A CLASS OF OPERATORS FROM A BANACH LATTICE INTO A BANACH SPACE

by

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ABSTRACT:

In this paper we study a class of operators from a Banach lattice X into a Banach space B. These operators map positive sequences in weak-l^p-spaces with values in X into sequences in l^q-spaces with values in B. We obtain some different characterizations of them an we consider, in particular, the case $X = l^r$.

The present paper is devoted to the study of certain class of operators from a Banach lattice X into a Banach space B. These operators map positive sequences in weak-l^p-spaces with values in X into sequences in l^q-spaces with values in B. They are near the (p, q)-absolutely summing operators $\Pi_{pq}(X,B)$ [4] and also the operators of type $\leqslant (p, q)$ defined by Maurey [7].

We shall obtain different characterizations of such operators and we shall study them for sequence Banach lattices. Finally we shall relate this operators to the Orlicz property on a Banach lattice.

Throughout this paper X will denote a Banach lattice and B a Banach space. Given $x_1, x_2, \ldots, x_n$ in X and $1 \leqslant p < \infty$ we shall use the following notations

(1) $$w_p(x_i) = \sup_{\|\xi\|_{X^*} \leqslant 1} \left(\sum_{1=i}^{n} |<\xi, x_i>|^p \right)^{1/p}$$

(2) $$w_p^+(x_i) = \sup_{\|\xi\|_{X^*} \leqslant 1,\, \xi \geqslant 0} \left(\sum_{i=1}^{n} <\xi, |x_i|>^p \right)^{1/p}$$

(3) $$l^p(B) = \{ (x_n) \subset B : \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p} < +\infty \}$$

(4) $l^p[B] = \{(x_n) \subset B : w_p(x_n) < +\infty\}$

(5) $l^p_+[X] = \{(x_n) \subset X : w^+_p(x_n) < +\infty\}$

All this terminology enables us to introduce the following.

DEFINITION 1. Let $1 \leqslant p,q < \infty$. An operator $T:X \to B$ is said to be positive (p,q)-summing if there exists a constant $C > 0$ such that for every $x_1, x_2, \dots, x_n \geqslant 0$ in X we have

(6) $$\left(\sum_{i=1}^{n} \|Tx_i\|_B^p\right)^{1/p} \leqslant C\, w_q(x_i)$$

For $q < p = \infty$,

(6′) $$\sup_{1 \leqslant i \leqslant n} \|Tx_i\|_B \leqslant C\, w_q(x_i)$$

We shall denote by $\Lambda_{pq}(X,B)$ (or $\Lambda_p(X,B)$ if $p = q$) the space of such operators. This becames a Banach space with the norm $\| \ \|_{\Lambda_{pq}}$ given by the infimum of the constants verifying (6) or (6′).

Observe that $\Lambda_{\infty,q}(X,B) = L(X,B)$ for all $1 \leqslant q < \infty$.

The positive (1,1)-summing operator are already known. They are called order summing or cone absolutely summing [10].

PROPOSITION 1. Let $1 \leqslant p,q \leqslant \infty$.

(a) If $p < q$ then $\Lambda_{pq}(X,B) = \{0\}$.

(b) If $q \geqslant r$ then $\Lambda_{pq}(X,B) \subseteq \Lambda_{pr}(X,B)$.

(c) If $p \leqslant r$ and $\frac{1}{p} + \frac{1}{s} = \frac{1}{q} + \frac{1}{r}$ then

$\Lambda_{pq}(X,B) \subseteq \Lambda_{rs}(X,B)$ (In particular $\Lambda_p(X,B) \subseteq \Lambda_r(X,B)$ if $p \leqslant r$).

(d) If $X_2 \xrightarrow{S} X_1 \xrightarrow{T} B_1 \xrightarrow{R} B_2$, $S \geqslant 0$ and $T \in \Lambda_{pq}(X_1,B)$ then $T.S \in \Lambda_{pq}(X_2,B_1)$ and $R.T \in \Lambda_{pq}(X_1,B_2)$.

(e) If $X_1 \subseteq X_2$ (for $x \in X_1$, $\|x\|_{X_2} \leqslant \|x\|_{X_1}$) and $\overline{X}_1 = X_2$ (X_1 dense in X_2) then $\Lambda_{pq}(X_2,B) \subseteq \Lambda_{pq}(X_1,B)$.

Proof.

(a) It is sufficient to take $x_n = n^{-1/p}x$ for some positive x in X to get a contradiction.

(b) It is clear since $1^r[X] \subset 1^q[X]$ for $q \geqslant r$.

(c) The argument used by Kwapien [4] can be reproduced in our case to prove the statement.

(d) It is straightforward to show $R \cdot T \in \Lambda_{pq}(X_1, B_2)$. To see that $T \cdot S \in \Lambda_{pq}(X_2, B_1)$ let us notice that if $x_1, x_2, \ldots, x_n \geqslant 0$ and $S \geqslant 0$ then

$$w_q(Sx_i) =$$

$$= \sup_{\|\xi\|_{X_1^*} \leqslant 1} \left(\sum_{i=1}^{n} |< \xi, Sx_i >|^q \right)^{1/q} =$$

$$\|S^*\| \sup_{\|\xi\|_{X_1^*} \leqslant 1} \left(\sum_{i=1}^{n} \left|< \frac{S^*\xi}{\|S^*\|}, x_i >\right|^q \right)^{1/q} \leqslant \|S\| \cdot w_q(x_i).$$

(e) It is obvious since if $\xi \in X_2^*$ and $\|\xi\|_{X_2^*} \leqslant 1$ then $\xi \in X_1^*$ and $\|\xi\|_{X_1^*} \leqslant 1$.

PROPOSITION 2. Let $1 \leqslant q \leqslant q \leqslant \infty$. The following statements are equivalent

(a) $T \in \Lambda_{pq}(X,B)$.

(b) There exists a constant $C > 0$ such that for every $x_1, x_2, \ldots, x_n \geqslant 0$ in X we have

$$(7) \quad \left(\sum_{i=1}^{n} \|Tx_i\|_B^p \right)^{1/p} \leqslant C \sup_{\Sigma \alpha_i^{q'} = 1} \left\| \sum_{i=1}^{n} \alpha_i x_i \right\|_X \quad (1 < q \leqslant \infty)$$

$$(7') \quad \left(\sum_{i=1}^{n} \|Tx_i\|_B^p \right)^{1/p} \leqslant C \left\| \sum_{i=1}^{n} x_i \right\|_X \qquad q = 1$$

(c) T maps positive sequences (x_n) in $\ell^q[X]$ in sequences (Tx_n) in $\ell^p(B)$.

(d) $\hat{T} : \ell^q_+[X] \to \ell^q(B)$, defined by $\dot{T}(x_n) = (Tx_n)$, is continuous.

Proof. We shall show the equivalence of each statement with (a).

(a) <==> (b) It follows from the following fact: For $1 < q \leqslant \infty, \frac{1}{q} + \frac{1}{q'} = 1$

$$w_q(x_i) = \sup_{\|\xi\|_{X^*} \leqslant 1} \left(\sum_{i=1}^{n} |< \xi, x_i >|^q \right)^{1/q} =$$

$$= \sup_{\|\xi\|_{X^*} \leqslant 1} \; \sup_{\Sigma \alpha_i^{q'} = 1} \left| \sum_{i=1}^{n} < \xi, \alpha_i x_i > \right| =$$

$$= \sup_{\Sigma \alpha_i^{q'} = 1} \; \sup_{\|\xi\|_{X^*} \leqslant 1} \left| < \xi, \sum_{i=1}^{n} \alpha_i x_i > \right| =$$

$$= \sup_{\Sigma \alpha_i^{q'} = 1} \left\| \sum_{i=1}^{n} \alpha_i x_i \right\|_X$$

For $q = 1$ and $x_1, x_2, \ldots, x_n \geqslant 0$ then $w_1(x_i) = \|\Sigma x_i\|_X$ ([10]).

(a) <==> (c). The direct implication is obvious. To see the converse let us suppose for every C these exist $x_1, x_2, \ldots, x_{N(C)} \geqslant 0$ such that

$$\left(\sum_{i=1}^{N(C)} \|Tx_i\|^p \right)^{1/p} \leqslant C\, w_q(x_i)$$

Let us take $C = n.2^n$, then there will be $x_1^n, x_2^n, \ldots, x_{m_n}^n \geqslant 0$ verifying $w_q(x_i^n) \leqslant 1/2^n$ and $\left(\sum_{i=1}^{m_n} \|Tx_i^n\|^p \right)^{1/p} \geqslant n$. By considering the sequence $x_1^1, x_2^1, \ldots, x_{m_1}^1, x_1^2, \ldots, x_{m_2}^2, \ldots$ we have that this positive sequence belongs to $l^q[X]$ and its image does belong to $l^p(B)$.

(a) <==> (d). It follows from the next simple fact:

For $(x_n) \geqslant 0$, $w_q(x_n) = w_q^+(x_n)$.

Obviously d) implies a). Now let us suppose $T \in \Lambda_{pq}(X,B)$ and $(x_n) \in l_+^p[X]$.

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p \right)^{1/p} \leqslant \left(\sum_{i=1}^{n} \|Tx_i^+\|^p \right)^{1/p} + \left(\sum_{i=1}^{n} \|Tx_i^-\|^p \right)^{1/p} =$$

$$\leqslant C.w_q(x_i^+) + C.w_q(x_i^-) \leqslant 2Cw_q^+(x_i).$$

Therefore $\|\hat{T}(x_n)\|_{l^p(B)} \leqslant 2Cw_q^+(x_n)$. #

Remark. By condition (b) we notice that a Banach lattice satisfy a lower p-estimate [6] if and only if the identity I belongs to $\Lambda_{p,1}(X,X)$.

This is the first connection of these spaces with some classical spaces. Another operators very related to these are the (p,q)-absolutely summing operators (cf. [4], [3]), as easily follows from both definitions. Let us recall another concept quite similar to these above. The (p,q)-concave operators, called in [7] operators of "type mixte" $\leqslant$ (p,q).

An operator $T \in L(X,B)$ is said to be (p,q)-concave if there exists a constant C such that for every $x_1, x_2, \ldots, x_n$ in X we have

$$(8)\quad \left(\sum_{i=1}^{n} \|Tx_i\|_B^p\right)^{1/p} \leqslant C \left\| \left(\sum_{i=1}^{n} |x_i|^q\right)^{1/q} \right\|_X$$

where $\left(\sum_{i=1}^{n} |x_i|^q\right)^{1/q}$ is the element in X given by $\sup\limits_{\Sigma\alpha_i^{q'} \leqslant 1} \sum_{i=1}^{n} \alpha_i x_i$.

We shall denote by $\mathcal{C}_{pq}(X,B)$ the espace of (p,q)-concave operators endowed with the usual norm $\| \quad \|_{\mathcal{C}_{pq}}$ given by the infimum of the constants verifying (8).

With all this terminology we have

PROPOSITION 3. For $1 \leqslant q \leqslant p \leqslant \infty$

$$\Pi_{pq}(X,B) \subseteq \Lambda_{pq}(X,B) \subseteq \mathcal{C}_{pq}(X,B).$$

Proof. The first inclusion is obvious and the second one is a simple consequence of Prop. 2 (d) and the fact that $\left\| \left(\sum_{i=1}^{n} |x_i|^q\right)^{1/q} \right\| \geqslant w_q^+(x_i)$ as it can be seen by using the homogeneus calculus on lattices [6]. #

In general the inclusions are strict as it will be shown later, but for q = 1 we have the following.

PROPOSITION 4.

(a) $\Lambda_{p1}(X,B) = \mathcal{C}_{p1}(X,B)$

(b) $\Lambda_{p1}(X,B) \subseteq \mathcal{C}_{qq}(X,B)$ for all $q > p$.

Proof. It is inmediate since for $x_1, x_2, \ldots, x_n \geqslant 0$ then

$$\left\| \sum_{i=1}^{n} |x_i| \right\|_X = \left\| \sum_{i=1}^{n} x_i \right\|_X = w_1(x_i).$$

Now a result of Maurey [7] shows $\mathcal{C}_{p1}(X,B) \subset \mathcal{C}_{qq}(X,B)$ if $q > p$ and then (b) follows from (a). #

There is a way of using operators in Λ_{pq} to deduce semothing about Π_{pq} and $\mathcal{C}_{pq}$ as the following result shows.

PROPOSITION 5.

(a) $T \in \Pi_{p1}(X,B)$ if and only if for every operator S in $L(c_o,X)$, $T . S \in \Lambda_{p1}(c_o,B)$ and $\| T . S \|_{\Lambda_{p1}} \leqslant C . \| S \|$.

(b) $T \in \Lambda_{p1}(X,B)$ if and only if for every positive operator S in $L(c_o,X)$, , $T . S \in \Lambda_{p1}(c_o,B)$ and $\| T . S \|_{\Lambda_{p1}} \leqslant C . \| S \|$.

(c) $T \in \mathcal{C}_{pq}(X,B)$ if and only if for every positive operator S in $L(C(\Omega),X)$ for a compact space Ω, $T . S \in \Lambda_{pq}(C(\Omega),B)$ and $\| T . S \|_{\Lambda_{pq}} \leqslant C . \| S \|$.

Proof. Parts (a) and (b) can be verified in the same way. We only show part (a). If $T \in \Pi_{p1}(X,B)$ and $S \in L(c_o,X)$ then $T . S \in \Pi_{p1}(c_o,B)$ and of course $T . S \in \Lambda_{p1}(c_o,B)$ and $\| T . S \|_{\Lambda_{p1}} \leqslant \| T \|_{\Lambda_{p1}} . \| S \|$.

To see the converse, let us take $x_1, x_2, \ldots, x_n$ in X and let us consider $S: c_o \longrightarrow X$ defined by $S(\xi_n) = \sum_{i=1}^{n} \xi_i . x_i$.

Obviously $\| S \| \leqslant \sup_{\| \xi \|_{X*} \leqslant 1} \sum_{i=1}^{n} |<\xi, x_i>|$.

Now denoting by e_i the basis in c_o, since $e_i \geqslant 0$ prop 3 (b) implies that

$$\left(\sum_{i=1}^{n} \| Tx_i \|^p \right)^{1/p} = \left(\sum_{i=1}^{n} \| T . Se_i \|_B^p \right)^{1/p} \leqslant$$

$$\leqslant C \| S \| . \| \sum_{i=1}^{n} e_i \|_{c_o} \leqslant C . \sup_{\| \xi \|_{X*} \leqslant 1} \sum_{i=1}^{n} |<\xi, x_i>|$$

The demostration of (c) can be done with a slight modification of the argument in page 56, [6]. #

A study of positive p-summing operators $\Lambda_p(X,B)$ when X is a L_p-space was done by the author in [1]. Here we shall deal with 1_p-spaces.

PROPOSITION 6. Let $1 \leqslant q \leqslant p \leqslant \infty$.

(a) $\Lambda_{pq}(\ell^1,B) = L(\ell^1,B)$.

(b) $\Lambda_{pq}(c_o,B) = \Pi_{pq}(c_o,B) = \mathcal{C}_{pq}(c_o,B)$.

Proof.

(a) Given T in $L(\ell^1,B)$ and $x_1, x_2, \ldots, x_n \geqslant 0$ in l^1 we have

$$\sum_{i=1}^{n} \|Tx_i\| \leqslant \|T\| . \sum_{i=1}^{n} \|x_i\|_{l^1} = \|T\| . \| \sum_{i=1}^{n} x_i\|_{l^1}$$

So $\Lambda_1(l^1,B) = L(l^1,B)$ and therefore the same for $1 \leqslant q < p = \infty$.
(b) It is a simple consequence of the following fact:
For $x_1, x_2, x_3, \ldots, x_n \in c_o$ we have

$$\|(\sum_{i=1}^{n} |x_i|^q)^{1/q}\|_\infty = \sup_k (\sum_{i=1}^{n} |x_{ik}|^q)^{1/q} =$$

$$= \sup_k \sup_{\Sigma\alpha_i^{q'}=1} |\sum_{i=1}^{n} x_{ik}\alpha_i| =$$

$$= \sup_{\Sigma\alpha_i^{q'}=1} \|\sum_{i=1}^{n} x_i\alpha_i\|_\infty =$$

$$= \sup_{\|\xi\|_{l^1} \leqslant 1} (\sum_{i=1}^{n} |<\xi, x_i>|^q)^{1/q}$$

The last inequality is obtained by duality ($(c_o)^* = l^1$) and by interchanging the supremums. #

Remark. Kwapien in [4] showed that $\Pi_{r,1}(l^1,l^p) \neq L(l^1,l^p)$ for $r < r(p)$ being $\frac{1}{r(p)} = 1 - |\frac{1}{p} - \frac{1}{2}|$.

So we notice that in general the spaces $\Lambda_{pq}(X,B)$ are larger than $\Pi_{pq}(X,B)$.

PROPOSITION 7. Let $1 < p \leqslant \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$

$$\Lambda_p(\ell^{p'},B) = \Lambda_1(\ell^{p'},B) = \ell^p(B).$$

Proof. We shall prove that

$$l^p(B) \subseteq \Lambda_1(l^{p'},B) \subseteq \Lambda_p(l^{p'},B) \subseteq l^p(B).$$

Given a sequence (x_n) belonging to $l^p(B)$, we consider the operator $T: l^{p'} \longrightarrow B$ defined by $T(\xi_n) = \sum_{n\in N} \xi_n x_n$. Let us take $\xi_1, \xi_2, \ldots, \xi_m \geqslant 0$ in $l^{p'}$ then

$$\sum_{i=1}^{m} \|T(\xi_i)\| \leqslant \sum_{i=1}^{m} \sum_{n\in N} \xi_{in} \|x_n\| =$$

$$= \sum_{n\in N} \left(\sum_{i=1}^{m} \xi_{in}\right) \|x_n\| \leq \left(\sum_{n\in N} \|x_n\|^p\right)^{1/p} . \left\|\sum_{i=1}^{n} \xi_i\right\|_{l^{p'}}$$

Therefore T belongs to $\Lambda_1(l^{p'},B)$ and $\|T\|_{\Lambda_1} \leq (\Sigma \|x_n\|^p)^{1/p}$. Given now an operator T in $\Lambda_p(l^{p'},B)$ we consider the sequence $T(e_n) = x_n$ being e_n the canonic basis in $l^{p'}$. Let us prove that $(x_n) \in l^p(B)$

$$(\Sigma \|x_n\|^p)^{1/p} = (\Sigma \|T(e_n)\|^p)^{1/p} \leq$$

$$\leq \|T\|_{\Lambda_p} \sup_{\|\xi\|_{l^p} \leq 1} (\Sigma |<\xi,e_n>|^p)^{1/p} \leq \|T\|_{\Lambda_p} \qquad \#$$

COROLLARY 1. For $1 \leq p,r \leq 2$ $\Lambda_r(\ell^p,B) = \Lambda_1(\ell^p,B)$.

Denoting by σ_2 the Hilbert-Schmidt operators then we get the following

COROLLARY 2. $\Lambda_1(\ell^2,\ell^2) = \sigma_2$.

Proof. It is obvious since $\Lambda_2(l^2,l^2) = \sigma_2$ as it can be shown easily.

Remark. For $1 \leq p \leq 2$ $\Lambda_p(l^p,B) = l^{p'}(B)$.

For p- concave operators we have

$$\mathcal{C}_p(l^p,B) = L(l^p,B) = l^{p'}[B] \qquad (9)$$

Indeed, for T in $L(l^p,B)$ and $\xi_1, \xi_2, \ldots, \xi_n \in l^p$ we have

$$\left(\sum_{i=1}^{n} \|T\xi_i\|^p\right)^{1/p} \leq \|T\| \left(\sum_{i=1}^{n} \|\xi_i\|_{l^p}^p\right)^{1/p} =$$

$$= \left(\sum_{i=1} \sum_{m} |\xi_{i,m}|^p\right)^{1/p} = \left\|\left(\sum_{i=1}^{n} |\xi_i|^p\right)^{1/p}\right\|_{l^p}$$

From this it follows that, in general, $\Lambda_p(X,B)$ is smaller than $\mathcal{C}_p(X,B)$.

In [1] we found a relationship between the Radon-Nikodym property on B and the positive p-summing operators. Here we shall deal with the Orlicz property.

Let us recall that a Banach space B is said to have the Orlicz property if every sequence x_n such that $\Sigma \|x_n\|^2 < \infty$ is an unconditionally convergent series in B.

PROPOSITION 8. Let X be a Banach lattice. The following statements are equivalent:

(a) X has the Orlicz property.

(b) $\Lambda_{2,1}(X,B) = L(X,B)$ for every Banach space B.

(c) For every Banach lattice Y, every regular operator T, $T:Y \longrightarrow X$ is positive (2,1)-summing.

Proof. This is an easy consequence of Prop. 1 (d) since the identity $I:X \longrightarrow X$ belongs to $\Lambda_{21}(X,X)$ and each regular operator $T = T_1 - T_2$ being T_1 and T_2 positive ones.

In [3] it was shown that $\Pi_2(c_0,X) = L(c_0,X)$ implies the Orlicz property on X, now we are able to approach to the converse.

PROPOSITION 9. If X has the Orlicz property then

(a) $\Pi_q(c_0,X) = L(c_0,X)$ for all $q > 2$.

(b) $\Pi_p(X,B) = \Pi_1(X,B)$ for all Banach space B and $1 \leqslant p < 2$.

Proof. It is sufficient to apply successively Prop. 8(b), 6(b) and 4(b) to solve part (a).

Part (b) is a simple consequence of a Rosenthal's result [9] which assures that $\Pi_q(c_0,X) = L(c_0,X)$ if and only if $\Pi_{q'}(X,B) = \Pi_1(X,B)$ for all Banach space B and $\frac{1}{q} + \frac{1}{q'} = 1$. #

As l^p $(1 \leqslant p \leqslant 2)$ has the Orlicz property, we can prove an analogous result to Corollary 1, proved by Kwapien in [5].

COROLLARY 3. For $1 \leqslant r < 2$ and $1 \leqslant p \leqslant 2$ $\Pi_r(\ell^p,B) = \Pi_1(\ell^p,B)$. #

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