently in agreement with (8.10).

For \( \alpha = \beta = 0 \), (10.6) becomes

(10.8) $$\sum_{k=0}^{\infty} \frac{\nu^k}{k!} \sum_{i=0}^{k} B_{k,i}^{(0,0)}(x, y) (x - y)^i \nu^i = \left( \frac{1 - xu}{1 - yu} \right)^{-\nu}.$$

Since

$$\left( \frac{1 - xu}{1 - yu} \right)^{-\nu} = \sum_{r=0}^{\infty} \frac{(\nu)_r}{r!} x^r y^r \sum_{s=0}^{\infty} \frac{(-\nu)_s}{s!} y^s \nu^s,$$

we get

(10.9) $$\sum_{j=0}^{k} B_{k,j}^{(0,0)}(x, y) (x - y)^j \nu^i = \sum_{r=0}^{\infty} \frac{(\nu)_r}{r!} \left( -\nu \right)_s (\nu)_{k-r}.$$  

The general result is only slightly more complicated, namely

(10.10) $$\sum_{j=0}^{k} B_{k,j}^{(\alpha,\beta)}(x, y) (x - y)^j \nu^i = \sum_{r=0}^{\infty} \frac{(\nu)_r}{r!} (x + \nu, (\beta - \nu)_{k-r}.$$
It follows from (10.6), (10.7) and (10.8) that
\[
\sum_{j=0}^{k} B_{k,j}^{(n,\theta)} (x, y) \psi^j = \sum_{r=0}^{k} \binom{k}{r} B_{k-r,0}^{(n,\theta)} (x, y) \sum_{j=0}^{r} B_{r,j}^{(0,\theta)} (x, y) \psi^j
\]
and therefore
\[
B_{k,1}^{(n,\theta)} (x, y) = \sum_{r=0}^{k} B_{r,0}^{(0,\theta)} (x, y) B_{k-r,0}^{(n,\theta)} (x, y). \tag{10.11}
\]
Comparing coefficients of $\psi^j$ on both sides of (10.8), we get
\[
\sum_{k=j}^{\infty} \frac{u^k}{k!} B_{k,j}^{(0,\theta)} (x, y) = (-1)^j \frac{j!}{j!} \left( \log \frac{1-xu^j}{1-yu^j} \right).
\]
Differentiation with respect to $u$ gives
\[
\sum_{k=j}^{\infty} \frac{u^k}{k!} B_{k+1,j}^{(0,\theta)} (x, y) = \frac{(-1)^{j-1}}{(j-1)!} (x-y)^{-j+1} \left( \log \frac{1-xu^{j-1}}{1-yu^{j-1}} \right) \left( \frac{1}{1-xu} \frac{1}{1-yu} \right)
\]
which is equivalent to (9.8).
REFERENCES


