SOME POLYNOMIALS RELATED TO THE ULTRA-SPHERICAL POLYNOMIALS (*)

BY

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1. INTRODUCTION. In a previous paper [4] the writer has defined a set of polynomials $A_n^{(\lambda)}(x)$ such that

$$\sum_{r=0}^{n} A_r^{(\lambda)}(x)\, C_{n-r}^{(\lambda+r)}(x) = \begin{cases} 1 & (n=0) \\ 0 & (n \geq 1), \end{cases} \tag{1.1}$$

where $C_n^{(\lambda)}(x)$ is the ultraspherical polynomial defined by

$$(1 - 2xt + t^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x) t^n . \tag{1.2}$$

It was shown that

$$A_n^{(\lambda)}(x) = (-1)^n \sum_{2r \leq n} \frac{(\lambda)_n (2x)^{n-2r}}{r!(\lambda+1)_r (n-2r)!}, \tag{1.3}$$

from which it follows that

$$A_n^{(\lambda)}(x) = \frac{\lambda}{\lambda+n} C_n^{(-\lambda-n)}(x) . \tag{1.4}$$

Making use of $A_n^{(\lambda)}(x)$ the inverses of a number of formulas involving $C_n^{(\lambda)}(x)$ were derived.

In the present paper we obtain a number of additional formulas involving $A_n^{(\lambda)}(x)$.

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2. Preliminaries. It will be convenient to recall a number of formulas from [4]. In the first place (1.1) is a special case of

$$C_n^{(\mu-\lambda)}(x) = \sum_{r=0}^{n} A_r^{(\lambda)}(x)\, C_{n-r}^{(\mu+r)}(x), \tag{2.1}$$

where λ, μ are arbitrary. Next we have the generating relation

$$(1 - 2xt + t^2)^{\lambda} = \sum_{n=0}^{\infty} z^n A_n^{(\lambda)}(x), \tag{2.2}$$

where

$$z = \frac{t}{1 - 2xt + t^2}; \tag{2.3}$$

also

$$\sum_{n=0}^{\infty} t^n A_n^{(\lambda)}(x) = (1 + 2xt)^{-\lambda} F\left[\frac{\lambda}{2}, \frac{\lambda+1}{2}; \lambda + 1; \frac{4t^2}{(1+2xt)^2}\right], \tag{2.4}$$

where F denotes the hypergeometric function. It is easily verified that (2.4) is equivalent to (1.3). Another generating relation is given by

$$\sum_{n=0}^{\infty} \frac{t^n}{(\lambda)_n} A_n^{(\lambda)}(x) = \Gamma(\lambda + 1) t^{-\lambda} e^{-2xt} I_{\lambda}(2t). \tag{2.5}$$

3. Some expansion formulas. We show first that

$$(2x)^n = \sum_{2s \leq n} (-1)^{n-s} \frac{n!}{s!\,(\lambda+1)_s\,(\lambda+s)_{n-2s}} A_{n-2s}^{(\lambda+s)}(x). \tag{3.1}$$

Indeed, using (1.3), we have

$$\sum_{2s \leq n} (-1)^{n-s} \frac{n!}{s!\,(\lambda+1)_s\,(\lambda+s)_{n-2s}} A_{n-2s}^{(\lambda+s)}(x)$$

$$= \sum_{2s \leq n} (-1)^s \frac{n!}{s!\,(\lambda+1)_s} \sum_{2r \leq n-2s} \frac{(2x)^{n-2r-2s}}{r\,(\lambda+s-1)_r\,(n-2r-2s)!}$$

$$= \sum_{2k \leq n} \frac{n!\,(2x)^{n-2k}}{k!\,(\lambda+1)_k\,(n-2k)!} \sum_{s=0}^{k} (-1)^s \binom{k}{s}.$$

Thus (3.1) follows at once.

Next using (1.3) and (3.1) we get

$$A_n^{(\lambda)}(xy) = (-1)^n \sum_{2r\le n} \frac{(\lambda)_n y^{n-2r}}{r!\,(\lambda+1)_r} \frac{(2x)^{n-2r}}{(n-2r)!}$$

$$= \sum_{2r\le n} \frac{(\lambda)_n y^{n-2r}}{r!\,(\lambda+1)_r} \sum_{2s\le n-2r} (-1)^s \frac{A_{n-2r-2s}^{(\mu+r+s)}(x)}{s!\,(\mu+r+1)_s\,(\mu+r+s)_{n-2r-2s}}$$

$$= \sum_{2k\le n} \frac{(\lambda)_n A_{n-2k}^{(\mu+k)}}{k!\,(\mu+1)_k\,(\mu+k)_{n-2k}} \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \frac{(\mu+1)_r}{(\lambda+1)_r} y^{n-2r}.$$

We have therefore

(3.2)

$$A_n^{(\lambda)}(xy) = \sum_{2k\le n} (-1)^k \frac{(\lambda)_n y^n A_{n-2k}^{(\mu+k)}}{k!\,(\mu+1)_k\,(\mu+k)_{n-2k}} \cdot F(-k, \mu+1; \lambda+1; y^{-2}).$$

In particular for $\lambda = \mu$, (3.2) becomes

$$A_n^{(\lambda)}(xy) = \sum_{2k\le n} (-1)^k \frac{(\lambda)_n y^{n-2k}\,(y^2-1)^k}{k!\,(\lambda+1)_k\,(\lambda+k)_{n-2k}} A_{n-2k}^{(\lambda+k)}(x), \tag{3.3}$$

while for $y = 1$ we get

$$A_n^{(\lambda)}(x) = \sum_{2k\le n} (-1)^k \frac{(\lambda)_n\,(\lambda-\mu)_k}{k!\,(\lambda+1)_k\,(\mu+1)_k\,(\mu+k)_{n-2k}} A_{n-2k}^{\mu+k}(x). \tag{3.4}$$

We remark that (3.1) and (2.5) yield

$$e^{-2xt} = \sum_{n=0}^{\infty} t^n \sum_{2s\le n} (-1)^s \frac{A_{n-2s}^{(\lambda+s)}(x)}{s!\,(\lambda+1)_s\,(\lambda+s)_{n-2s}}$$

$$= \sum_{s=0}^{\infty} (-1)^s \frac{t^{2s}}{s!\,(\lambda+1)_s} \sum_{n=0}^{\infty} \frac{t^n}{(\lambda+s)_n} A_n^{(\lambda+s)}(x)$$

$$= \sum_{s=0}^{\infty} (-1)^s \frac{t^{2s}}{s!\,(\lambda+1)_s} \cdot \Gamma(\lambda+s+1) t^{-\lambda-s} e^{-2xt} I_{\lambda+s}(2t)$$

$$= \Gamma(\lambda + 1) \sum_{s=0}^{\infty} (-1)^s \frac{t^{s-\lambda}}{s!} I_{\lambda+s}(2t)\, e^{-2xt},$$

so that

$$\frac{t^\lambda}{\Gamma(\lambda+1)} = \sum_{s=0}^{\infty} (-1)^s \frac{t^s}{s!} I_{\lambda+s}(2t). \tag{3.5}$$

Similarly from (3.2)

$$\sum_{n=0}^{\infty} \frac{t^n}{(\lambda)_n} A^{(\lambda)}(xy) = \sum_{k=0}^{\infty} (-1)^k y^{2k} \frac{F(-k, \mu+1; \lambda+1; y^{-2})}{k!\,(\mu+1)_k} \times$$

$$\times \sum_{n=0}^{\infty} \frac{(yt)^n}{(\mu+k)_n} A^{(\mu+k)}(x).$$

Using (2.5) this becomes

$$\Gamma(\lambda+1) t^{-\lambda} e^{-2xyt} I_\lambda(2t) = \sum_{k=0}^{\infty} (-1)^k (yt)^{2k} \frac{F(-k, \mu+1; \lambda+1; y^{-2})}{k!\,(\mu+1)_k}$$

$$\cdot\, \Gamma(\mu+k+1)\,(ty)^{-\mu-k} e^{-2xyt} I_{\mu+k}(2yt),$$

so that

$$\frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)} t^{\mu-\lambda} I_\lambda(2t) = y^{-\mu} \sum_{k=0}^{\infty} (-1)^k \frac{(yt)^k}{k!} \tag{3.6}$$

$$F(-k, \mu+1; \lambda+1; y^{-2})\, I_{\mu+k}(2yt).$$

The formula (3.5) is well known ; also the case $\lambda = \mu$ of (3.6) is familiar [6, p. 142].

4. Other expansion formulas. Analogous to (3.1) we have a second expansion

$$(2x)^n = (-1)^n \frac{n!}{(\lambda)_n} \sum_{2r \leq n} (-1)^r \frac{(\lambda)_r}{r!} A_{n-2r}^{(\lambda+2r)}(x), \tag{4.1}$$

which can be verified using (1.3). Note that (4.1) is equivalent to

$$e^{-2xt} = \sum_{r=0}^{\infty} (-1)^r \frac{t^{2r}}{r!\,(\lambda+r)_r} \sum_{n=0}^{\infty} \frac{t^n}{(\lambda+2r)_n} A_n^{(\lambda+2r)}(x)$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{t^{2r}}{r!(\lambda+r)_r} \cdot \Gamma(\lambda + 2r + 1) t^{-\lambda-2r} e^{-2xt} I_{\lambda+2r}(2t),$$

that is, to the familiar expansion [6, p. 138]

$$t^{\lambda} = \sum_{r=0}^{\infty} \frac{(\lambda + 2r)\,\Gamma(\lambda + r)}{r!} I_{\lambda+2r}(2t).$$

Using (4.1) and (1.3) we get

$$A_n^{(\lambda)}(xy) = \sum_{2r \leq n} \frac{(\lambda)_n y^{n-2r}}{r!(\lambda+1)_r(\mu+2r)_{n-2r}}$$

$$\cdot \sum_{2s \leq n-2r} (-1)^s \frac{(\mu+2r)_s}{s!} A_{n-2r-2s}^{(\mu+2r+2s)}(x)$$

$$= \frac{(\lambda)_n}{(\lambda)_n} \sum_{2k \leq n} \frac{1}{k!} A_{n-2k}^{(\mu+2k)}(x) \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \frac{(\mu)k + r}{(\lambda+1)_r} y^{n-2r},$$

so that

$$A_n^{(\lambda)}(xy) = \frac{(\lambda)_n}{(\mu)_n} y^n \sum_{2k \leq n} (-1)^k \frac{(\mu)_k}{k!} A_{n-2k}^{(\mu+2k)}(x) \tag{4.2}$$

$$\cdot\ F(-k,\ \mu + k\ ;\ \lambda + 1\ ;\ y^{-2}).$$

In particular for $\lambda = \mu$, (4.2) becomes

$$A_n^{(\lambda)}(xy) = y^n \sum_{2k \leq n} (-1)^k \frac{(\lambda)_k}{k!} A_{n-2k}^{(\lambda+2k)}(x) \tag{4.3}$$

$$\cdot\ F(-k,\ \lambda + k\ ;\ \lambda + 1\ ;\ y^{-2}).$$

Since

$$F(-k,\ \mu + k\ ;\ \lambda + 1\ ;\ 1) = \frac{(\lambda - \mu - k + 1)_k}{(\lambda + 1)_k} = (-1)^k \frac{(\mu - \lambda)_k}{(\lambda + 1)_k},$$

(4.2) reduces for $y = 1$ to

$$A_n^{(\lambda)}(x) = \frac{(\lambda)_n}{(\mu)_n} \sum_{2k \leq n} \frac{(\mu)_k (\mu - \lambda)_k}{k!(\lambda + 1)_k} A_{n-2k}^{(\mu+2k)}(x). \tag{4.4}$$

Making use of (1.4), the formulas (4.1), (4.2), (4.3), (4.4) can also be expressed compactly in terms of $C_n^{(\lambda)}(x)$.

We remark that (4.2) is equivalent to the familiar expansion [6, p. 140]

$$\Gamma(\lambda+1) y^{\mu} t^{\mu-\lambda} I_\lambda(2t)$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{(\mu+2n)\,\Gamma(\mu+n)}{n\,!} I_{\mu+2n}(2t) \cdot F(-n,\, \mu+n;\, \lambda+1;\, y^{-2}).$$

5. Generating functions. We now consider some extensions of (2.4). Using (1.3) we get

$$\sum_{n=0}^{\infty} \frac{(\mu)_n}{(\lambda)_n} t^n A_n^{(\lambda)}(x) = \sum_{n=0}^{\infty} (-1)^n t^n \sum_{2r \leq n} \frac{(2x)^{n-2r}}{r\,!\,(\lambda+1)_r\,(n-2r)\,!}$$
$$= \sum_{r=0}^{\infty} \frac{(\mu)_{2r} t^n}{r\,!\,(\lambda+1)_r} \sum_{n=0}^{\infty} (-1)^n \frac{(\mu+2r)_n}{n\,!} (2xt)^n$$
$$= \sum_{r=0}^{\infty} \frac{(\mu)_{2r}}{r\,!\,(\lambda+1)_r} t^{2r} (1+2xt)^{-\mu-2r}.$$

We have therefore

$$\sum_{n=0}^{\infty} \frac{(\mu)_n}{(\lambda)_n} t^n A_n^{(\lambda)}(x) \tag{5.1}$$
$$= (1+2xt)^{-\mu} F\left[\frac{\mu}{2}, \frac{\mu+1}{2};\, \lambda+1;\, \frac{4t^2}{(1+2xt)^2}\right].$$

In particular for $\mu = 2\lambda+1$, the right side becomes

$$(1+2xt)^{-2\lambda+1}\left\{1 - \frac{4t^2}{(1+2xt)^2}\right\}^{-\lambda-\frac{1}{2}} = \{1+4xt+4(x^2-1)t^2\}^{-\lambda-\frac{1}{2}},$$

so that

$$\sum_{n=0}^{\infty} \frac{(2\lambda+1)_n}{(\lambda)_n} t^n A_n^{(\lambda)}(x) = \{1+4xt+4(x^2-1)t^2\}^{-\lambda-\frac{1}{2}}. \tag{5.2}$$

Similarly for $\mu = 2\lambda+2$, (5.1) reduces to

$$\sum_{n=0}^{\infty} \frac{(2\lambda+2)_n}{(\lambda)_n} t^n A_n^{(\lambda)}(x) = (1+2xt)\{1+4xt+4(x^2-1)t^2\}^{-\lambda-\frac{3}{2}}. \tag{5.3}$$

From (5.2) we evidently get

$$(5.4) \qquad \frac{(2\lambda+1)_n}{(\lambda)_n} A_n^{(\lambda)}(x) = (-1)^n 2^n (x^2-1)^{\frac{n}{2}} C_n^{\left(\lambda+\frac{1}{2}\right)} \left(\frac{x}{(x^2-1)^{\frac{1}{2}}}\right),$$

while (5.3) implies an equivalent result, as can be verified without much difficulty.

When $\lambda = \mu$, it is evident that (5.1) reduces to (2.4). Also if we replace t by t/μ and let $\mu \to \infty$ then formally (5.1) reduces to (2.5).

6. The polynomial $S_n^{(\lambda,\mu)}(y, x)$. We now consider

$$(6.1) \qquad S_n = S_n^{(\lambda,\mu)}(y, x) = \sum_{r=0}^{n} C_r^{(\lambda)}(y) A_{n-r}^{(\mu+r)}(x).$$

Then if

$$(6.2) \qquad z = \frac{t}{1-2xt+t^2},$$

it follows from (2.2) and (2.3) that

$$\begin{aligned} \sum_{n=0}^{\infty} S_n z^n &= \sum_{r=0}^{\infty} C_r^{(\lambda)}(y) z^r \sum_{n=0}^{\infty} A_n^{(\mu+r)}(x) z^n \\ &= \sum_{r=0}^{\infty} C_r^{(\lambda)}(y) z^r (1-2xt+t^2)^{\mu+r} \\ &= (1-2xt+t^2)^{\mu} \sum_{r=0}^{\infty} C_r^{(\lambda)}(y) t^r \\ &= (1-2xt+t^2)^{\mu}(1-2yt+t^2)^{-\lambda}, \end{aligned}$$

so that

$$(6.3) \qquad \sum_{n=0}^{\infty} S_n^{(\lambda,\mu)}(y, x) z^n = (1-2xt+t^2)^{\mu}(1-2yt+t^2)^{-\lambda}.$$

In particular, when $x = y$, since

$$(1-2xt+t^2)^{\mu-\lambda} = \sum_{n=0}^{\infty} z^n A_n^{(\mu-\lambda)}(x),$$

it follows that

$$(6.4) \qquad S_n^{(\lambda,\mu)}(x, x) = A_n^{(\mu-\lambda)}(x).$$

We show next that

$$(6.5)\qquad S_n^{(\lambda,\lambda)}(y,x)=\frac{2^n(\lambda)_n}{n!}(y-x)^n.$$

This can be proved rapidly by means of the degenerate addition theorem [6, p. 369]:

$$e^{2yt}=\Gamma(\lambda)\sum_{r=0}^{\infty}(\lambda+r)\frac{I_{\lambda+r}(2t)}{t}C_r^{(\lambda)}(y).$$

Then, employing (2.5), we have

$$\begin{aligned}e^{2(y-x)t}&=\sum_{r=0}^{\infty}\frac{t^r}{(\lambda)_r}\sum_{s=0}^{\infty}A_s^{(\lambda+r)}(x)\frac{t^s}{(\lambda+r)_s}C_r^{(\lambda)}(y)\\&=\sum_{n=0}^{\infty}\frac{t^n}{(\lambda)_n}\sum_{r=0}^{n}C_r^{(\lambda)}(y)A_{n-r}^{(\lambda+r)}(x)\\&=\sum_{n=0}^{\infty}\frac{t^n}{(\lambda)_n}S_n^{(\lambda,\lambda)}(y,x),\end{aligned}$$

which is evidently equivalent to (6.5).

If we take $\lambda=\mu$ in (6.3) and compare with (6.5) it is clear that

$$(6.6)\qquad (1-2xt+t^2)^{\lambda}(1-2yt+t^2)^{-\lambda}=\{1-2(y-x)z\}^{-\lambda},$$

where z is defined by (6.2). If now we rewrite (6.3) as

$$\begin{aligned}\sum_{n=0}^{\infty}S_n^{(\lambda,\mu)}(y,x)z^n&=(1-2xt+t^2)^{\mu-\lambda}\cdot(1-2xt+t^2)^{\lambda}(1-2yt+t^2)^{\lambda}\\&=\sum_{r=0}^{\infty}A_r^{(\mu-\lambda)}(x)z^r\sum_{s=0}^{\infty}\frac{(\lambda)_s}{s!}2^s(y-x)^s z^s,\end{aligned}$$

it follows at once that

$$(6.7)\qquad S_n^{(\lambda,\mu)}(y,x)=\sum_{s=0}^{n}\frac{(\lambda)_s}{s!}2^s(y-x)^sA_{n-s}^{(\mu-\lambda)}(x).$$

This formula evidently includes both (6.4) and (6.5).

We note that (6.3) yields

(6.8) $$\sum_{r=0}^{n} S_r^{(\lambda,\mu)}(y,\ x)\, S_{n-r}^{(\lambda',\mu')}(y,\ x) = S_n^{(\lambda+\lambda',\,\mu+\mu')}(y,\ x).$$

Also it can be shown that

(6.9) $$\sum_{r=0}^{n} S_r^{(\lambda,\mu)}(y,\ x)\, S_{n-r}^{(\lambda'+r,\,\mu'+r)}(x,\ y) = S_n^{(\lambda'-\mu,\,\mu'-\lambda)}(x,\ y).$$

Indeed the left side of (6.9) is equal to

$$\sum_{r+s+j+k=n} C_r^{(\lambda)}(y)\, A_s^{(\mu+r)}(x)\, C_j^{(\lambda'+r+s)}(x)\, A_k^{(\mu'+r+s+j)}(y)$$

$$= \sum_{r+s+k=n} C_r^{(\lambda)}(y)\, C_s^{(\lambda'-\mu)}(x)\, A_k^{(\mu'+r+s)}(y)$$

$$= \sum_{s=0} C_s^{(\lambda'-\mu)}(x)\, A_{n-s}^{(\mu'-\lambda+s)}(y) = S_n^{(\lambda'-\mu,\,\mu'-\lambda)}(x,\ y),$$

where we have used (2.1) and (6.4).

7. THE POLYNOMIAL $T_n^{(\lambda,\mu)}(x,\ y)$. In the next place we put

(7.1) $$T_n = T_n^{(\lambda.\,\mu)}(x,\ y) = \sum_{r=0}^{n} A_r^{(\lambda)}(x)\, C_{n-r}^{(\mu+r)}(y).$$

Then

$$\sum_{n=0}^{\infty} T_n t^n = \sum_{r=0}^{\infty} A_r^{(\lambda)}(x)\, t^r \sum_{n=0}^{\infty} C_n^{(\mu+r)}(y)\, t^n$$

$$= \sum_{r=0}^{\infty} A_r^{(\lambda)}(x)\, t^r (1 - 2yt + t^2)^{-\mu-r}.$$

By (2.4)

$$\sum_{r=0}^{\infty} A_r^{(\lambda)}(x)\, t^r (1 - 2yt + t^2)^{-\mu-r}$$

$$= \left(1 + \frac{2xt}{1-2yt+t^2}\right)^{-\lambda} F\left[\frac{\lambda}{2},\ \frac{\lambda+1}{2};\ \lambda+1;\ \frac{4t^2}{(1-2(y-x)t+t^2)^2}\right],$$

so that

$$\sum_{n=0}^{\infty} T_n t^n = (1 - 2yt + t^2)^{\lambda-\mu}\,(1 - 2(y-x)t + t^2)^{-\lambda}$$

$$\cdot F\left[\frac{\lambda}{2},\ \frac{\lambda+1}{2};\ \lambda+1;\ \frac{4t^2}{(1-2(y-x)t+t^2)^2}\right]$$

$$= \sum_{r=0}^{\infty} C_r^{(\mu-\lambda)}(y)\, t^r \sum_{s=0}^{\infty} \frac{(\lambda)_{2s}}{s\,!\,(\lambda+1)_s} t^{2s} \sum_{j=0}^{\infty} C_j^{(\lambda+2s)}(y-x) t^j.$$

It follows that

$$T_n^{(\lambda,\,\mu)}(x,\ y) = \sum_{r+2s+j=n} \frac{(\lambda)_{2s}}{s\,!\,(\lambda+1)_s} C_r^{(\mu-\lambda)}(y)\, C_j^{(\lambda+2s)}(y-x). \tag{7.2}$$

Comparing (7.1) with (2.1) it is clear that

$$T_n^{(\lambda,\,\mu)}(x,\ x) = C_n^{(\mu-\lambda)}(x). \tag{7.3}$$

Also, for $\lambda = \mu$, (7.2) reduces to

$$T_n^{(\lambda,\,\lambda)}(x,\ y) = \sum_{2s \le n} \frac{(\lambda)_{2s}}{s\,!\,(\lambda+1)_s} C_{n-2s}^{(\lambda+2s)}(y-x). \tag{7.4}$$

On the other hand, by (1.4), we have

$$\begin{aligned} T_n^{(\lambda,\,\lambda)}(x,\ y) &= \sum_{r=0}^{n} \frac{\lambda}{\lambda+r} C_r^{(-\lambda-r)}(x) \frac{\lambda+r}{\lambda+n} A_{n-r}^{(-\lambda-n)}(y) \\ &= \frac{\lambda}{\lambda+n} \sum_{r=0}^{n} A_r^{(-\lambda-n)}(y)\, C_{n-r}^{(-\lambda-n+r)}(x) \\ &= \frac{\lambda}{\lambda+n} T_n^{(-\lambda-n,\ -\lambda-n)}(y,\ x) \end{aligned}$$

so that

$$T_n^{(\lambda;\,\lambda)}(x,\ y) = \frac{\lambda}{\lambda+n} T_n^{(-\lambda-n,\ -\lambda-n)}(y,\ x). \tag{7.5}$$

We notice also that

$$\sum_{r=0}^{n} T_r^{(\lambda,\,\mu)}(y,\ x)\, T_{n-r}^{(\lambda'+r,\ \mu'+r)}(x,\ y) = T_n^{(\lambda'-\mu,\ \mu'-\lambda)}(y,\ x). \tag{7.6}$$

The proof is similar to the proof of (6.9).

8. FORMULAS FOR THE PRODUCT $A_m^{(\lambda)}(x)\, A_n^{(\lambda)}(x)$. In the addition theorem for $I_\lambda(z)$:

$$\frac{I_\lambda(w)}{w^\lambda} = \frac{2^\lambda\, \Gamma(\lambda)}{(uv)^\lambda} \sum_{r=0}^{\infty} (-1)^r (\lambda+r)\, I_{\lambda+r}(u)\, I_{\lambda+r}(v) \cdot C_r^\lambda(\cos \Phi), \tag{8.1}$$

where

$$w = (u^2 + v^2 - 2\,uv\ \cos\ \Phi)^{\frac{1}{2}},$$

take $\Phi = \pi$. Since

$$C_r^{\lambda}(-1) = (-1)^r \frac{(2\,\lambda)_r}{r!},$$

(8.1) implies

$$\frac{I_\lambda\,(2\,(u+v))}{(u+v)^\lambda} = \frac{\Gamma(\lambda)}{(uv)^\lambda} \sum_{r=0}^{\infty} (\lambda + r)\, \frac{(2\,\lambda)_r}{r!}\, I_{\lambda+r}(2\,u)\, I_{\lambda+r}(2\,v).$$

Then using (2.5) we get

$$\sum_{n=0}^{\infty} \frac{A_n^{(\lambda)}(x)}{(\lambda)_n}\,(u+v)^n = \frac{1}{\lambda} \sum_{r=0}^{\infty} \frac{(\lambda+r)\,(2\,\lambda)_r}{r!\,(\lambda+1)_r\,(\lambda+1)_r}\,(uv)^r$$

$$\cdot \sum_{m=0}^{\infty} A_m^{(\lambda+r)}(x)\, \frac{u^m}{(\lambda+r)_m} \sum_{n=0}^{\infty} A_n^{(\lambda+r)}(x)\, \frac{v^n}{(\lambda+r)_n}.$$

It follows that

$$(8.2)\quad \binom{m+n}{m} \frac{(\lambda)_m\,(\lambda)_n}{(\lambda)_{m+n}}\, A_{m+n}^{(\lambda)}(x) = \lambda \sum_{r=0}^{\min(m,n)} \frac{(2\,\lambda)_r}{r!\,(\lambda+r)}\, A_{m-r}^{(\lambda+r)}(x)\, A_{n-r}^{(\lambda+r)}(x).$$

In terme of $C_n^{(\lambda)}(x)$ (8.2) becomes

$$(8.3)\quad \binom{m+n}{m} \frac{(\lambda)_{m+1}\,(\lambda)_{n+1}}{(\lambda)_{m+n+1}}\, C_{m+n}^{(-\lambda-m-n)}(x)$$

$$= \sum_{r=0}^{\min(m,n)} \frac{(\lambda+r)\,(2\lambda)_r}{r!}\, C_{m-r}^{(-\lambda-m)}(x)\, C_{n-r}^{(-\lambda-n)}(x).$$

Again, by formula (37) of [4]

$$I_\lambda\,(2\,u)\, I_\lambda\,(2\,v) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(\lambda+r+1)} \binom{2\,\lambda+2\,r}{r} \left(\frac{uv}{u+v}\right)^{\lambda+r} \cdot I_{\lambda+r}\,(2\,(u+v)).$$

It follows on applying (2.5) that

$$(8.4)\quad \frac{A_m^{(\lambda)}(x)\, A_n^{(\lambda)}(x)}{(\lambda)_m\,(\lambda)_n} = \sum_{r=0}^{\min(m,n)} (-1)^r \frac{\lambda}{(\lambda+r)\,(\lambda+1)_r} \binom{2\,\lambda+2\,r}{r} \binom{m+n-2r}{m-r}$$

$$\cdot \frac{A_{m+n-2r}^{(\lambda+r)}(x)}{(\lambda)_{m+n-r}}.$$

It can be checked directly that (8.4) is indeed the inverse of (8.2).

In terms of $C_n^{(\lambda)}(x)$, (8.4) becomes

$$(8.5)\qquad \frac{C_m^{(-\lambda-m)}(x)\, C_m^{(-\lambda-n)}(x)}{(\lambda)_{m+1}\,(\lambda)_{n+1}}$$

$$= \frac{1}{\lambda}\sum_{r=0}^{\min(m,n)} (-1)^r \frac{1}{(\lambda+1)_r}\binom{2\lambda+2r}{r}\binom{m+n-2r}{m-r}\frac{C_{m+n-2r}^{(-m-n-\lambda+r)}(x)}{(\lambda)_{m+n-r+1}}.$$

We remark that (8.5) does not contain the formulas found by BAILEY [2] for the product of two associated LEGENDRE polynomials. Also (8.3) does not contain the inverse formula found by AL-SALAM [1].

9. SOME ADDITIONAL RESULTS. We conclude with some formulas of a different kind. For brevity put

$$(9.1)\qquad \bar{A}_n^{(\lambda)}(x) = \frac{n!}{(\lambda)_n} A_n^{(\lambda)}(x).$$

Then by (1.3)

$$\sum_{r=0}^{n} (-1)^{n-r}\binom{n}{r} u^r \bar{A}_r^{(\lambda)}(x)\, v^{n-r} \bar{A}_{n-r}^{(\mu)}(y)$$

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} u^r v^{n-r} \sum_{2s\le r} \frac{r!\,(2x)^{r-2s}}{s!\,(\lambda+1)_s\,(r-2s)!}$$

$$\cdot \sum_{2j\le n-r} \frac{(n-r)!\,(2y)^{n-r-2j}}{j!\,(\mu+1)_j\,(n-r-2j)!}$$

$$= n! \sum_{2s+2j\le n} \frac{u^{2s} v^{2j}}{s!\,j!\,(\lambda+1)_s\,(\mu+1)_j} \sum_{2s\le r\le n-2j} (-1)^r \frac{(2ux)^{r-2s}\,(2vy)^{n-r-2j}}{(r-2s)!\,(n-r-2j)!}$$

$$= n! \sum_{2s+2j\le n} \frac{2^{n-2s-2j}\, u^{2s} v^{2j}\,(vy-ux)^{n-2s-2j}}{s!\,j!\,(\lambda+1)_s\,(\mu+1)_j\,(n-2s-2j)!}.$$

This may be written in the following form.

$$(9.2)\qquad \left(u\bar{A}^{(\lambda)}(x) - v\bar{A}^{(\mu)}(y)\right)^n$$

$$= n! \sum_{2k\le n} \frac{2^{n-2k}\,(vy-ux)^{n-2k}}{k!\,(n-2k)!} \sum_{j=0}^{k} \binom{k}{j} \frac{u^{2k-2j} v^{2j}}{(\lambda+1)_{k-j}\,(\mu+1)_j}.$$

In particular, for $u = y$, $v = x$, (9.2) reduces to

$$(9.3)\qquad \left(y\,\bar{A}^{(\lambda)}(x) - x\,\bar{A}^{\mu}(y)\right)^n$$

$$= \begin{cases} 0 & (n \text{ odd}) \\ \dfrac{n\,!\,y^n}{m\,!\,(\lambda+1)}\, F\left(-m,\ -\lambda-m\ ;\ \mu+1\ ;\ \dfrac{x^2}{y^2}\right) & (n = 2m). \end{cases}$$

For $u = v = 1$, (9.2) becomes

$$(9.4)\qquad \left(\bar{A}^{(\lambda)}(x) - \bar{A}^{(\mu)}(y)\right)^n$$

$$= n\,! \sum_{2k \le n} \frac{2^{n-2k}(y-x)^{n-2k}}{k\,!\,(n-2k)\,!}\ \frac{(\lambda+\mu+k)_k}{(\lambda+1)_k\,(\mu+1)_k}$$

and in particular

$$(9.5)\qquad \left(\bar{A}^{(\lambda)}(x - \bar{A}^{(\mu)}(x)\right)^n = \begin{cases} 0 & (n \text{ odd}) \\ \dfrac{n\,!\,(\lambda)+\mu+m)_m}{m\,!\,(\lambda+1)_m\,(\mu+1_m)} & (n = 2m). \end{cases}$$

Formulas of a similar kind can be found for

$$(9.6)\qquad \bar{C}_n^{(\lambda)}(x) = \frac{n\,!}{(2\lambda)_n}\, C_n^{(\lambda)}(x).$$

Making use of the formula

$$(9.7)\qquad C_n^{(\lambda)}(x) = \sum_{2r \le n} \frac{(2\lambda)_n\, x^{n-2r}\,(x^2-1)^r}{2^{2r}\, r\,!\left(\lambda+\frac{1}{2}\right)_r (n-2r)\,!}$$

we get

$$\sum_{r=0}^{n} (-1)^{n-r}\binom{n}{r} u^r\, \bar{C}_r^{(\lambda)}(x)\, v^{n-r}\, C_{n-r}^{(\mu)}(y)$$

$$= \sum_{r=0}^{n} (-1)^{n-r}\binom{n}{r} u^r v^{n-r} \sum_{2s \le r} \frac{r\,!\,x^{r-2s}(x^2-1)^s}{2^{2s}\, s\,!\left(\lambda+\frac{1}{2}\right)_s (r-2s)\,!}$$

$$\cdot \sum_{2j \le n-r} \frac{(n-r)\,!\,y^{n-r-2j}(y^2-1)^j}{2^{2j}\, j\,!\left(\mu+\frac{1}{2}\right)_j (n-r-2j)\,!}$$

$$= n\,! \sum_{2s+2j \le n} \frac{(x^2-1)^s\,(y^2-1)^j\, u^{2s}\, v^{2j}}{2^{2s+2j}\, s\,!\, j\,!\left(\lambda+\frac{1}{2}\right)_s \left(\mu+\frac{1}{2}\right)_j}$$

$$\cdot \sum_{2s \leq r \leq n-2j} (-1)^s \frac{(ux)^{r-2s} (vy)^{n-r-2j}}{(r-2s)!\,(n-r-2j)!}$$

$$= n! \sum_{2s+2j \leq n} \frac{(x^2-1)^s (y^2-1)^j u^{2s} v^{2j} (vj-ux)^{n-2s-2j}}{2^{2s+2j} s!\, j! \left(\lambda+\frac{1}{2}\right)_s \left(\mu+\frac{1}{2}\right)_j (n-2s-2j)!}.$$

We have therefore

(9.8)
$$(u\bar{C}^{(\lambda)}(x) - v\bar{C}^{(\mu)}(y))^n$$

$$= n! \sum_{2k \leq n} \frac{(vy-ux)^{n-2k}}{2^{2k} k!\,(n-2k)!} \sum_{j=0}^{k} \binom{k}{j} \frac{(x^2-1)^{k-j}(y^2-1)^j u^{2(k-j)} v^{2j}}{\left(\lambda+\frac{1}{2}\right)_{k-j} \left(\mu+\frac{1}{2}\right)_j}.$$

In particular, for $u = y$, $v = x$, (9.8) becomes

(9.9)
$$(y\bar{C}^{(\lambda)}(x) - x\bar{C}^{(\mu)}(y))^n$$

$$= \begin{cases} 0 & (n \text{ odd}) \\ \dfrac{n!}{m!} \dfrac{(x^2-1)^m y^{2m}}{\left(\lambda+\frac{1}{2}\right)_m} F\left(-m,\ -\lambda-m+\frac{1}{2};\ \mu+\frac{1}{2};\ \dfrac{(y^2-1)x^2}{(x^2-1)y^2}\right) & (n=2m), \end{cases}$$

while for $u^2(x^2-1) = v^2(y^2-1)$, (9.8) reduces to

(9.10)
$$(u\bar{C}^{(\lambda)}(x) - v\bar{C}^{(\mu)}(y))^n$$

$$= n! \sum_{2k \leq n} \frac{(vy-ux)^{n-2k}(x^2-1)^k u^{2k}}{2^{2k} k!\,(n-2k)!} \quad \frac{(\lambda+\mu+m)_m}{\left(\lambda+\frac{1}{2}\right)_k \left(\mu+\frac{1}{2}\right)_k}.$$

In particular for $u = v = 1$, $x = y$, (9.10) becomes

(9.11)
$$(\bar{C}^{(\lambda)}(x) - \bar{C}^{(\mu)}(x))^n$$

$$= \begin{cases} 0 & (n \text{ odd}) \\ \dfrac{n!\,(\lambda+\mu+m)_m (x^2-1)^m}{2^{2m} m! \left(\lambda+\frac{1}{2}\right)_m \left(\mu+\frac{1}{2}\right)_m} & (n = 2m), \end{cases}$$

a result due to BURCHNALL [3, p. 239].

In connection with the above we remark that it follows easily from (2.5) that

$$(2y + \bar{A}^{(\lambda)}(x))^n = \bar{A}_n^{(\lambda)}(x-y) \tag{9.12}$$

and in particular

$$(4x + \bar{A}^{(\lambda)}(x))^n = (-1)^n \bar{A}_n^{(\lambda)}(x). \tag{9.13}$$

Also, using (9.7), we can show that

$$(y - \bar{C}^{(\lambda)}(x))^n = \varrho^n \bar{C}_n^{(\lambda)}\left(\frac{y-x}{\varrho}\right), \tag{9.14}$$

where

$$\varrho = (1 - 2xy + y^2)^{1/2}.$$

In particular for $y = 2x$, (9.14) becomes

$$(2x - \bar{C}^{(\lambda)}(x))^n = \bar{C}_n^{(\lambda)}(x), \tag{9.15}$$

while for $y = x$ we get

$$(x - \bar{C}^{(\lambda)}(x))^n = \begin{cases} 0 & (n \text{ odd}) \\ \dfrac{\left(\frac{1}{2}\right)_m}{\left(\lambda + \frac{1}{2}\right)_m}(x^2-1)^m & (n = 2m). \end{cases} \tag{9.16}$$

For $y = 1/(2x)$, (9.14) reduces to

$$(2x\bar{C}^{(\lambda)}(x) - 1)^n = \bar{C}_n^{(\lambda)}(2x^2 - 1). \tag{9.17}$$

Also for

$$y = x + \frac{x^2}{(x^2+1)}$$

we can verify that (9.14) becomes

$$(y - \bar{C}^{(\lambda)}(x))^n = (x^2+1)^{-\frac{n}{2}} \bar{C}_n^{(\lambda)}(x^2). \tag{9.18}$$

The formula (9.14) is equivalent to formula (22) on p. 280 of RAINVILLE's book [5]; also the case $\lambda = \frac{1}{2}$ of (9.15) and (9.17) occurs in RAINVILLE [5, p. 347].

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