A DIRECT PROOF OF GODEL'S INCOMPLETENESS THEOREMS

by

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ABSTRACT

An unusual enumeration of partial recursive functions is defined starting from the equivalence between recursiveness and representability in Peano arithmetic. This allows us to obtain by a double diagonalization an undecidable formula expressing a property of Peano arithmetic, equivalent to its consistency.

The basic idea in the proof of the Gödel's incompleteness theorem is the posibility of defining primitive recursive functions in the Peano arithmetic. This allows us to construct a formula which in some way expresses its own indemostrability, so that —under the assumption of the ω-consistency of the arithmetic— this formula cannot be proved, nor refuted.

Starting from a stronger representation theorem, we difine in this paper an enumeration of the partial recursive functions from an enumeration of the arithmetical formulae. The we construct by diagonalization a partial recursive function and by a new diagonalization over its index we obtain an undecidable formula asserting the consistency of the Peano arithmetic in a rather unnatural way.

Let Π be an axiomatization (without induction) of the Peano arithmetic. For each natural number m let $\underline{m}$ be the numeral $s^m 0$ ($\underline{0} = 0$, $\underline{1} = s0$, $\underline{2} = ss0$, etc . . .).

Def.: A partial function $\psi\colon \omega^n \to \omega$ is definable in the Peano arithmetic if a formula α with the free variables $x_0, x_1, \ldots, x_n$ exists, such that for every $a_1, \ldots, a_n$, $b \in \omega$ the following is verified:

(a) if $\psi(a_1, \ldots, a_n) = b$, then $\Pi \vdash \alpha(x_0, \underline{a}_1, \ldots, \underline{a}_n) \longleftrightarrow x_0 \equiv \underline{b}$
(b) if $\psi(a_1, \ldots, a_n)\uparrow$, then $\Pi \nvdash \exists x_0 \alpha(x_0, \underline{a}_1, \ldots, \underline{a}_n)$.

TH 1 (representation theorem) If Π is ω-consistent, then a partial function is recursive if and only if it is definable in Π.

A proof of TH 1 can be obtained by induction on the definition of partial recursive functions (cfr. (Prd 1982), pp. 202-209).

Let $\alpha_0^n, \alpha_1^n, \alpha_2^n, \ldots$ be an effective enumeration of the aritmetical formulae with the free variables $x_0, x_1, \ldots, x_n$ and let $\phi_k^n: \omega^n \to \omega$ be the partial function

$$\{<a_0, \ldots, a_n, b>: \Pi \vdash \alpha_k^n (x_0, \underline{a}_1, \ldots, \underline{a}_n) \longleftrightarrow x_0 \equiv \underline{b}\}$$

It follows inmediatly from TH 1:

TH 2 (enumeration theorem) If Π is ω-consistent, then a partial function $\psi: \omega^n \to \omega$ is recursive if ad only if a $k\epsilon\omega$ exists, such that $\psi = \phi_k^n$.

From this characterization of the partial recursive functions Gödel's incompleteness theorem follows inmediatly. Indeed, let r be an index of the partial recursive function ψ such that for every $m\epsilon\omega$

$$\psi(m) = \begin{cases} 0 & \text{if } \Pi \vdash \neg\exists x_0 \alpha_m^1 (x_0, \underline{m}) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The following propositions are obviously equivalent (second diagonalization!):

i) $\Pi \vdash \neg\exists x_0 \alpha_r^1 (x_0, \underline{r})$
ii) $\phi_r^1 (r) \downarrow$
iii) $\Pi \vdash \exists x_0 \alpha_r^1 (x_0, \underline{r})$.

It follows inmediatly from the equivalence of i) and iii):

TH 3 (Gödel's first incompleteness theorem) If Π is ω-consistent, then neither $\Pi \vdash \exists x_0 \alpha_r^1 (x_0, \underline{r})$ nor $\Pi \vdash \neg\exists x_0 \alpha_r^1 (x_0, \underline{r})$.

The second incompleteness therem can be obtained from the

LEMMA: The following are equivalent:
(a) Π is inconsistent
(b) $\Pi \vdash \neg\exists x_0 \alpha_r^1 (x_0, \underline{r})$
(c) $\phi_r^1 (r) \downarrow$.

In fact, obviously (a) implies (b), (b) implies (c) and (c) implies (a).

It follows from the definition of ϕ_a^1 that the formula $\exists x_0 \alpha_a^1 (x_0, \underline{b})$ asserts in the language of the arithmetic that ϕ_a^1 (b) is defined. Therefore the formula $\neg\exists x_0 \alpha_r^1 (x_0, \underline{r})$ asserts that $\phi_r^1 (r) \uparrow$ and hence, according with the lemma, that Π is consistent. Let us call this formula "CONS".

Because of the equivalece of (a) and (b) we have:

TH 4 (Gödel's second incompleteness theorem) $\Pi \vdash$ CONS if and only if Π is inconsistent.

BIBLIOGRAFIA

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