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A CHARACTERIZATION OF THE DUALS OF SOME ECHELON KÖTHE SPACES OF BANACH VALUED FUNCTIONS

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ABSTRACT. In [8] Nyugen Phuong-Cac considers Köthe spaces of vector valued functions in a Banach space X. In this work we improve the duality result of [6], [8], restricting ourselves to an echelon Köthe space. We prove that the topological dual of $\Lambda^p(X)$ is the same as its α–dual if and only if X' has the Radon-Nikodym property.

1. Echelon Köthe spaces of Banach valued functions

Let (E, Σ, μ) be an arbitrary finite measure space where Σ is a σ–algebra of subsets of E and μ is a positive, σ-additive measure. The vector spaces we use here are defined over the real field $\mathbb{R}$ and we use the standard notation of the theory of locally convex spaces (see [5]). $\mathbb{N}$ will be the set of natural numbers and X will be a Banach space. A function $f : E \to X$ is strongly measurable (or simply measurable) if there is a sequence $(f_n)_{n=1}^{\infty}$ of simple functions such that $\lim_n \|f_n - f\| = 0$ μ–almost everywhere (a.e.). We denote by $\Omega(X)$ the set of all X–valued measurable functions on E. We will identify two functions f_1 and f_2 of $\Omega(X)$ if $f_1(x) = f_2(x)$ almost everywhere on E. The quotient set will be denoted by $\Omega_0(X)$. We will use the same symbol to denote the elements of $\Omega(X)$ and their equivalence classes, when there is no risk of confusion. Given the function $f \in \Omega(X)$ (or any other in its class) we define the support of f as

$$S(f) = \{x \in E : f(x) \neq 0\}.$$

Let $(g_k)_{k=1}^{\infty}$ be an increasing sequence of measurable functions such that $g_k(x) \geq 0$ for every $x \in E$, $k \in \mathbb{N}$ and

$$\mu\left(\bigcap_{k=1}^{\infty}\{x \in E : g_k(x) = 0\}\right) = 0.$$

If $p \in \mathbb{R}$, $p \geq 1$, we define the echelon Köthe space of order p as the space $\Lambda^p = \Lambda^p(E, \Sigma, \mu, g_k)$ of all measurable functions $f : E \to \mathbb{R}$ such that

$$\|f\|_k^p = \int_E |f|^p \, g_k \, d\mu < \infty \qquad \text{for every } k \in \mathbb{N}.$$

We also define $\Lambda_k^p = \Lambda_k^p(E, \Sigma_k, \mu_k, g_k)$ of all measurable functions $f : S(g_k) \to \mathbb{R}$ such that

$$\|f\|_k^p = \int_{S(g_k)} |f|^p \, g_k \, d\mu < \infty,$$

where Σ_k is the restriction of Σ to $S(g_k)$, μ_k is the restriction of μ to Σ_k and the restriction of g_k to $S(g_k)$ is denoted in the same way.

We will write Λ and Λ_k instead of Λ^1 and Λ_k^1. We will always consider Λ^p endowed with the toplogy defined by the collection of seminorms $\{\|\cdot\|_k : k \in \mathbb{N}\}$. Λ_k^p will be endowed with the toplogy defined by the norm $\|\cdot\|_k$.

The α–duals of these spaces are the space $(\Lambda^p)^\alpha$ of all measurable functions $f : E \to \mathbb{R}$ such that

$$\int_E |f|\,|g| \, d\mu < \infty \qquad \text{for every } g \in \Lambda^p$$

and the space $(\Lambda_k^p)^\alpha$ of all measurable functions $f : S(g_k) \to \mathbb{R}$ such that

$$\int_{S(g_k)} |f|\,|g| \, d\mu < \infty \qquad \text{for every } g \in \Lambda_k^p.$$

The formula

$$\langle f, h\rangle = \int_E f\,h \, d\mu \qquad \text{for } f \in \Lambda^p,\; h \in (\Lambda^p)^\alpha$$

defines a canonical bilinear form on the cartesian product $\Lambda^p \times (\Lambda^p)^\alpha$.

Analogously given a Banach space X we define $\Lambda^p(X) = \Lambda^p(E, \Sigma, \mu, g_k, X)$ as the space of all measurable functions $f : E \to X$ such that

$$\|f\|_k^p = \int_E \|f\|^p \, g_k \, d\mu < \infty \qquad \text{for every } k \in \mathbb{N}$$

endowed with the topology defined by the collection of seminorms $\{\|\cdot\|_k : k \in \mathbf{N}\}$ and $(\Lambda^p(X))^\alpha$ as the space of all measurable functions $f : E \to X'$ such that

$$\int_E \|f\|\,\|g\|\,d\mu < \infty \qquad \text{for every } g \in \Lambda^p(X).$$

The formula

$$\langle f, g\rangle = \int_E \langle f, g\rangle\,d\mu \qquad \text{for } f \in \Lambda^p(X),\ g \in (\Lambda^p(X))^\alpha$$

defines a canonical bilinear form on the cartesian product $\Lambda^p(X) \times (\Lambda^p(X))^\alpha$.

We also define the space $\Lambda_k^p(X) = \Lambda_k^p(S(g_k), \Sigma_k, \mu_k, g_k, X)$ of all measurable functions $f : S(g_k) \to X$ such that

$$\|f\|_k^p = \int_{S(g_k)} \|f\|^p\, g_k\, d\mu < \infty$$

endowed with the topology defined by the norm $\|\cdot\|_k$. Then $\Lambda_k^p(X)$ is a Banach space since the map

$$\varphi_k : \Lambda_k^p(X) \longrightarrow L^p(S(g_k), \mu_k, X)$$

defined by

$$\varphi_k(f) = f\, g_k^{1/p}$$

is an isometry. Furthermore, $\Lambda_k^p(X)$ inherits from $L^p(S(g_k), \mu_k, X)$ the well known theorem which states that every τ_k convergent sequence in $\Lambda_k^p(X)$ contains a μ_k a.e. convergent subsequence.

It is simply checked that $\Lambda^p(X)$ is a Fréchet space. Moreover if $(f_n)_{n=1}^\infty$ converges to f in $\Lambda^p(X)$ then $(f_n \chi_{S(g_k)})_{n=1}^\infty$ converges to $f\chi_{S(g_k)}$ in $\Lambda_k^p(X)$. Thus by a diagonal procedure we obtain an increasing subsequence $(f_{n_k})_{k=1}^\infty$ convergent to f μ–a.e..

It is also interesting to note that an echelon Köthe space Λ^p contains a lot of characteristic functions and consequently a lot of simple functions (since by [6], pp. 161, given $\epsilon > 0$ and a set $B \in \Sigma$ of positive measure, there is a subset M so that $\chi_M \in \Lambda^p$ and $\mu(B - M) < \epsilon$).

Proposition 1. *The set of simple functions in $\Lambda^p(X)$ with support in $S(g_k)$ is dense in $\Lambda_k^p(X)$. Consequently the simple functions in $\Lambda^p(X)$ determine a dense subset in $\Lambda^p(X)$.*

Proof. Let us consider $f \in \Lambda_k^p(X)$ and $\epsilon > 0$. For each $m \in \mathbf{N}$ we can find $B_m \subset S(g_k)$ so that $\mu(S(g_k) - B_m) < 1/m$ and $0 \neq \chi_{B_m} \in \Lambda^p$. By [1] II.2.(4)

$$\lim_m \int_{S(g_k) - B_m} \|f\|^p\, g_k\, d\mu = 0.$$

Then there exists m_0 so that

$$\int_{S(g_k)-B_{m_0}} \|f\|^p g_k \, d\mu < \epsilon/3.$$

Moreover by [1] II.1.3 we contruct a series

$$S' = \sum_{n=1}^{\infty} x_n \chi_{A_n}$$

where $\{A_n\}_{n=1}^{\infty}$ is a partition of B_{m_0}, so that

$$\operatorname{ess\,sup} \|f - S'\|^p < \frac{\epsilon}{3 \int_{B_{m_0}} g_k \, d\mu} .$$

Therefore

$$\int_{B_{m_0}} \|f - S'\|^p g_k \, d\mu < \epsilon/3.$$

Now by [1] II.2.4 there is $p_0 \in \mathbf{N}$ such that

$$\int_{\bigcup_{n=p_0}^{\infty} A_n} \|f\|^p g_k \, d\mu < \epsilon/3.$$

The function

$$f_1 = \sum_{n=1}^{p_0-1} x_n \chi_{A_n}$$

verifies that

$$\begin{aligned}
\int_{S(g_k)} \|f - f_1\|^p g_k \, d\mu &= \int_{S(g_k)-B_{m_0}} \|f - f_1\|^p g_k \, d\mu + \int_{\bigcup_{n=1}^{p_0-1} A_n} \|f - f_1\|^p g_k \, d\mu \\
&\quad + \int_{\bigcup_{n=p_0}^{\infty} A_n} \|f - f_1\|^p g_k \, d\mu \\
&= \int_{S(g_k)-B_{m_0}} \|f\|^p g_k \, d\mu + \int_{\bigcup_{n=1}^{p_0-1} A_n} \|f - S'\|^p g_k \, d\mu \\
&\quad + \int_{\bigcup_{n=p_0}^{\infty} A_n} \|f\|^p g_k \, d\mu \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

This completes the proof.

2. α–duality

Let T be a set of measurable functions. We define T^α as the set of all measurable functions $g : E \to \mathbb{R}$ such that

$$\int_E |f|\,|g|\,d\mu < \infty \qquad \text{for all } f \in T.$$

If X is a Banach space we define $T(X)$ as the set of all measurable functions $f : E \to X$ such that $\|f\| \in T$, and also $(T(X))^\alpha$ as the set of all measurable functions $g : E \to X'$ such that

$$\int_E \|f\|\,\|g\|\,d\mu < \infty \qquad \text{for all } f \in T(X).$$

It is easy to prove that $(T(X))^\alpha = T^\alpha(X')$. In fact if $g \in (T(X))^\alpha$ and $f \in T$, then $xf \in T(X)$ for each $x \in X$ with $\|x\| = 1$; as

$$\int_E \|g\|\,\|xf\|\,d\mu = \int_E \|g\|\,|f|\,d\mu$$

is finite we have $\|g\| \in T^\alpha$. It follows that $g \in T^\alpha(X')$. On the other hand direct verifications show that $T^\alpha(X') \subset (T(X))d^\alpha$.

In particular if $T = L_p(\mu)$ and $p > 1$, then $(L_p(\mu))^\alpha = L_q(\mu)$ with $1/p + 1/q = 1$; if $p = 1$ then $(L_1(\mu))^\alpha = L_\infty(\mu)$, [10], pp. 366. Hence $T(X) = L_p(\mu, X)$ and $(L_p(\mu, X))^\alpha = L_q(\mu, X') = (L_p(\mu, X))'$ for every $p \geq 1$.

The next lemma is proved in [8], pp. 605, for locally integrable functions defined in a locally compact space with a Radon measure.

Lemma 1. *If $f \in T(X)$ and $g \in T^\alpha$, $g \geq 0$, then*

$$\int_E g\,\|f\|\,d\mu = \sup\left\{\left|\int_E \langle f(t), h(t)\rangle\,d\mu\right| : h \in M'\right\}$$

where M' is the set of all measurable functions $h : E \to X'$ such that

$$\|h(t)\| \leq g(t) \quad \mu-\text{a.e.}.$$

Proof. Since

$$\left|\int_E \langle f(t), h(t)\rangle \, d\mu\right| \leq \int_E \|f(t)\| \, \|h(t)\| \, d\mu$$
$$< \int_E \|f(t)\| \, g(t) \, d\mu,$$

it is enough to prove that given

$$c < \int_E \|f(t)\| \, g(t) \, d\mu$$

there is $h \in M'$ so that

$$c < \left|\int_E \langle f(t), h(t)\rangle \, d\mu\right| .$$

Let $\epsilon < 0$ be such that

$$c + \epsilon < \int_E \|f(t)\| \, g(t) \, d\mu.$$

By the Lebesgue monotone convergence theorem there is a non negative simple function

$$S = \sum_{i=1}^{n} a_i \chi_{H_i}$$

such that the sets H_i, $i = 1, 2, \ldots, n$, are pairwise disjoint and

$$c + \epsilon < \int_E \|f(t)\| \, s(t) \, d\mu$$
$$= \sum_{i=1}^{n} a_i \int_{H_i} \|f(t)\| \, d\mu$$
$$< \int_E \|f(t)\| \, |g(t)| \, d\mu.$$

Now there is no loss of generality in assuming that $a_i > 0$, $i = 1, 2, \ldots, n$.

Since the last inequality guarantees that f is Bochner integrable in H_i, $i = 1, 2, \ldots, n$, we find a partition $\{G_i^j : 1 \leq j \leq n_i\}$ of H_i and a collection of vectors $\{c_i^j : 1 \leq j \leq n_i\}$ in X, so that

$$\int_{H_i} \left\| f(t) - \sum_{j=1}^{n_i} c_i^j \chi_{G_i^j} \right\| d\mu < \frac{\epsilon}{2na_i} .$$

By the Hahn-Banach theorem there is $f_i^j \in X'$ so that $\|f_i^j\| = 1$ and $\langle f_i^j, c_i^j\rangle = \|c_i^j\|$. Let us see that

$$h = \sum_{i=1}^{n} a_i \sum_{j=1}^{n_i} f_i^j \chi_{G_i^j}$$

is the required function.

It is clear that $h \in M'$. Then it only rests to verify that

$$c < \left| \int_E \langle f(t), h(t) \rangle \, d\mu \right| .$$

To this end

$$\begin{aligned}
\left| \int_E \langle f(t), h(t) \rangle \, d\mu \right| &= \left| \sum_{i=1}^{n} \int_{H_i} \left\langle \sum_{j=1}^{n_i} e_i^j \chi_{G_i^j} + f - \sum_{j=1}^{n_i} e_i^j \chi_{G_i^j}, h \right\rangle d\mu \right| \\
&\geq \left| \sum_{i=1}^{n} \int_{H_i} \left\langle \sum_{j=1}^{n_i} e_i^j \chi_{G_i^j}, h \right\rangle d\mu \right| \\
&\quad - \left| \sum_{i=1}^{n} \int_{H_i} \left\| f - \sum_{j=1}^{n_i} e_i^j \chi_{G_i^j} \right\| \|h\| \, d\mu \right| \\
&> \left| \sum_{i=1}^{n} a_i \int_{H_i} \sum_{j=1}^{n_i} \|e_i^j\| \chi_{G_i^j} \, d\mu \right| - \frac{\epsilon}{2} \\
&= \left| \sum_{i=1}^{n} a_i \int_{H_i} \|f\| \, d\mu - \sum_{i=1}^{n} a_i \int_{H_i} \left(\|f\| - \sum_{j=1}^{n_i} \|e_i^j\| \chi_{G_i^j} \right) d\mu \right| - \frac{\epsilon}{2} \\
&\geq \sum_{i=1}^{n} a_i \int_{H_i} \|f\| \, d\mu - \left| \sum_{i=1}^{n} a_i \int_{H_i} \left(\|f\| - \sum_{j=1}^{n_i} \|e_i^j\| \chi_{G_i^j} \right) d\mu \right| - \frac{\epsilon}{2} \\
&> c + \epsilon - \sum_{i=1}^{n} a_i \int_{H_i} \left\| f - \sum_{j=1}^{n_i} e_i^j \chi_{G_i^j} \right\| d\mu - \frac{\epsilon}{2} \\
&> c + \epsilon - \frac{\epsilon}{2} - \frac{\epsilon}{2} \\
&= c.
\end{aligned}$$

where we have repeatedly used the triangle inequality and the fact that sums

$$\sum_{j=1}^{n_i} e_i^j \chi_{G_i^j}$$

are reduced in each point only to one term because the sets G_i^j are pairwise disjoint.

Theorem 2. *Let $\Lambda^p(X)$ be an echelon Köthe space. Then $(\Lambda^p(X), (\Lambda^p(X))^\alpha)$ is a dual pair with respect to the bilinear form*

$$\langle f, g \rangle = \int_E \langle f(x), g(x) \rangle \, d\mu.$$

Proof. By [6], prop. 1, if $\varphi \neq 0$, $\varphi \in \Lambda^p(X)$, there is $g \in (\Lambda^p)^\alpha$ so that $\langle g, \|\varphi\|\rangle \neq 0$. Since $g = g^+ - g^-$ we have $\langle g^+, \|\varphi\|\rangle \neq 0$ or $\langle g^-, \|\varphi\|\rangle \neq 0$. Hence we can suppose $g \geq 0$. Then by lemma above, we get $\psi \in (\Lambda^p(X))^\alpha$ such that

$$\int_E \langle \varphi, \psi\rangle \, d\mu \neq 0.$$

Suppose now $0 \neq \phi \in (\Lambda^p(X))^\alpha = (\Lambda^p)^\alpha(X')$. Since Λ^p is perfect we know that there is $0 \neq \chi_A \in \Lambda^p = ((\Lambda^p)^\alpha)^\alpha$ where $A \subset S(\|\phi\|)$ (see the proof of prop. 1 in [6], pg. 161). Therefore

$$\int_E \chi_A \|\phi\| \, d\mu \neq 0.$$

Then by lemma 1 there is $\varphi \in \Lambda^p(X'')$ such that

$$\int_E \langle \varphi, \phi\rangle \, d\mu \neq 0 \qquad \text{and } \|\varphi(t)\| < \chi_A(t) \; \mu - \text{a.e.}.$$

Now there is no loss of generality in assuming the existence of a positive number $\epsilon > 0$ such that

$$\int_E \langle \varphi, \phi\rangle \, d\mu = \int_A \langle \varphi, \phi\rangle \, d\mu > \epsilon.$$

The function φ being measurable is the limit μ a.e. of a sequence of X'' valued simple functions $(S_n)_{n=1}^\infty$ vanishing out of A. By Egorov's theorem given the natural number m, there is an element $B_m \in \Sigma$, $B_m \subset A$, so that $(S_n)_{n=1}^\infty$ is uniformly convergent to φ in B_m, and $\mu(A - B_m) < 1/m$. [1] II.2.4 justifies the existence of an $m \in \mathbb{N}$ so that

$$\left| \int_{A - B_m} \langle \varphi, \phi\rangle \, d\mu \right| < \frac{\epsilon}{2}.$$

Then, as $\chi_A \in \Lambda^p$, by the uniform convergence

$$\frac{\epsilon}{2} < \int_{B_m} \langle \varphi, \phi\rangle \, d\mu = \lim_n \int_{B_m} \langle S_n, \phi\rangle \, d\mu.$$

Hence there is a simple function still denoted by S_n,

$$S_n = \sum_{i=1}^{k_n} x_i^{**} \chi_{E_i},$$

with $E_i \subset B_m$, so that

$$\frac{\epsilon}{2} < \int_{B_m} \langle S_r, \phi\rangle \, d\mu.$$

Furthermore

$$\int_{E_i} \|\phi\|\, d\mu < \int_E \chi_A \|\phi\|\, d\mu < \infty,$$

i.e., ϕ is Bochner integrable in E_i and therefore by [1].11.2(6)

$$\frac{\epsilon}{2} < \int_{B_m} \langle S_n, \phi\rangle\, d\mu = \sum_{i=1}^{k_n} \left\langle x_i^{**}, \int_{E_i} \phi\, d\mu \right\rangle.$$

Finally by $\sigma(X'', X')$ density of X in X'', there is a finite set of vectors $(y_i)_{i=1}^p$ in X so that

$$\frac{\epsilon}{2} < \sum_{i=1}^{p} \left\langle y_i, \int_{E_i} \phi\, d\mu \right\rangle = \int_{B_m} \langle S, \phi\rangle\, d\mu$$

where

$$S = \sum_{i=1}^{p} y_i \chi_{E_i} \in \Lambda^p(X).$$

This function satisfies

$$\int_E \langle S, \phi\rangle\, d\mu = \int_{B_m} \langle S, \phi\rangle\, d\mu \neq 0.$$

3. Duality and α duality

Theorem 1. *Let $\Lambda^p(X)$ be an echelon Köthe space. If $h \in (\Lambda^p(X))^\alpha$, $p > 1$, then the linear form φ_h defined on $\Lambda^p(X)$ by $h \mapsto \varphi_h$ is an immersion of $(\Lambda^p(X))^\alpha$ into $(\Lambda^p(X))'$.*

Proof. Since $\Lambda^p(X)$ is metrizable, it is enough to prove that φ_h is locally bounded.

To this end, proceeding by contradiction, we suppose the existence of a bounded set $B \subset \Lambda^p(X)$ and a sequence $(f_n)_{n=1}^\infty$ in B, so that for each $n \in \mathbb{N}$

$$\left| \int_E \langle f_n(x), h(x)\rangle\, d\mu \right| \geq n^3.$$

Proceeding as in [6], pg. 167, we show that the sequence

$$S_n = \sum_{i=1}^{n} \frac{\|f_i\|}{i^2}\, z,$$

where $z \in X$, $\|z\| = 1$, is a Cauchy sequence in $\Lambda^p(X)$. Therefore S_n converges to a certain function φ in $\Lambda^p(X)$ and there is a subsequence in $(S_n)_{n=1}^{\infty}$ convergent to φ μ-a.e.. Then

$$\varphi(x) = \sum_{i=1}^{\infty} \frac{\|f_i(x)\|}{i^2}\, z \qquad \mu - \text{a.e.}.$$

Now

$$\int_E \|\varphi\|\,\|h\|\,d\mu = \infty$$

which contradicts the fact that $h \in \left(\Lambda^p(X)\right)^{\alpha}$.

Finally we infer that if $h \neq h'$ then $\varphi_h \neq \varphi_{h'}$ by theorem 2.(2).

Theorem 2. *Let $\Lambda^p(X)$ be an echelon Köthe space, $p \geq 1$. If X' verifies the Radon-Nikodym property and for each k the scalar functions $g_k \neq 0$ μ a.e., we have that given $\varphi \in (\Lambda^p(X))^{\alpha}$ there is a uniquely determined function h in $(\Lambda^p(X))^{\alpha}$ such that*

$$\varphi(f) = \int_E \langle f(x), h(x)\rangle\, d\mu, \qquad \text{for } f \in \Lambda^p(X).$$

Proof. It is easy to see that there is $k_0 \in \mathbb{N}$ such that φ is continuous with the induced topology of $\Lambda^p_{k_0}(X)$. By proposition 1.(1), φ can be extended in a continuous way to $\Lambda^p_{k_0}(X)$, still denoted by φ.

Since $g_{k_0} \neq 0$ μ-a.e., the map from $\Lambda^p_{k_0}(X)$ into $L^p(\mu, X)$ such that it assigns to f the function $f g_{k_0}^{1/p}$ is an isometry and then we can define a continuous linear form on $L^p(\mu, X)$ such that

$$\hat{\varphi}\left(f g_{k_0}^{1/p}\right) = \varphi(f).$$

Therefore there is a function h' in $L^q(\mu, X')$ so that

$$\begin{aligned}\varphi(f) &= \hat{\varphi}\left(f g_{k_0}^{1/p}\right)\\ &= \int_E \left\langle f g_{k_0}^{1/p}, h'\right\rangle d\mu\\ &= \int_E \left\langle f, g_{k_0}^{1/p} h'\right\rangle d\mu \qquad ([1], \S 4)\end{aligned}$$

According to the equality $L^q(\mu, X') = (L^p(\mu, X))^{\alpha}$

$$\int_E \left\| f g_{k_0}^{1/p}\right\| \|h'\|\, d\mu = \int_E \|f\| \left\| g_{k_0}^{1/p} h'\right\| d\mu < \infty \qquad \text{for } f \in \Lambda^p(X).$$

Consequently

$$h = g_{k_0}^{1/p} h' \in (\Lambda^p(X))^{\alpha}$$

and evidently it satisfies the required conditions.

The uniqueness of h follows directly from theorem 2.(2).

Note that h is in the α–dual of certain $\Lambda^p_{k_0}(X)$.

Theorem 3. *Let $\Lambda^p(E,\Sigma,\mu,g_k,X)$ be an echelon Köthe space, $p \geq 1$. If X' satisfies the Radon-Nikodym property, given a linear continuous form φ in $\Lambda^p(X)$, there is a uniquely determined function h in $(\Lambda^p(X))^\alpha$ so that*

$$\varphi(f) = \int_E \langle f(x), h(x)\rangle \, d\mu \qquad \text{for } f \in \Lambda^p(X)$$

and there is an index k_0 such that $S(h) \subset S(g_{k_0})$ a.e..

Proof. Let $\Gamma^p_k(X) = \Gamma^p_k(S(g_k),\Sigma_k,\mu_k,\varphi_r,X)$ be the echelon Köthe space where Σ_k and μ_k are the σ algebra and measure induced by Σ and μ on $S(g_k)$ and $\varphi_r(x) = g_{k+r-1}(x)$ for every x in $S(g_k)$ and $r \in \mathbb{N}$.

The mapping $i_k : \Gamma^p_k(X) \to \Lambda^p(X)$ defined by $i_k(f) = 0$ on $E - S(g_k)$ and $i_k(f) = f$ on $S(g_k)$ is continuous. The composition $\varphi \circ i_k \in \left(\Gamma^p_k(X)\right)'$.

Then by the above theorem there is $h_k \in \left(\Gamma^p_k(X)\right)^\alpha$ so that

$$\varphi(i_k(f)) = \int_{S(g_k)} \langle h_k, f\rangle \, d\mu \qquad \text{for } f \in \Gamma^p_k(X).$$

Proceeding as in [6], pg. 169, we have

$$\int_{S(g_k)} \langle h_k, f\rangle \, d\mu = \int_{S(g_k)} \langle h_{k+1}, f\rangle \, d\mu \qquad \text{for } f \in \Gamma^p_k(X).$$

This equality combined with the fact that the restriction of h_{k+1} to $S(g_k)$ is an element of $(\Gamma^p_k(X)7bigr)^\alpha$ and theorem 2.(2, shows that $h_{k+1}(x) = h_k(x)$ μ–a.e. on $S(g_k)$. Then we can define a function h on E, $h(x) = h_k(x)$ if $x \in S(g_k)$ and $h(x) = 0$ if $x \in E - \bigcup_{k=1}^\infty S(g_k)$ changing the values of h_k on a set of zero measure if it is necessary.

If $f \in \Lambda^p(X)$, it is easy to see that

$$f = \lim f\chi_{S(g_k)}.$$

Then

$$\varphi(f) = \lim_k \int_{S(g_k)} \langle f, h_k\rangle \, d\mu.$$

According to the fact that φ is continuous, there is an $\epsilon > 0$ and a $k_0 \in \mathbb{N}$ so that if $f \in \Lambda^p(X)$ and

$$\int_E \|f\|^p \, g_{k_0} \, d\mu < \epsilon,$$

then $|\varphi(f)| \leq 1$.

We consider $B = E - S(g_{k_0})$ and $\alpha = \mu(B)$. If $\alpha = 0$ directly $h = 0$ on $E - S(g_{k_0})$. Then if $\alpha > 0$, as we did in proposition 1, we can construct a sequence $(A_n)_{n=1}^{\infty}$ of subsets of B pairwise disjoint such that

$$A_n \subset B - \bigcup_{i=1}^{n-1} A_i,$$

$$\mu\left(B - \bigcup_{i=1}^{n} A_i\right) < \frac{\alpha}{2^n}$$

and $\chi_{A_n} \in \Lambda^p$. Then

$$\mu\left(B - \bigcup_{n=1}^{\infty} A_n\right) = 0.$$

Given n_0 we write

$$A_{n_0} = \bigcup_{r=1}^{\infty} \{t \in A_{n_0} : \|h(t)\| \leq r\}$$

where we call B_{r,n_0} each of these subsets.

Let r be a fixed natural number. Given $x_0 \in X$ and $m \in \mathbb{N}$ and $M \subset B_{r,n_0}$, $M \in \Sigma$ we have $m\chi_M \, x_0 \in \Lambda^p(X)$. Then by Hille's theorem ([1], pg. 47)

$$\begin{aligned}
1 &\geq |\varphi(m\chi_M \, x_0| \\
&= \left| \int_E \langle m\chi_M \, x_0, h \rangle \, d\mu \right| \\
&= m \left| \int_M \langle x_0, h(t) \rangle \, d\mu \right| \\
&= m \left| \left\langle x_0, \int_E \chi_M h(t) \, d\mu \right\rangle \right| .
\end{aligned}$$

Therefore

$$\left| \left\langle x_0, \int_M h(t) \, d\mu \right\rangle \right| = 0 \qquad \text{for } x_0 \in X.$$

Consequently

$$\int_M h(t)\,d\mu = 0 \qquad \text{for } M \subset B_{r,n_0},$$

and, by [2], pg. 175, $h(t) = 0$ μ–a.e. on B_{r,n_0}. Then we obtain $h(t) = 0$ μ–a.e. on $E - S(g_{k_0})$.

It follows that

$$\varphi(f) = \int_{S(g_{k_0})} \langle f, h_{k_0}\rangle\,d\mu = \int_E \langle f, h\rangle\,d\mu \qquad \text{for } f \in \Lambda^p(X),$$

and since $h_{k_0} \in (\Gamma^p_{k_0}(X))^\alpha$ we easily obtain $h \in (\Lambda^p(X))^\alpha$.

Finally proposition 2.(2) justifies the uniqueness of h.

Theorem 4. *Let $\Lambda^p(E, \Sigma, \mu, g_k, X)$ be an echelon Köthe space. If the α–dual of $\Lambda^p(X)$ is the topological dual of $\Lambda^p(X)$ then X' has the Radon-Nikodym property.*

Proof. Suppose $\left(\Lambda^p(X)\right)' = \left(\Lambda^p(X)\right)^\alpha$ and let $G : \Sigma \to X'$ be a μ-continuous vector measure of bounded variation. We shall show that if $E_0 \subset \Sigma$ has positive μ–measure, then G has a Bochner integrable Radon-Nikodym derivative on a set $B \in \Sigma$, $B \subset E_0$, with $\mu(B) > 0$. Then, by [1] III.2.5, the proof will be complete. Thus let $E_0 \subset \Sigma$ have positive μ–measure. Applying the Hahn decomposition theorem to the scalar measures $|G|$ and $M\mu$ for a large enough positive integer M produces a subset B' of E_0, $B' \in \Sigma$, $\mu(B') > 0$, such that $|G|(E) \leq M\mu(E)$ for all $E \in \Sigma$ with $E \subset B'$. Then we can choose a positive integer k so that $\mu\big(B' \cap S(g_k)\big) > 0$. Then as we can write

$$B' \cap S(g_k) = \bigcup_{r=1}^{\infty} \big\{t \in B' \cap S(g_k) : g_k(t) \subset [1/r, r]\big\}$$

there is an $r \in \mathbb{N}$ such that

$$\mu(B'_r) = \mu\big\{t \in B' \cap S(g_k) : g_k(t) \in [1/r, r]\big\} > 0$$

and we can choose a subset B of B'_r, $B \in \Sigma$, $\mu(B) > 0$, such that $\chi_B \subset \Lambda^p$.

Now we define for a simple function

$$f = \sum_{i=1}^{n} x_i \chi_{E_i},$$

where $x_i \in X$, $E_i \in \Sigma$ and $E_i \cap E_j = \emptyset$ for $i \neq j$,

$$l(f) = \sum_{i=1}^{n} \big\langle G(E_i \cap B), x_i \big\rangle.$$

Then using

$$\int_{E_i\cap B} g_k\,d\mu \geq \int_{E_i\cap B} \inf\{g_k(t) : t \in B_r'\}\,d\mu \geq (1/r)\,\mu(E_i \cap B)$$

we have

$$\begin{aligned}
|l(f)| &= \left|\sum_{i=1}^{n}\langle G(E_i \cap B), x_i\rangle\right| \\
&= \left|\sum_{i=1}^{n}\left\langle \frac{1}{\int_{E_i\cap B} g_k\,d\mu}\, G(E_i\cap B), \left(\int_{E_i\cap B} g_k\,d\mu\right) x_i\right\rangle\right| \\
&\leq \sum_{i=1}^{n} \frac{M\mu(E_i\cap B)}{\int_{E_i\cap B} g_k\,d\mu} \left\|\left(\int_{E_i\cap B} g_k\,d\mu\right) x_i\right\| \\
&\leq Mr\sum_{i=1}^{n}\left\|\left(\int_{E_i\cap B} g_k\,d\mu\right) x_i\right\| \\
&= Mr\int_B \left\|\sum_{i=1}^{n} x_i\chi_{E_i}\right\| g_k\,d\mu \\
&\leq Mr\left(\int_B \|f\|^p g_k\,d\mu\right)^{1/p}\left(\int_B g_k\,d\mu\right)^{1/q} \\
&\leq Mr\left(\sup_{t\in B_r'} g_k(t)\right)^{1/q} \mu(E)^{1/q}\,\|f\|_k.
\end{aligned}$$

Since l is evidently linear on the simple functions in $\Lambda^p(X)$ this shows that l is continuous on the simple functions in $\Lambda^p(X)$ and therefore has a bounded linear extension to $\Lambda^p(X)_{|B'}$ and by Hahn-Banach theorem to $\Lambda^p(X)$.

By hypothesis, there is $g \in \Lambda^p(X)^\alpha$ such that

$$l(f) = \int_E \langle f, g\rangle\,d\mu \qquad \text{for } f \in \Lambda^p(X).$$

But

$$\begin{aligned}
G(E\cap B)(x) &= l(x\chi_E) \\
&= \int_E \langle x, g\rangle\,d\mu \\
&= \left(\int_E g\,d\mu\right)(x)
\end{aligned}$$

for all $x \in X$ and $E \in \Sigma$, $E \subset B$.

Consequently

$$G(E \cap B) = \int_E g \, d\mu \qquad \text{for } E \in \Sigma.$$

This completes the proof.

Then we have shown

Theorem 5. $(\Lambda^p(X))^\alpha = (\Lambda^p(X))'$ *if and only if* X' *has the Radon-Nikodym property.*

Now we characterize the equicontinuous sets of $((\Lambda^p(X))^\alpha$.

In $(\Lambda^p(X))^\alpha$ we show the equivalence

$$\|f(x)\| \leq \|g(x)\| \qquad \mu-\text{a.e.} \qquad\qquad \text{if and only if}$$

$$\int_E \|f\| \, \|h\| \, d\mu \leq \int_E \|g\| \, \|h\| \, d\mu \qquad \text{for } h \in \Lambda^p(X).$$

In fact if we suppose the second inequality and we consider

$$\mu(A) = \mu\{x \in E : \|f(x)\| > \|g(x)\|\} > 0,$$

with the function $h = \chi_{A'}x$, $A' \subset A$, $A' \in \Sigma$, where $\chi_{A'} \in \Lambda^p$ and $\|x\| = 1$ we have

$$\int_A \|f\| \, \|h\| \, d\mu \leq \int_A \|g\| \, \|h\| \, d\mu$$

which contradicts the fact that, in A, $\|f(x)\| > \|g(x)\|$.

By theorems above $(\Lambda^p(X))' = (\Lambda^p(X))^\alpha$ when X' has the Radon-Nikodym property. Then in that case, as $I_k : \Lambda^p(X) \to \Lambda^p_k(X)$ defined by $I_k(f) = f_{|S(g_k)}$ is continuous, its transposed I^t_k maps $(\Lambda^p(X))^\alpha$ into $(\Lambda^p(X))^\alpha$ injectively because

$$\overline{I_k(\Lambda^p(X))} = \Lambda^p_k(X).$$

Let us see that, if $f \in (\Lambda^p_k(X))^\alpha$, $I^t_k(f)$ is the function $\hat{f}$ equal to f on $S(g_k)$ and equal to zero on $E - S(g_k)$. If $h \in \Lambda^p(X)$

$$\begin{aligned}
\langle I^t_k(f), h \rangle &= \langle f, I_k(h) \rangle \\
&= \int_{S(g_k)} \langle f, I_k(h) \rangle \, d\mu \\
&= \int_{S(g_k)} \langle f, h \rangle \, d\mu \\
&= \langle \hat{f}, h \rangle.
\end{aligned}$$

Then $I_k^t(f) = \hat{f}$.

Theorem 6. *Let M be a subset of $(\Lambda^p(X))^\alpha$. We suppose that X' has the Radon-Nikodym property.*

Then if $p = 1$ the following conditions are equivalent:

1) *M is equicontinuous.*
2) *There are $k \in \mathbb{N}$ and an equicontinuous set M' of $(\Lambda_k^p(X))^\alpha$ so that $M = I_k^t(M')$.*
3) *There are $k \in \mathbb{N}$ and $C > 0$ so that $\|f\| < Cg_k$ for every $f \in M$.*

If $p > 1$ the following conditions are equivalent:

1) *M is τ equicontinuous.*
2) *There are $k \in \mathbb{N}$ and an equicontinuous set M' of $(\Lambda_k^p(X))^\alpha$ so that $M = I_k^t(M')$.*
3) *There are $k \in \mathbb{N}$ and $\alpha > 0$ so that for all $f \in M$, $S(f) \subset S(g_k)$ and if $1/p + 1/q = 1$*

$$\sup_{f\in M}\left(\int_{S(g_k)} \|f\|^q\, g_k^{-q/p}\, d\mu\right)^{1/q} = \alpha < \infty.$$

Proof. 1)$\longrightarrow$2) If $p \geq 1$ by 1) there are $\epsilon > 0$ and $k \in \mathbb{N}$ so that $M \subset V^\circ$ where

$$V = \left\{ f \in \Lambda^p(X) : \int_E \|f\|^p\, g_k\, d\mu < \epsilon \right\}.$$

We consider an element z of X so that $\|z\| = 1$. If $h \in M$ and $A = E - S(g_k)$, we have $h\chi_A = 0$. In fact, if h is not zero μ a.e. on A in the same way as in prop. 1 we have that there is $D \subset A$, $D \in \Sigma$, such that

$$\mu(D) > 0, \qquad \chi_D\, h \neq 0, \qquad \text{and } \chi_D \in \Lambda^p.$$

Then if we consider

$$D = \bigcup_{m=1}^{\infty} \{x \in D : 1/m \leq \|h(x)\| \leq m\} = \bigcup_{m=1}^{\infty} D_m$$

we obtain $D_m \in \Sigma$, $D_m \subset D$, such that h is Bochner integrable in D_m. Hence for every $n \in \mathbb{N}$, $nz\chi_{D_m} \in V$. But $h \in V^\circ$. Then $|\langle h, z\chi_{D_m}\rangle|$ must be zero for every $m \in \mathbb{N}$. Then by [2], pg. 175, $h\chi_D = 0$ which is a contradiction.

Now it is enough to prove that $M' \subset W^\circ$, where M' is the set of restrictions to $S(g_k)$ of the elements of M, and

$$W = \left\{ f \in \Lambda_k^p(X) : \int_{S(g_k)} \|f\|^p g_k \, d\mu < \epsilon \right\}.$$

We consider $h \in M'$, $h \neq 0$, and $f \in W$. By the theorem (1.1) there is a sequence $(f_n) \subset \Lambda^p(X)$ so that

$$\lim_n I_k(f_n) = f \qquad \mu - \text{a.e. on } S(g_k)$$

and

$$\lim_n I_k(f_n) = f \qquad \text{in } \left[\Lambda_k^p(X), \tau_k\right].$$

Then there is n_0 so that if $n \geq n_0$ by Minkowski inequality

$$\begin{aligned}\left(\int_{S(g_k)} \|f_n\|^p g_k \, d\mu\right)^{1/p} &\leq \left(\int_{S(g_k)} \|f_n - f\|^p g_k \, d\mu\right)^{1/p} + \left(\int_{S(g_k)} \|f\|^p g_k \, d\mu\right)^{1/p} \\ &< \epsilon - \left(\int_{S(g_k)} \|f\|^p g_k \, d\mu\right)^{1/p} + \left(\int_{S(g_k)} \|f\|^p g_k \, d\mu\right)^{1/p} \\ &= \epsilon.\end{aligned}$$

Hence $f_n \in V$.

By Fatou's lemma

$$\begin{aligned} 0 &\leq |\langle h, f\rangle| \\ &< \int_{S(g_k)} |\langle f, h\rangle| \, d\mu \\ &= \int_{S(g_k)} \liminf_n |\langle h, f_n\rangle| \, d\mu \\ &\leq \liminf_n \int_{S(g_k)} |\langle f_n, h\rangle| \, d\mu \\ &\leq 1. \end{aligned}$$

Hence by lemma 1 $h \in \left(\Lambda_k^p(X)\right)^\alpha$ and $h \in W^\circ$, because every $r \in \Lambda_k^p(X)$ is absorbed by W. Then $M' \subset W^\circ$.

2)⟶3) If $h \in M$, h is zero on $E - S(g_k)$, and there are $k \in \mathbf{N}$ and $\epsilon > 0$ so that $M' \subset W^\circ$, where

$$W := \left\{ f \in \Lambda_k^p(X) : \left(\int_{S(g_k)} \|f\|^p g_k \, d\mu\right)^{1/p} < \epsilon \right\}$$

for every $p \geq 1$. Let us suppose $p = 1$. If $f \in \Lambda_k^p(X)$, $f \neq 0$, the function

$$\frac{\epsilon}{\int_{S(g_k)} \|f\|^p \, g_k \, d\mu} f$$

belongs to W. Hence if $h \in M'$, and we consider an element z of X, so that $\|z\| = 1$, we have by lemma 1

$$\begin{aligned}\int_{S(g_k)} \|h\| \, \|f\| \, d\mu &\leq \frac{1}{\epsilon} \int_{S(g_k)} \|f\|^p \, g_k \, d\mu \\ &= \frac{1}{\epsilon} \int_{S(g_k)} \|f\|^p \, \|g_k z\| \, d\mu,\end{aligned}$$

for all $f \in \Lambda^p(X)$, $f \neq 0$. Then

$$\|h\| < \frac{1}{\epsilon} \|g_k z\|$$

for every $h \in M$.

If $p > 1$, let h be in M'. If $r \in L^p(S(g_k), \Sigma, \mu, X)$ then $r g_k^{-1/p} \in \Lambda_k^p(X)$. As for every r of the unit ball of $L^p(S(g_k), \Sigma, \mu, X)$, $\epsilon r g_k^{-1/p} \subset W$ and $h \in M' \subset W^\circ$, we see that $\epsilon h g_k^{-1/p}$ is an element of the unit ball of $(L^p(S(g_k), \Sigma, \mu, X))'$. Then

$$\left(\int_{S(g_k)} \|h\|^q \, g_k^{-q/p} \, d\mu \right)^{1/q} \leq \frac{1}{\epsilon} \qquad \text{for } h \in M'.$$

3)——1) If $p = 1$ it is clear that

$$M \subset \left\{ f \in \Lambda^p(X) : \int_E \|f\| \, g_k \, d\mu \leq 1/\alpha \right\}^\circ.$$

If $p > 1$, by Hölder's inequality, if $h \in M$ and $f \in V$, where

$$V = \left\{ f \in \Lambda^p(X) : \left(\int_E \|f\|^p \, g_k \, d\mu \right)^{1/p} < 1/\alpha \right\},$$

we have, proceeding as in [6], pg. 187,

$$\begin{aligned}\langle h, f \rangle &= \int_E \langle h, f \rangle \, d\mu \\ &\leq \int_{S(g_k)} \left\| h g_k^{-1/p} \right\| \left\| f g_k^{1/p} \right\| \, d\mu \\ &\leq \left(\int_{S(g_k)} \|h\|^q \, g_k^{-q/p} \, d\mu \right)^{1/q} \left(\int_E \|f\|^p \, g_k \, d\mu \right)^{1/p} \\ &< 1.\end{aligned}$$

Hence $M \subset V^\circ$.

Corollary 7. *If* $\Lambda^p(E,\Sigma,\mu,g_k,X)$ *is an echelon Köthe space,*

$$(\Lambda^p(X))^\alpha = \bigcup_{k=1}^{\infty} I_k'((\Lambda_k^p(X))^\alpha)$$

if X' *has the Radon-Nikodym property.*

Proof. It is immediate because $(\Lambda^p(X))^\alpha$ is the dual of $\Lambda^p(X)$.

Finally we study a property that characterizes the reflexive echelon Köthe spaces.

Proposition 8. *Let* $\Lambda^p(E,\Sigma,\mu,g_k,X)$ *be an echelon Köthe space,* $p \geq 1$. $\Lambda^p(X)$ *is reflexive if and only if* Λ^p *and* X *are reflexive.*

Proof. For the sufficiency, if $p = 1$ by [7], pg. 226, Λ^1 is a reflexive Montel space. Then by [9] $\Lambda^1(X)$ is reflexive.

If $p > 1$, $L^p(\mu, X)$ is reflexive by [1], pg. 100. Then so is $\Lambda_k^p(X)$ by the isometry with $L^p(S(g_k),\mu_k,X)$, and thus the projective limit $\Lambda^p(X)$ is also reflexive.

For the necessity, suppose that $\Lambda^p(X)$ is reflexive. We can obtain an element B of Σ, $\mu(B) > 0$, such that $\chi_B \in \Lambda^p$.

We consider now the closed subspace $\{x\chi_B : x \in X\}$ of $\Lambda^p(X)$, that is trivially isomorphic to X. Then X is reflexive.

In addition, if $p > 1$, by the first part $\Lambda^p(X)$ will be reflexive and so Λ_k^p by the isometry with the closed subspace $\{x_0 f : f \in \Lambda_k^p\}$ of $\Lambda^p(X)$, where $x_0 \in X$ is fixed and $\|x_0\| = 1$. As Λ^p is the projective limit of the Λ_k^p, we have that Λ^p is reflexive. If $p = 1$, it is easy to see by standard methods (see [4], pg. 340) that $\Lambda^p(X) \cong \Lambda\hat{\otimes}_\pi X$. Hence, if $\Lambda(X)$ is reflexive, Λ and X are also reflexive.

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