A GENUS FOR N-DIMENSIONAL KNOTS AND LINKS*

by

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ABSTRACT:

The concept of regular genus for n-dimensional links is introduced. It extends the classical genus of one-dimensional links. Some characterization theorems of the trivial knot are given. In particular, the only genus zero n-dimensional knot is proved to be homeomorphic with the trivial knot. Then the regular genus of a knot is proved to be related to the one-dimensional homology of the universal abelian covering of its complement. Partial extensions for links of these results are also obtained. Some applications to low-dimensional links and a final section about connected sums of links complete the paper.

1. DEFINITIONS AND NOTATIONS.

Throughout this paper, we work in the piecewise-linear (PL) category in the sense of [Gℓ]. All manifolds will be compact. If M^n is an n-manifold with spherical boundary components, then $\hat{M}^n$ denotes the closed n-manifold obtained from M by capping off each component of ∂M with an n-ball.

For graph theory see [Ha]. As general reference about crystallizations, we refer to [FGG]. We shall always use the term *graph* instead of finite multigraph without loops. Given a graph Γ, $V(\Gamma)$ and $E(\Gamma)$ denote the sets of vertices and edges of Γ respectively. By $g(\Gamma)$ we mean the number of connected components of Γ.

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An *edge-coloration on* Γ is a map $\gamma: E(\Gamma) \longrightarrow \Delta_n = \{0,1, \ldots ,n\}$ such that $\gamma(e) \neq \gamma(f)$ for any two adjacent edges $e,f \in E(\Gamma)$.

An *(n+1)-coloured graph with boundary* is a pair (Γ,γ) where Γ is a graph and $\gamma: E(\Gamma) \longrightarrow \Delta_n$ is and edge-coloration on Γ. Note that each vertex of Γ has degree $\leqslant n+1$. By definition, a *boundary vertex* of Γ is a vertex of degree $< n+1$. If Γ has no boundary vertices (i.e. Γ is regular of degree $n+1$), then (Γ,γ) is simply called an *(n+1)-coloured graph.* For every $B \subseteq \Delta_n$, Γ_B denotes the subgraph $(V(\Gamma), \gamma^{-1}(B))$; for any $c \in \Delta_n$, we set $\hat{c} = \Delta_n - \{c\}$. By $g_{\{\alpha,\beta\}}$ $(\alpha,\beta \in \Delta_n, \alpha \neq \beta)$, we denote the number of cycles of $\Gamma_{\{\alpha,\beta\}}$.

(Γ,γ) is said to be *regular with respect to the colour c* if $\Gamma_{\hat{c}}$ is regular of degree n.

Two (n+1)-coloured graphs with boundary (Γ,γ) and (Γ',γ') are said to be *colour isomorphic* if there exist a graph isomorphism $\Phi: \Gamma \longrightarrow \Gamma'$ and a bijection $f: \Delta_n \longrightarrow \Delta_n$ such that $\gamma' \circ \Phi = f \circ \gamma$.

Now let G_{n+1} be the set of all (n+1)-coloured graphs with boundary, regular with respect to the colour n.

For each $(\Gamma,\gamma) \in G_{n+1}$, the *boundary graph* $(\partial\Gamma, \partial\gamma)$ of (Γ,γ) is defined (see [CG]) by the following rules: 1) the vertices of $\partial\Gamma$ are the boundary vertices of Γ; 2) two vertices v,w of $\partial\Gamma$ are joined by an i-coloured edge $(i \in \Delta_{n-1})$ iff v,w belong to the same connected component of $\Gamma_{\{i,n\}}$.

It is proved in [CG] that $(\partial\Gamma, \partial\gamma)$ is a (possibly non connected) n-coloured graph, regular of degree n, and whence $\partial\partial\Gamma$ is void.

Obviously, (Γ,γ) is an (n+1)-coloured graph iff $(\partial\Gamma,\partial\gamma)$ is void.

Given $(\Gamma,\gamma) \in G_{n+1}$, an n-dimensional pseudocomplex (see [HW]), written $K(\Gamma)$, can be associated with (Γ,γ) as follows: 1′) take an n-simplex $\sigma^n(v)$ for each vertex v of Γ and label its vertices by Δ_n; 2′) if v,w are joined in Γ by an edge $e \in \gamma^{-1}(c)$, then identify the (n−1)-faces of $\sigma^n(v)$, $\sigma^n(w)$ which do not contain the vertex labelled by c, so that equally labelled vertices coincide.

By construction, there is a bijection between the connected components of Γ_B (for each $B \subseteq \Delta_n$ with card $(B) = r \leqslant n$) and the set of (n−r)-balls of $K(\Gamma)$ whose vertices are labelled by $\Delta_n - B$. We shall also call *simplexes* the balls of $K(\Gamma)$.

If $|K(\Gamma)|$ is an n-manifold M^n, then (Γ,γ) is said to *represent* M and every homeomorphic manifold. In this case, $\partial\Gamma$ represents the closed (n−1)-manifold ∂M since $\partial K(\Gamma) = K(\partial\Gamma)$.

A graph (Γ,γ) representing an n-manifold with non void boundary is said to be *∂-contracted* iff $\Gamma_{\hat{n}}$ is connected and $g(\Gamma_{\hat{c}}) = g(\partial\Gamma)$ for every $c \in \Delta_{n-1}$. Then $K(\Gamma)$ has only one vertex in its interior and each component of $\partial K(\Gamma)$ has exactly n vertices because the number of c-labelled vertices of $K(\Gamma)$ is equal to $g(\Gamma_{\hat{c}})$, for every $c \in \Delta_n$.

An (n+1)-coloured graph (Γ,γ) representing a closed n-manifold is said to be *contracted* iff $\Gamma_{\hat{i}}$ is connected for every $i \in \Delta_n$. In this case, the pseudocomplex $K(\Gamma)$ has exactly n+1 vertices.

A *crystallization* of an n-manifold M^n with non void boundary (resp. a closed n-manifold) is defined to be a ∂-contracted (resp. contracted) graph which represents M.

Every closed connected n-manifold can be represented by a crystallization [P]. Further , let M^n be a connected n-manifold with h ($h > 0$) boundary components $\partial M_1, \dots, \partial M_h$ and let (Γ_i, γ_i) be a crystallization of ∂M_i ($i \in \Delta_h - \{0\}$). Then there exists a crystallization (Γ, γ) of M such that $(\partial\Gamma, \partial\gamma)$ is colour isomorphic to $\bigcup_{i=1}^{h} (\Gamma_i, \gamma_i)$ (see [CG], [G_1], [CG_r]).

2. THE REGULAR GENUS OF AN N-MANIFOLD WITH BOUNDARY.

For each $(\Gamma, \gamma) \in G_{n+1}$, we construct a graph (Γ^*, γ^*), called the *extended (n+1)-coloured graph* of Γ (see [G_2]), by adding one vertex v* for each boundary vertex v of Γ and an n-coloured edge between v, v*. By V* we denote the set $V(\Gamma^*) - V(\Gamma)$.

An imbedding $j: |\Gamma^*| \longrightarrow F$ of (Γ^*, γ^*) on a bordered surface F is called a *2-cell imbedding* (see [G_2]) iff 1) $\partial F \cap j(|\Gamma^*|) = j(V^*)$; 2) $(\text{Int } F) - j(|\Gamma^*|)$ has open balls (named *regions of* j) as connected components; 3) if R is any such region, then either ∂R is the image of a cycle of Γ^* (R *internal region*) or $\partial R = \alpha(R) \cup \beta(R)$, where $\alpha(R)$ is the image of a walk of Γ^*, $\beta(R)$ is an arc of ∂F and $\alpha(R) \cap \beta(R)$ consists of two (possibly coincident) vertices of V* (R *boundary region*). Moreover, j is said to be ***regular*** iff there exists a cyclic permutation $\epsilon = (\epsilon_0, \epsilon_1, \dots, \epsilon_n)$ of Δ_n, such that, for each internal (resp. boundary) region R, the edges of ∂R (resp. of $\alpha(R)$) are alternatively coloured by ϵ_i, ϵ_{i+1}, i being an integer mod n+1.

Definition 1. – By the *regular genus* $\rho(\Gamma^*)$ (resp. the *hole number* $\lambda(\Gamma^*)$) of (Γ^*, γ^*), we mean the smallest integer r (resp. s) such that Γ^* regularly imbeds on a bordered surface of genus r (resp. a bordered surface with s spherical boundary components).

The above definition is well-posed as shown in [G_2].

Definition 2. – If M^n is a connected n-manifold with boundary ∂M, then the *regular genus* G(M) and the *hole number* L(M) of M are defined as follows

$$G(M) = \min \{ \rho(\Gamma^*) \ / \ (\Gamma, \gamma) \text{ is a crystallization of } M \}$$
$$L(M) = \min \{ \lambda(\Gamma^*) \ / \ (\Gamma, \gamma) \text{ is a crystallization of } M \}.$$

These invariants are proved in [G_2] to coincide with the classical ones in dimension two, and to be related to Heegaard-like handlebody decomposition in dimension three. If ∂M is void, then G(M) gives the analogous concept of [G_3] since $(\Gamma,\gamma) = (\Gamma^*,\gamma^*)$.

Let $\check{S}^n_h$ ($h \geqslant 0$) be the n-manifold with boundary, called *punctured n-sphere*, obtained by taking the interiors of h disjoint n-balls out of the n-sphere S^n. Finally, we state the following propositions proved in [FG]:

Theorem 1.– *Let M^n be a connected n-manifold with (possibly void) boundary. Then M^n is homeomorphic with $\check{S}^n_h$ iff $G(M) = 0$ and $L(M) = h$.*

Proposition 2.– *Let M^n be an n-manifold whose boundary ∂M is the disjoint union of h ($h \geqslant 0$) ($n-1$)-spheres. Then $G(M) = G(\hat{M})$ and $L(M) = h$.*

3. THE REGULAR GENUS OF AN N-DIMENSIONAL KNOT OR LINK.

Let L^n be a knot or link in an (n+2)-sphere S^{n+2}. A *Seifert surface* for L is a connected bicollared (n+1)-manifold $M^{n+1} \subset S^{n+2}$ such that $\partial M = L$. Note that M must be orientable.

Throughout the paper, we shall restrict our attention to knots or links which admit tubular neighbourhoods in S^{n+2}. Under this condition (always satisfied in dimension one), it is proved that each L^n bounds a Seifert surface (see [R]). Thus the following definition is well-posed:

Definition 3.– By the *regular genus* of L^n, we mean the integer

$$g(L) = \min \{ G(M^{n+1}) \ / \ M^{n+1} \text{ is a Seifert surface for } L \} .$$

This concept is clearly a *link invariant.* Since the regular genus of a bordered surface coincides with its genus, the definition 3.– extends the classical genus of a polygonal knot or link L^1 in S^3 (or R^3) (see [R], [Se], [Sc], [Ki], [Ka]).

A Seifert surface M^{n+1} for L^n will be said to be *minimal* iff G(M) = g(L).

Theorem 3.– *Let K^n be an n-dimensional Knot in an (n+2)-sphere S^{n+2}. Then K is equivalent to the trivial knot $S^n \subset S^{n+2}$ iff $g(K) = 0$.*

Proof.– C.N.– If K is equivalent to the trivial knot $S^n \subset S^{n+2}$, then K is the boundary of a flat (n+1)-ball B^{n+1} in S^{n+2} (*basic unknotting theorem,* see [R]). Since B is a flat ball in S^{n+2}, then B is bicollared in S^{n+2}, whence it is a Seifert surface for K. By theorem 1.–, the regular genus of B is zero; thus $0 \leqslant g(k) \leqslant$ $\leqslant G(B) = 0$.

C.S.– If the regular genus of K is zero, let $M^{n+1} \subset S^{n+2}$ be a minimal Seifert surface for K. Since $\partial M = K$ is connected and $G(M) = g(K) = 0$, then M is homeomorphic with an (n+1)-ball $B^{n+1} \subset S^{n+2}$ (see theorem 1.–). Obviously, B is a flat ball of S^{n+2} as M is bicollared. Thus K is equivalent to the trivial knot by the basic unknotting theorem.

The above result can be partially extended to the n-dimensional trivial link with h (h > 1) components, i.e. the disjoint union of h n-spheres standardly imbedded into an (n+2)-sphere.

Corollary 4.– *Let L^n be an n-dimensional link in an (n+2)-sphere S^{n+2}. If L is equivalent to the trivial link, then $g(L) = 0$.*

Proof.– If L is equivalent to the trivial link, then L bounds a flat punctured (n+1)-sphere $\check{S}_h^{n+1}$ in S^{n+2}. By theorem 1.–, we have $0 \leq g(L) \leq G(\check{S}_h^{n+1}) = 0$.

Remark.– Note that the converse of corollary 4.– is false (even in classical dimensions): consider the following non trivial link with genus zero in S^3 (see [R], p. 121).

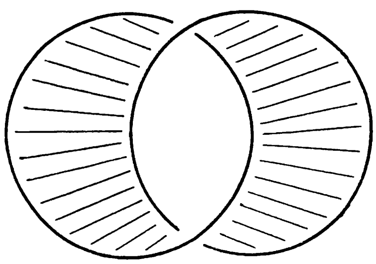


Fig. 1.

Theorem 5.– *Let $L^n \subset S^{n+2}$ be an n-dimensional link with h components. Then, for each minimal Seifert surface M^{n+1} for L, there exist a crystallization (Γ,γ) of M and cyclic permutation $\epsilon = (\epsilon_0, \epsilon_1, \ldots, \epsilon_n, n+1)$ of Δ_{n+1} such that*

$$g(L) = 1 - (1/2) \sum_i g_{\{\epsilon_i, \epsilon_{i+1}\}} + (n\,p)/4 - \bar{p}/4 - h/2 \quad (i \bmod n+1),$$

where $p, \bar{p}$ *are the orders of* Γ, $\partial\Gamma$ *respectively.*

Proof.– Let $M^{n+1} \subset S^{n+2}$ be a minimal Seifert surface for L. Since M^{n+1} has exactly h spherical boundary components, it follows that $L(M) = h$ (see proposition 2.–, sec. 2). By corollary 2 of [FG] and proposition 4.– of [G_2], there exist a crystallization (Γ,γ) of M, a cyclic permutation $\epsilon = (\epsilon_0, \epsilon_1, \ldots, \epsilon_n, n+1)$ of

Δ_{n+1} and a regular imbedding j_ϵ: $|\Gamma^*| \dashrightarrow F_\epsilon$ of Γ^* into the orientable surface F_ϵ with $\lambda(\Gamma^*) = L(M) = h$ holes and genus $\rho(\Gamma^*) = G(M) = g(L)$. By means of a direct computation, it is easy to see that the Euler characteristic $\chi(F_\epsilon)$ of F_ϵ is

$$\sum_i g_{\{\epsilon_i,\epsilon_{i+1}\}} - (n\,p)/2 + \bar{p}/2 \quad (i \bmod n+1).$$

Thus we have $g(L) = G(M) = \rho(\Gamma^*) = 1 - (1/2)\,\chi(F_\epsilon) - h/2$ as requested.

Corollary 6.– a) *A link L^n in S^{n+2} with h components has regular genus zero iff there exist a crystallization (Γ,γ) of a minimal Seifert surface for L and a cyclic permutation $\epsilon = (\epsilon_0, \epsilon_1, \ldots, \epsilon_n, n+1)$ of Δ_{n+1} such that*

$$2 = \sum_i g_{\{\epsilon_i,\epsilon_{i+1}\}} - (n\,p)/2 + \bar{p}/2 + h \quad (i \bmod n+1).$$

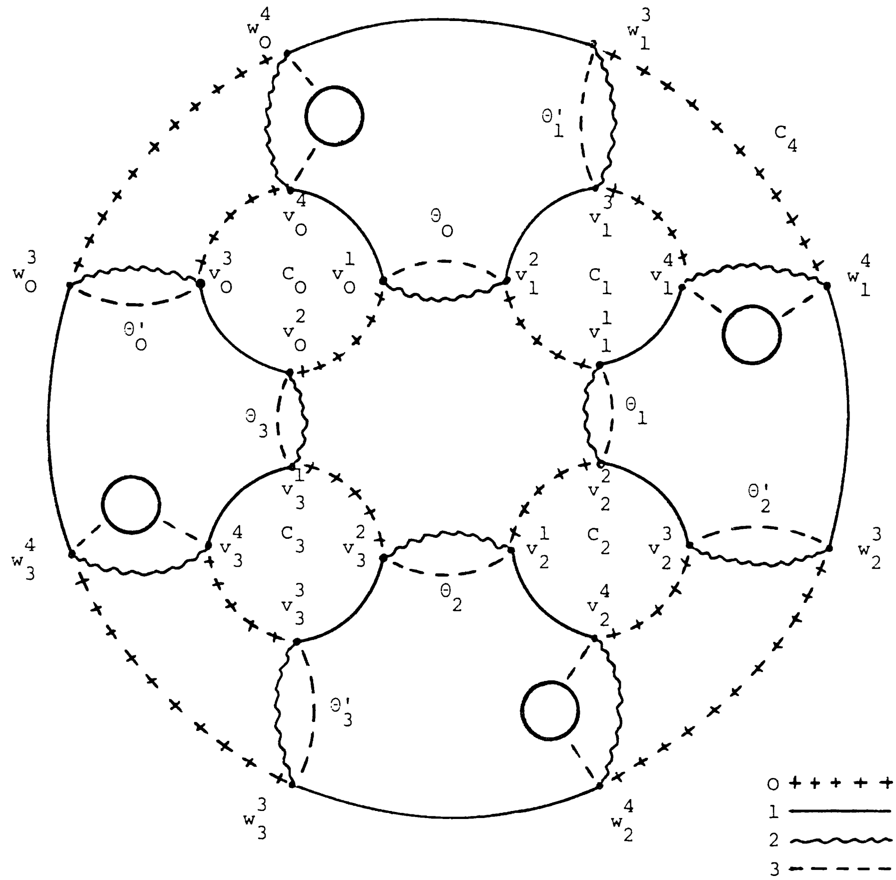
b) *A knot K^n in S^{n+2} is equivalent to the trivial knot iff there exist a crystallization (Γ,γ) of a minimal Seifert surface for K and a cyclic permutation $\epsilon = (\epsilon_0, \epsilon_1, \ldots, \epsilon_n, n+1)$ of Δ_{n+1} such that*

$$1 = \sum_i g_{\{\epsilon_i,\epsilon_{i+1}\}} - (n\,p)/2 + \bar{p}/2 \quad (i \bmod n+1).$$

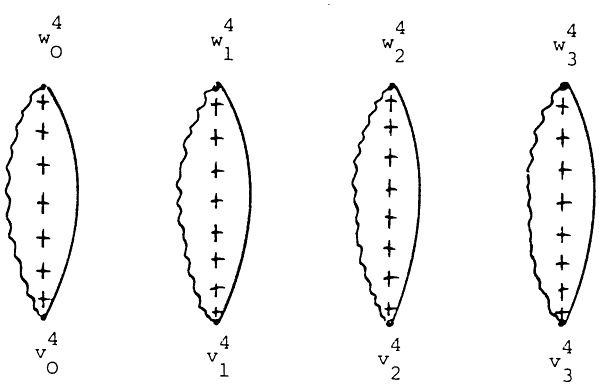
Example: Given an (n+1)-coloured graph (Γ, γ), a subgraph Θ of Γ formed by two vertices X,Y joined by r edges $(1 \leqslant r \leqslant n)$ with colours $c_1, c_2, \ldots, c_r$ will be called an *r-dipole* iff X and Y belong to distinct components of $\Gamma_{\Delta_n - \{c_1,\ldots,c_r\}}$ (see [FGG]).

Now we construct a special crystallization of the (n+1)-sphere S^{n+1}. It will be used to produce a crystallization of genus zero representing a minimal Seifert surface of the n-dimensional trivial link with h components.

Let $(\Gamma_h^{n+1}, \gamma_h^{n+1})$ be the (n+2)-coloured graph defined as follows: take h cycles C_i of length 4 $(i \in \Delta_{h-1})$, cyclically set in the plane and clockwise numbered $0,1,\ldots,h-1$. If $v_i^1, v_i^2, v_i^3\ v_i^4$ are the vertices of C_i clockwise ordered, then colour the edges $v_i^1v_i^2$ and $v_i^3v_i^4$ (resp. $v_i^1v_i^4$ and $v_i^2v_i^3$) by 0 (resp. 1). Let C_h be a cycle of length 2h containing each C_i in its interior and let $w_0^3, w_0^4, w_1^3, w_1^4, \ldots, w_{h-1}^3, w_{h-1}^4$ be its vertices clockwise ordered. Then put an n-dipole Θ_i, with edges labelled by $\Delta_{n+1} - \{0,1\}$, between v_i^1 and v_{i+1}^2 (i mod h) and put an n-dipole Θ_i' (resp. Θ_i''), with edges labelled by $\Delta_{n+1} - \{0,1\}$, between v_i^3 and w_i^3 (resp. v_i^4 and w_i^4) for each $i \in \Delta_{h-1}$. By cancelling dipoles, the graph $(\Gamma_h^{n+1}, \gamma_h^{n+1})$ becomes the standard crystallization of S^{n+1}, which consists of two vertices joined by n+2 edges labelled by Δ_{n+1}. A crystallization $(\check{\Gamma}_h^{n+1}, \check{\gamma}_h^{n+1})$ of $\check{S}_h^{n+1}$ is obtained from $(\Gamma_h^{n+1}, \gamma_h^{n+1})$ by deleting the (n+1)-coloured edge of

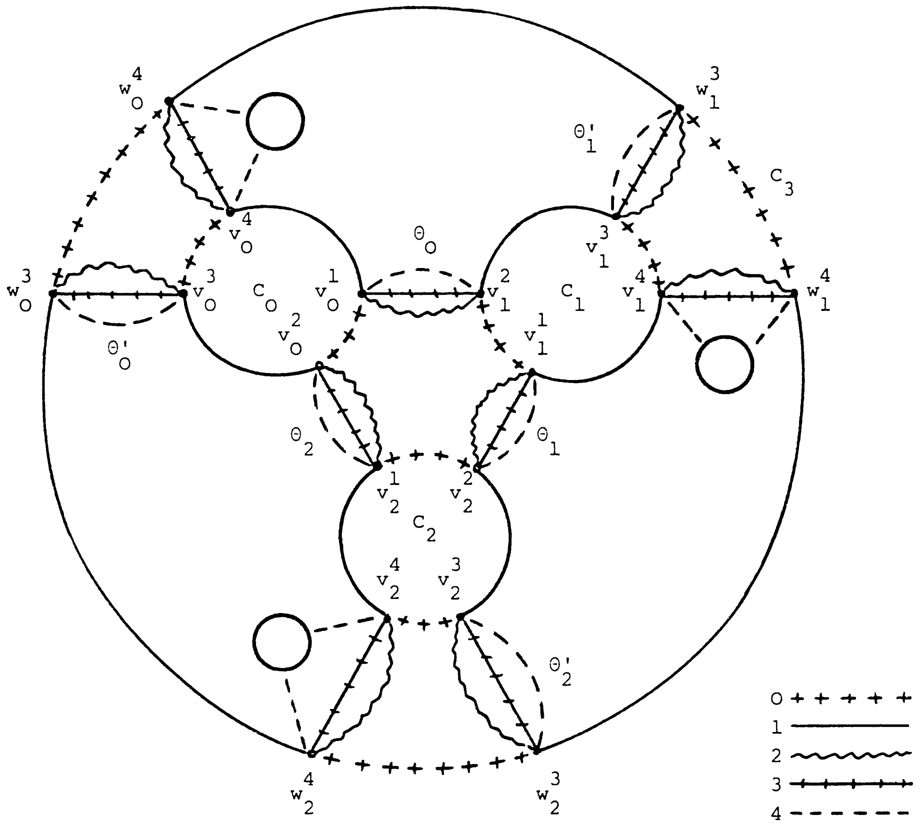


A regular imbedding of $((\check{\Gamma}_4{}^4)^*, (\check{\gamma}_4{}^4)^*)$ into the punctured 2-sphere with 4 holes.

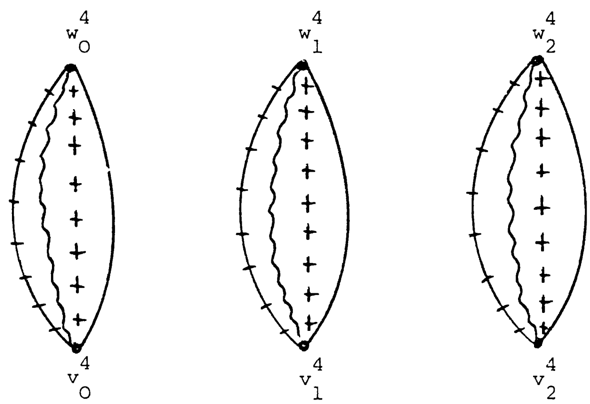


The boundary graph $(\partial\check{\Gamma}_4{}^4, \partial\check{\gamma}_4{}^4)$ of $(\check{\Gamma}_4{}^4, \check{\gamma}_4{}^4)$

Fig. 2.



A regular imbedding of $((\check{\Gamma}_3{}^5)^*, (\check{\gamma}_3{}^5)^*)$ into the punctured 2-sphere with 3 holes.



The boundary graph $(\partial\check{\Gamma}_3{}^5, \partial\check{\gamma}_3{}^5)$ of $(\check{\Gamma}_3{}^5, \check{\gamma}_3{}^5)$.

Fig. 3.

Θ_i'' for every $i \in \Delta_{h-1}$. It is an easy exercise to prove that $((\check{\Gamma}_h^{n+1})^*, (\check{\gamma}_h^{n+1})^*)$ regularly imbeds into the punctured 2-sphere with h holes, so that $\rho((\check{\Gamma}_h^{n+1})^*) = 0$ and $\lambda((\check{\Gamma}_h^{n+1})^*) = h$. In fig. 2 and fig. 3 we illustrate the above construction for $\check{S}_4{}^3$ and $\check{S}_3{}^4$ respectively.

***Theorem** 7.– Let $L^1 \subset S^3$ be a one-dimensional link with h components. Then, for each minimal Seifert surface M^2 for L, there exists a crystallization (Γ, γ) of M such that*

$$g(L) = 1/2 + p/4 - h .$$

Proof. – Let M^2 be an arbitrary minimal Seifert surface for L and let (Γ, γ) be a crystallization of M^2 which satisfies the property of theorem 5. Since every component of $\partial\Gamma$ is a crystallization of a one-dimensional sphere, it only consists of two vertices joined by two different coloured edges. Thus the order $\bar{p}$ of $\partial\Gamma$ is 2h.

Recall that there is a bijection between the vertices of the interior of $K(\Gamma)$ and the set of all bicoloured cycles of Γ since Γ is a 3-coloured graph with boundary. Then the fact that $K(\Gamma)$ has exactly one vertex in its interior implies the relation

$$g_{\{\epsilon_0,\epsilon_1\}} + g_{\{\epsilon_1,\epsilon_2\}} + g_{\{\epsilon_2,\epsilon_0\}} = 1 .$$

Finally the genus of L^1 is $g(L^1) = 1 - (1 - p/2 + 2h)/2 = 1/2 + p/4 - h$.

***Corollary** 8.– A knot $K^1 \subset S^3$ is equivalent to the trivial knot iff there exists a two order crystallization (Γ, γ) of a minimal Seifert surface for K:*

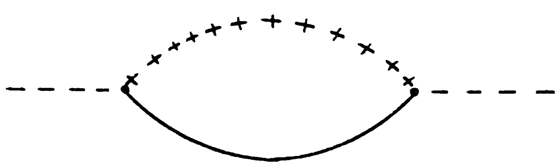


Fig. 4.

A new problem arises naturally from the above results: the study of the relations between a crystallization of the complement of a knot or link L^n in S^{n+2} and a crystallization of a suitable Seifert surface for L. In particular, it would be interesting to construct a graph-theoretical algorithm to obtain the latter crystallization from the former one.

4. MINIMAL SEIFERT SURFACES AND REGULAR GENUS.

If R is a ring and A is a finitely generated R-module, let rk(A) be the minimum number of generators of A.

Recently Bracho and Montejano ([BM]) have proved that $rk(\Pi_1(M)) \leqslant G(M)$ for any closed connected n-manifold M ($n \geqslant 3$). Here this result will be used to study some properties about minimal Seifert surfaces and their fundamental groups. We also relate the regular genus of a knot to the one-dimensional homology of the universal abelian covering of its complement.

In order to make the following proposition clear, we need some definitions listed in [L].

Let K^n be an n-dimensional knot in an (n+2)-sphere S^{n+2}. By $X = S^{n+2} - K$ we denote the *complement* of K in S^{n+2}. The *universal abelian covering* $\widetilde{X}$ of X is the covering associated with the commutator subroup $C = [G,G]$ of $G = \Pi_1(X)$. If Λ is the integral group ring of Z, then $H_*(\widetilde{X})$ becomes a finitely generated Λ-module. Similarly the rational homology of $\widetilde{X}$, i.e. $H_*(\widetilde{X}; Q) \simeq \simeq H_*(\widetilde{X}) \otimes_Z Q$, is a finitely generated module over the rational group ring $\widetilde{\Lambda} = \Lambda \otimes_Z Q$ of the integers.

Theorem 9. *– With the above notation and for each $n \geqslant 2$, we have*

1) $g(K) \geqslant rk(H_1(\widetilde{X}; Q))$ (as $\widetilde{\Lambda}$-module)

2) $g(K) \geqslant rk(\frac{C}{[C,C]} \otimes_Z Q)$ (as $\widetilde{\Lambda}$-module)

Proof. –

1) We recall the construction given in [L] (sec. 2.4) to obtain a presentation for $H_q(\widetilde{X}; Q)$ (as $\widetilde{\Lambda}$-module) by using an arbitrary Seifert surface $M^{n+1} \subset S^{n+2}$ of K. Let Y be the (n+2)-manifold obtained from S^{n+2} by cutting along M. Then ∂Y consists of two copies of M, say N_1 and N_2, identified along their boundaries. Let (Y_i, N_1^i, N_2^i) ($i \in Z$) be a countable number of copies of $(Y\text{-}K, N_1\text{-}K, N_2\text{-}K)$. Then $\widetilde{X}$ is obtained as a quotient space of the disjoint union of the Y_i' s by identifying N_2^i with N_1^{i+1} for every i. If $B_q(M)$ is the free abelian group image of $H_q(M) \longrightarrow H_q(M;Q)$, then choose a basis $\{a_i^q\}$ of $B_q(M)$ and a dual basis $\{b_i^p\}$ of $B_p(S^{n+2} - M) \simeq B_q(Y)$ ($p = n+1-q$).

If $i_1, i_2 : M \longrightarrow Y$ are defined by the identification of M with N_1 or N_2, then set $(i_2)_*(a_j^q) = \Sigma\, \lambda_{ij}^q\, b_i^q$ and $(i_1)_*(a_j^q) = \Sigma\, \mu_{ij}^q\, b_i^q$. It is proved in [L] that the square matrix $P_q(t) = \| t\mu_{ij}^q - \lambda_{ij}^q \|$ is a presentation matrix for the $\widetilde{\Lambda}$-module $H_q(\widetilde{X}; Q)$, where t is the variable of the Laurent polynomials in $\widetilde{\Lambda}$. As a direct consequence, we have $rk(H_q(\widetilde{X}; Q)$ as $\widetilde{\Lambda}$-module) $\leq rk(B_q(M)) \leq rk(H_q(M)$ as Z-module). For $q = 1$ and $n \geq 2$, it follows that $rk(H_1(M)$ as Z-module) $\leq$ $\leq rk(\Pi_1(M)) = rk(\Pi_1(\hat{M}) \leq G(\hat{M}) = G(M)$ (use the above mentioned result of [BM] and proposition 2. sec. 2).

Thus $g(K) = G(M) \geq rk(H_1(\widetilde{X}; Q)$ as $\widetilde{\Lambda}$-module), whenever M is an arbitrary minimal Seifert surface for K.

2) It is an easy corollary of 1) and of the Λ-isomorphism $H_1(\widetilde{X}) \simeq \dfrac{C}{[C,C]}$ given in [R] p. 174.

Remark.– The equalities are not generally true in the statement of theorem 9.–; In fact, for each $n \geq 3$, there exist non trivial knots K^n in S^{n+2} with infinite cyclic knot group $\Pi_1(S^{n+2} - K) \simeq Z$ (see [R]). Thus $\Pi_1(\widetilde{X}) = H_1(\widetilde{X}) = H_1(\widetilde{X}; Q) = 0$ but $g(K) > 0$.

Proposition 10,– *Let L^n be an n-dimensional link in an (n+2)-sphere $S^{n+2} (n \geq 2)$. If $g(L) = 1$, then the fundamental group of each minimal Seifert surface for L is cyclic. In general, if $g(L) = k$, then the fundamental group of each minimal Seifert surface for L is a quotient of the free group on h generators for some $h \leq k$.*

Proof.– Let $M^{n+1} \subset S^{n+2}$ be an arbitrary minimal Seifert surface for L. Then we have $k = G(M) = G(\hat{M}) \geq rk(\Pi_1(\hat{M})) = rk(\Pi_1(M))$ as required.

Proposition 11.– *Let K^n $(n \geq 2)$ be an n-dimensional knot in an (n+2)-sphere. If $g(K) = 1$ and $\Pi_1(S^{n+2} - K)$ is not infinite cyclic, then the fundamental group of each minimal Seifert surface for K is non-trivial cyclic.*

Proof.– Let $M^{n+1} \subset S^{n+2}$ be an arbitrary minimal Seifert surface for K. By proposition 10, the fundamental group $\Pi_1(M)$ is cyclic. Since $\Pi_1(M) = 0$ implies that $\Pi_1(S^{n+2} - K) \simeq Z$ (see [F]), the proof is completed.

Proposition 12.– *Let $L^2 \subset S^4$ be a 2-dimensional link with h components and let $M^3 \subset S^4$ be a minimal Seifert surface for L. If $g(L) = 1$, then M is homeomorphic to the connected sum $\# hB^3 \# N$, N being a lens space (different from S^3).*

Proof.– If M is minimal, then $G(\hat{M}) = G(M) = g(L) = 1$ (see proposition 2 sec. 2). Thus $\hat{M}$ is homeomorphic to a lens space N ($S^2 \times S^1$ is included among lens spaces) as it is a closed orientable 3-manifold with Heegaard genus one, whence $M \approx \# hB^3 \# N$ (where N cannot be homeomorphic to S^3).

Proposition 13. *– Let $L^3 \subset S^5$ be a 3-dimensional link with h components and let $M^4 \subset S^5$ be a minimal Seifert surface for K. If g(L) = 1, then M is homeomorphic to* $\# hB^4 \# (S^3 \times S^1)$.

Proof. – If M is minimal, then we have $G(\hat{M}) = G(M) = g(L) = 1$. The relation $G(\hat{M}) = 1$ implies that $\hat{M}$ is homeomorphic to $S^3 \times S^1$ since the unique closed orientable 4-manifold with regular genus one is $S^3 \times S^1$ (the proof will appear in [C]); therefore the statement follows.

The last two propositons allow us to determine the topological structure of a minimal Seifert surface for a low-dimensional link L by starting from the regular genus of L. This fact suggests to study analogous relations in higher dimensions.

5. Connected sums.

Let M_i (i = 1,2) be a connected n-manifold with h_i boundary components $\partial_j M_i$ (j = 1,2, . . . ,h_i). The *boundary connected sum of* M_1 *and* M_2 *with respect to* $\partial_r M_1$ *and* $\partial_s M_2$, written $M_1 \#_{\partial(r,s)} M_2$, is the n-manifold obtained by identifying two standard (n−1)-balls B_1 and B_2 contained in $\partial_r M_1$ and $\partial_s M_2$ respectively (also see [R]).

Given two (n+1)-coloured graphs (Γ_1, γ_1), (Γ_2, γ_2) with boundary and two boundary vertices $P_1 \in V(\Gamma_1)$, $P_2 \in V(\Gamma_2)$, we define the *boundary connected sum of* Γ_1 *and* Γ_2 *with respect to* P_1 *and* P_2 as the (n+1)-coloured graph with boundary $(\Gamma_1 \#_{\partial(P_1,P_2)} \Gamma_2, \#_{\partial\gamma})$ obtained by deleting P_1 and P_2 from Γ_1 and Γ_2 and pasting together the pairs of free edges (the ones that had an end-point in the deleted vertices) with the same colour.

If (Γ_i, γ_i) is a crystallization of M_i (i = 1,2), then let $\partial_r \Gamma_1$ (resp. $\partial_s \Gamma_2$) be the connected component of $\partial\Gamma_1$ (resp. $\partial\Gamma_2$) representing $\partial_r M_1$ (resp. $\partial_s M_2$). By construction, it is easily seen that a crystallization of the boundary connected sum $M_1 \#_{\partial(r,s)} M_2$ is given by the boundary connected sum of (Γ_1, γ_1) and (Γ_2, γ_2) with respect two arbitrarly chosen boundary vertices $P_1 \in V(\partial_r \Gamma_1)$ and $P_2 \in V(\partial_s \Gamma_2)$.

If L_i^n (i = 1,2) is an n-dimensional link with h_i components $L_{i,1}, \ldots, L_{i,h_i}$, let $L_1^n \#_{r,s} L_2^n$ be the *connected sum of* L_1^n *and* L_2^n *with respect to* $L_{1,r}$ *and* $L_{2,s}$.

Theorem 14. *– With the above notation, the regular genus of links is subadditive, i.e., for each r,s, we have*

$$g(L_1^n \#_{r,s} L_2^n) \leqslant g(L_1^n) + g(L_2^n)$$

Proof. – Let M_i (i = 1,2) be a minimal Seifert surface for L_i and let (Γ_i, γ_i) be a crystallization of M_i such that $\rho(\Gamma_i^*) = G(M_i) = g(L_i^n)$ and $\lambda(\Gamma_i^*) = h_i$. We write $\partial_r M_1$ (resp. $\partial_s M_2$) for the component of ∂M_1 (resp. ∂M_2) that coincides with $L_{1,r}$ (resp. $L_{2,s}$) and let $\partial_r \Gamma_1$ (resp. $\partial_s \Gamma_2$) be the component of Γ_1 (resp. Γ_2) representing $\partial_r M_1$ (resp. $\partial_s M_2$). The (n+1)-manifold $M_1 \#_{\partial(r,s)} M_2$ is a Seifert surface for $L_1^n \#_{r,s} L_2^n$ (for each pair r,s) and $(\Gamma_1 \#_{\partial(P_1,P_2)} \Gamma_2, \#_{\partial\gamma})$ is a crystallization of $M_1 \#_{\partial(r,s)} M_2$, where $P_1 \in V(\partial_r \Gamma_1)$ and $P_2 \in V(\partial_s \Gamma_2)$. Let F_i^ϵ (i = 1,2) be the orientable bordered surface, with genus $\rho(\Gamma_i^*)$ and h_i holes, in which Γ_i^* regularly imbeds with respect to a cyclic permutation $\epsilon = (\epsilon_0, \epsilon_1, \ldots, \epsilon_n, n+1)$ of Δ_{n+1}. Therefore $(\Gamma_1 \#_{\partial(P_1,P_2)} \Gamma_2)^*$ admits a regular imbedding, associated with the same ϵ, into the orientable bordered surface $F_1^\epsilon \#_{\partial(r,s)} F_2^\epsilon$ with genus $\rho(\Gamma_1^*) + \rho(\Gamma_2^*)$ and $h_1 + h_2 - 1$ holes. Thus we have $g(L_1^n \#_{r,s} L_2^n) \leqslant G(M_1 \#_{\partial(r,s)} M_2) \leqslant \rho((\Gamma_1 \#_{\partial(P_1,P_2)} \Gamma_2)^*) = g(\Gamma_1^*) + \rho(\Gamma_2^*) = G(M_1) + G(M_2) = g(L_1) + g(L_2)$ (for each r,s) as requested.

Remark. – The regular genus of one-dimensional polygonal links is *additive* as it coincides with the usual genus (see [R]). In higher dimensions, we do not know whether such a result holds too. Nevertheless it would be interesting (if it is possible) to prove at least the weaker inequalities $g(K_i^n) \leqslant g(K_1^n \# K_2^n)$, i = 1,2, for n-dimensional knots ($n \geqslant 2$) becouse this fact implies the *non cancellation theorem* in higher dimensions (see [R] for the one-dimensional case), i.e. if $K_1^n \# K_2^n$ is a trivial knot, then both K_1 and K_2 are trivial knots.

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