GLOBAL STABILITY OF SETS
FOR SYSTEMS WITH IMPULSES*

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ABSTRACT. In the present paper the question of global stability of sets of sufficiently general type with respect to systems of differential equations with impulses is considered. It is proved that the existence of piecewise continuous functions of the type of Lyapunov's functions with certain properties is a sufficient condition for various types of global asymptotic stability.

1. Introduction

the recent years systems of differential equations with impulses have been an object of numerous investigations ([1]–[11]), related to the applications of these systems in physics, biology, control theory, etc. The necessity of consideration of such systems arises from the study of real processes and phenomena which during their evolution are subject to short-time perturbations in the form of impulses. The duration of the action of these perturbations is negligibly small in comparison with the total duration of the process.

In the present paper the problem of global stability of sets with respect to systems of differential equations with impulses is considered. In the investigations accomplished piecewise continuous auxiliary functions are used which are an analog of the classical Lyapunov's functions. It is proved that the existence of such functions with certain properties is a sufficient condition for various types of global asymptotic stability.

* Investigation accomplished with the financial support of the Committee for Science of the Council of Ministers of the People's Republic of Bulgaria, Grant No. 61
We shall note that results related to the stability and global stability of sets systems of ordinary differential equations without impulses have been obtained [12]–[16].

2. Preliminary notes and definitions

Let $\mathbb{R}^n$ be an $n$–dimensional Euclidian space with a norm $\| \cdot \|$, scalar product $\cdot$ and distance $d(\cdot, \cdot)$ and let $I = [0, \infty]$.

Consider the following system of differential equations with impulses

$$
\frac{dx}{dy} = f(t, x), \quad \text{for } (t, x) \in \tau_i;
$$

$$
\Delta x_{|(t, x) \in \tau_i} = I_i(x), \quad \text{for } i = 1, 2, \ldots,
$$

where $x \in \mathbb{R}^n$, $f : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $I_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\tau_i : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$
\sigma_i = \{(t, x) \in I \times \mathbb{R}^n : t = \tau_i(x)\},
$$

$$
\Delta x_{|(t, x) \in \sigma_i} = x(t + 0) - x(t - 0).
$$

Such systems are characterized by the fact that under the action of a short–time force (impact, impulse) the mapping point $(t, x(t))$ from the extended phase space $I \times \mathbb{R}^n$ by meeting some of the hypersurfaces $\sigma_i$ is transferred momentarily from position $(t, x(t))$ into the position $(t, x(t) + I_i(x(t)))$. Each solution $x(t)$ of system is a piecewise continuous function with points of discontinuity of first type at which it is left continuous, i.e. at the moment $t_i$ when the integral curve $(t, x(t))$ meets hypersurface $\sigma_i$ the following relations hold:

$$
x(t_i - 0) = x(t_i),
$$

$$
\Delta x_{|t=t_i} = x(t_i + 0) - x(t_i - 0) = I_i(x(t_i)).
$$

Let $M \subset I \times \mathbb{R}^n$. Introduce the notations:

$$
M(t) = \{x \in \mathbb{R}^n : (t, x) \in M\}, \quad \text{for } t \in I;
$$

$$
M(t, \epsilon) = \{x \in \mathbb{R}^n : d(x, M(t)) < \epsilon\}, \quad \text{for } \epsilon > 0,
$$

where

$$
d(x, M(t)) = \inf_{y \in M(t)} ||x - y||
$$
the distance between \( x \) and the set \( M(t) \);

\[
G_i = \{(t, x) \in I \times \mathbb{R}^n : \tau_{i-1}(x) < t < \tau_i(x)\}, \quad \text{for } i = 1, 2, \ldots;
\]

\[
(\tau_0(x) \equiv 0 \quad \text{for } x \in \mathbb{R}^n)
\]

\[
G = \bigcup_{i=1}^{\infty} G_i;
\]

\[
S_\alpha = \{x \in \mathbb{R}^n : \|x\| \leq \alpha\}, \quad \text{for } \alpha > 0;
\]

\[
D_{\alpha, t} = \{x \in \mathbb{R}^n : d(x, M(t)) \leq \alpha\}, \quad \text{for } \alpha > 0, t \in I.
\]

Let \( t_0 \in I, x_0 \in \mathbb{R}^n \). Denote by \( x(t; t_0, x_0) \) the solution of system (1) satisfying the initial condition \( x(t_0 + 0; t_0, x_0) = x_0 \) and by \( J^+(t_0, x_0) \) denote the maximal interval of the form \((t_0, \omega)\) in which the solution \( x(t; t_0, x_0) \) is defined.

We shall say that conditions (A) are satisfied if the following conditions hold:

A. The function \( f(t, x) \) is continuous on \( I \times \mathbb{R}^n \) and Lipschitz continuous with respect to its second argument.

A. \[ \|f(t, x)\| \leq L < \infty \text{ for } (t, x) \in I \times \mathbb{R}^n \quad (L \geq 0). \]

A. For any \( i = 1, 2, \ldots \) the following inequality holds

\[
\|I_i(x_1) - I_i(x_2)\| \leq d\|x_1 - x_2\|, \quad \text{for } x_1, x_2 \in \mathbb{R}^n \quad (d \geq 0).
\]

A. The functions \( \tau_i(x) \) \((i = 1, 2, \ldots)\) are continuous and the following relations hold

\[
0 < \tau_1(x) < \tau_2(x) < \cdots, \quad \text{for } x \in \mathbb{R}^n,
\]

\[
\lim_{i \to \infty} \tau_i(x) = \infty \quad \text{uniformly on } \mathbb{R}^n,
\]

\[
\inf_{\mathbb{R}^n} \tau_i(x) - \sup_{\mathbb{R}^n} \tau_{i-1}(x) \geq \theta > 0.
\]

We assume that for system (1) the phenomenon "beating" is not observed, i.e. at the following condition (B) holds:

B. The integral curve of each solution of system (1) meets each of the hypersurfaces \( \{\sigma_i\} \) at most once.

Effective sufficient conditions which guarantee the absence of the phenomenon "beating" are given in [10] and [11].

We shall note that condition (B) is satisfied, for instance, in the particular case when \( \tau_i(x) = t_i, i = 1, 2, \ldots, x \in \mathbb{R}^n \), i.e. the impulse effect is realized at fixed moments.
We shall say that conditions (C) are satisfied if the following conditions hold

C₁. For each $t \in I$ the set $M(t)$ is not empty.

C₂. There exists a compact set $Q \subset \mathbb{R}^n$ such that $M(t) \subset Q$ for any $t \in I$.

C₃. For any compact subset $F$ of $I \times \mathbb{R}^n$ there exists a constant $K > 0$ depend
on $F$ so that if $(t, x), (t', x) \in F$ then the following inequality holds

$$|d(x, M(t)) - d(x, M(t'))| \leq K |t - t'|.$$ 

We shall say that condition (D) is satisfied if the following condition holds:

D. Each solution $x(t; t_0, x_0)$ of system (1) which satisfies the estimate

$$d(x(t, t_0, x_0), M(t)) \leq h < \infty \quad \text{for } t \in J^+(t_0, x_0)$$

is defined in the interval $(t_0, \infty)$.

We shall give definitions of stability of the set $M$ with respect to system (1).

**Definition 1.** The set $M$ is called:

a) **Stable with respect to system (1)** if

$$(\forall t_0 \in I) (\forall \alpha > 0) (\forall \varepsilon > 0) (\exists \delta = \delta(t_0, \alpha, \varepsilon) > 0) \left( \forall x_0 \in S_\alpha \cap M(t_0, \delta) \right)$$

$$(\forall t \in J^+(t_0, x_0)): \quad x(t; t_0, x_0) \in M(t, \varepsilon).$$

b) $t$ (α)–uniformly stable with respect to system (1) if the number $\delta$ from point a) does not depend on $t_0$ (on $\alpha$).

c) Uniformly stable with respect to system (1) if the number $\delta$ from point depends only on $\varepsilon$.

We shall give definitions of boundedness of the solutions of system (1) with respect to the set $M$.

**Definition 2.** The solutions of system (1) are called:

a) **Equi–M–bounded (equi–bounded with respect to the set M)** if:

$$(\forall t_0 \in I) (\forall \alpha > 0) (\forall \eta > 0) (\exists \beta = \beta(t_0, \alpha, \eta) > 0)$$

$$(\forall x_0 \in S_\alpha \cap D_\eta, t_0) (\forall t \in J^+(t_0, x_0)): \quad x(t; t_0, x_0) \in M(t, \beta).$$

b) $t$ (α)–uniformly M–bounded if the number $\beta$ from point a) does not dep-
don on $t_0$ (on $\alpha$).
c) **Uniformly M–bounded** if the number \( \beta \) from point a) depends only on \( \eta \).

Finally we shall give definitions of global asymptotical stability of the set \( M \) with respect to system (1).

**Definition 3.** The set \( M \) is called:

a) **Globally equi-attractive with respect to system** (1) if

\[
(\forall t_0 \in I) (\forall \alpha > 0) (\forall \eta > 0) (\forall \epsilon > 0) (\exists \sigma = \sigma(t_0, \alpha, \eta, \epsilon) > 0)

(\forall x_0 \in S_\alpha \cap D_{\eta, t_0}) (\forall t \geq t_0 + \sigma, t \in J^+(t_0, x_0)) : x(t; t_0, x_0) \in M(t, \epsilon).
\]

b) **\( t (\alpha) \)-uniformly globally attractive with respect to system** (1) if the number \( \sigma \) from point a) does not depend on \( t_0 \) (on \( \alpha \)).

c) **Uniformly globally attractive with respect to system** (1) if the number \( \sigma \) from point a) depends only on \( \eta \) and \( \epsilon \).

**Definition 4.** The set \( M \) is called:

a) **Globally equi-asymptotically stable with respect to system** (1) if \( M \) is a stable set and a globally equi-attractive set of system (1) and the solutions of (1) are equi–M–bounded.

b) **\( t (\alpha) \)-uniformly globally asymptotically stable with respect to system** (1) if \( M \) is a \( t (\alpha) \)-uniformly stable and \( t (\alpha) \)-uniformly globally attractive set of system (1) and if the solutions of system (1) are \( t (\alpha) \)-uniformly M–bounded.

c) **Uniformly globally asymptotically stable with respect to system** (1) if \( M \) is a uniformly stable and uniformly globally attractive set of system (1) and if the solutions of system (1) are uniformly M–bounded.

d) **Exponentially globally asymptotically stable with respect to system** (1) if

\[
(\exists c > 0) (\forall \alpha, \eta > 0) (\exists K(\alpha, \eta) > 0) (\forall t_0 \in I) (\forall x_0 \in S_\alpha \cap D_{\eta, t_0})

(\forall t \in J^+(t_0, x_0)) : d(x(t; t_0, x_0), M(t)) \leq K(\alpha, \eta) d(x_0, M(t_0)) e^{-c(t-t_0)}
\]

**Remark 1.** If condition \( C_2 \) holds, then for any \( \alpha > 0 \) there exists a number \( \eta > 0 \) such that \( S_\alpha \subset D_{\eta, t} \) for all \( t \in I \) and, conversely, for any \( \eta > 0 \) one can find a number \( \beta > 0 \) such that \( D_{\eta, t} \subset S_\alpha \) for all \( t \in I \). This shows that the M–boundedness of the solutions of system (1) defined in definition 2 is equivalent to the \( I \times O \)-boundedness of these solutions (\( O \) is the origin in \( \mathbb{R}^n \)), i.e. to boundedness in the usual sense [11].
Remark 2. If condition $C_2$ holds, then the numbers $\delta$ from definition 1, a), from definition 2, a) and $\sigma$ from definition 3, a) can be chosen independent of $t$ number $\alpha$. Hence, for instance, if the set $M$ is globally equi-asymptotically stable with respect to system (1), then $M$ is an $\alpha$-uniformly globally asymptotically stable set of system (1).

In the further considerations we shall use the class $\mathcal{V}_0$ of piecewise continuous auxiliary functions $V : I \times \mathbb{R}^n \to \mathbb{R}$ which are an analogue of Lyapunov's functions [8].

Definition 5. We shall say that the function $V : I \times \mathbb{R}^n \to \mathbb{R}$ belongs to the class if the following conditions hold:

1. The function $V$ is continuous on $G$ and is locally Lipschitz continuous with respect to its second argument on each of the sets $G_i$.
2. $V(t, x) = 0$ for $(t, x) \in M$ and $V(t, x) > 0$ for $(t, x) \notin M$.
3. For each $i = 1, 2, \ldots$ and for any point $(t_0, x_0) \in \sigma_i$ the limits

$$V(t_0 - 0, x_0) = \lim_{(t, x) \to (t_0, x_0)} V(t, x),$$

$$V(t_0 + 0, x_0) = \lim_{(t, x) \to (t_0, x_0)} V(t, x),$$

exist and are finite, and the equality

$$V(t_0 - 0, x_0) = V(t_0, x_0)$$

holds.

4. For any point $(t, x) \in \sigma_i$ ($i = 1, 2, \ldots$) the following inequality holds

$$V(t + 0, x + I_i(x)) \leq V(t, x).$$

Let $V \in \mathcal{V}_0$. For $(t, x) \in G$ we set

$$\dot{V}_{(1)}(t, x) = \limsup_{h \to 0^+} h^{-1} \left[V(t + h, x + h f(t, x)) - V(t, x)\right].$$

From condition 1 of definition 5 it follows that if $x = x(t)$ is a solution of system (1), then the equality

$$V(t, x) = D^+ V(t, x)$$
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\( D^+ V(t, x) = \limsup_{h \to 0^+} h^{-1} \left[ V(t + h, x(t + h)) - V(t, x(t)) \right] \)

the upper right Dini derivative of the function \( V(t, x(t)) \).

Moreover, if the function \( V \in \mathcal{V}_0 \) satisfies the condition

\[ \dot{V}(t, x) \leq 0 \quad \text{for} \ (t, x) \in G \]

d if \( x(t; t_0, x_0) \) is a solution of system (1), then the function \( V(t, x(t; t_0, x_0)) \) is

monotonically decreasing on \( J^+(t_0, x_0) \).

Denote by \( K \) the class of all continuous and strictly increasing functions \( a : I \to I \)

that \( a(0) = 0 \).

3. Main results

the next considerations we shall use the following lemma.

Lemma 1. Let condition \( A_1 \) hold and let the function \( V : I \times \mathbb{R}^n \to \mathbb{R} \) belong to

e class \( \mathcal{V}_0 \). Then for any choice of the numbers \( t_0 \in I, \alpha > 0, \eta > 0 \), there exists a

number \( K(t_0, \alpha, \eta) > 0 \) such that for \( (t_0, x_0) \in S_0 \cap D_\eta \) the following inequality holds

\[ V(t_0 + 0, x_0) \leq K(t_0, \alpha, \eta). \quad (3) \]

Proof. Suppose that the assertion is not true. Then there exist numbers \( t_0 \in I, \alpha > 0, \eta > 0 \),

and a sequence \( \{x_k\}^\infty_{k=1} \subset \mathbb{R}^n \) so that \( (t_0, x_k) \in S_0 \cap D_\eta, t_0 \) for \( k = 1, 2, \ldots \) and

e following inequality holds

\[ V(t_0 + 0, x_k) > k, \quad \text{for} \ k = 1, 2, \ldots \quad (4) \]

Since the sequence \( \{x_k\} \) is bounded, then out of it we can choose a convergent

subsequence. We shall use the same notation for the members of this subsequence.

Let \( t_0, x_0 \in G_i \) for some \( i \in \mathbb{N} \), i.e.

\[ \tau_{i-1}(x_0) < t_0 < \tau_i(x_0). \]

Then since

\[ \tau_{i-1}(x_0) = \lim_{k \to \infty} \tau_{i-1}(x_k) \]
and
\[ \tau_i(x_0) = \lim_{k \to \infty} \tau_i(x_k), \]
for all sufficiently large values of \( k \) we have
\[ \tau_{i-1}(x_k) < t_0 < \tau_i(x_0) \]
which shows that \((t_0, x_k) \in G_i\). From the continuity of \( V(t, x) \) on \( G_i \) it follows that
\[ \lim_{k \to \infty} V(t_0 + 0, x_k) = \lim_{k \to \infty} V(t_0, x_k) = V(t_0, x_0) \]
which contradicts inequality (4).

Let \((t_0, x_0) \in \sigma_i\) for some \( i \in \mathbb{N} \), i.e. \( \tau_i(x_0) = t_0 \). Then the following relations hold
\[ \lim_{k \to \infty} \tau_{i-1}(x_k) = \tau_{i-1}(x_0) \]
\[ < \tau_i(x_0) \]
\[ = t_0 \]
\[ = \tau_i(x_0) \]
\[ < \tau_{i+1}(x_0) \]
\[ = \lim_{k \to \infty} \tau_{i+1}(x_k). \]

These relations show that the following three cases are possible:

1. For infinitely many members \( \{x_{k_j}\}_{j=1}^{\infty} \) of the sequence \( \{x_k\} \) we have \((t_0, x_k) \in G_i\). Then from the left continuity of the function \( V(t, x) \) we obtain
\[ \lim_{j \to \infty} V(t_0 + 0, x_{k_j}) = \lim_{j \to \infty} V(t_0, x_{k_j}) \]
\[ = V(t_0 - 0, x_0) \]
\[ = V(t_0, x_0), \]
which contradicts inequality (4).

2. For infinitely many members \( \{x_{k_j}\}_{j=1}^{\infty} \) of the sequence \( \{x_k\} \) we have \((t_0, x_k) \in G_{i+1}\). Then
\[ \lim_{j \to \infty} V(t_0 + 0, x_{k_j}) = \lim_{j \to \infty} V(t_0, x_{k_j}) \]
\[ = V(t_0 + 0, x_0) \]
\[ < \infty, \]
which contradicts inequality (4).

3. Only a finite number of members \( \{x_{k_j}\}_{j=1}^{\infty} \) of the sequence \( \{x_k\} \) are such that \((t_0, x_{k_j}) \in G_i\) or \((t_0, x_{k_j}) \in G_{i+1}\). Without loss of generality we can assume
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From inequality (4) and the left continuity of the function \( t, x \) at the points \((t_0, x_k)\) it follows that there exist sequences \(((t_n^{(k)}, y_n^{(k)}))_{n=1}^{\infty} = 1, 2, \ldots\) such that

\[
\lim_{n \to \infty} \left( t_n^{(k)}, y_n^{(k)} \right) = (t_0, x_k)
\]

and

\[
V \left( t_n^{(k)} + 0, y_n^{(k)} \right) \geq k \quad \text{for } k = 1, 2, \ldots.
\]

Hence

\[
\lim_{n \to \infty} V(t_n^{(n)} + 0, y_n^{(n)}) = \lim_{n \to \infty} V(t_n^{(n)}, y_n^{(n)})
\]

\[
= V(t_0 - 0, x_0)
\]

\[
= V(t_0, x_0),
\]

which contradicts the inequality

\[
V(t_n^{(n)} + 0, y_n^{(n)}) \geq n.
\]

This completes the proof of Lemma 1.

We shall find sufficient conditions for global asymptotical stability of the set \( M \) with respect to system (1).

**Theorem 1.** Let conditions (A), (B), (C_1), (C_3) and (D) hold and functions \( V \in \mathcal{V}_0 \) and \( a \in \mathcal{K} \) exist such that

(i) \( a(d(x, M(t))) \leq V(t, x) \) for \((t, x) \in I \times \mathbb{R}^n\) and \( a(r) \to \infty \) as \( r \to \infty \).

(ii) \( \dot{V}(t, x) \leq -c V(t, x) \) for \((t, x) \in G\) where \( c > 0 \) is a constant.

Then the set \( M \) is globally equi-asymptotically stable with respect to system (1).

**Proof.** Let \( \epsilon > 0, \alpha > 0, t_0 \in I \). From the condition \( V(t_0, x) = 0 \) for \( x \in M(t_0) \) it follows that there exists a number \( \delta = \delta(t_0, \alpha, \epsilon) > 0 \) such that if \( x \in S_\alpha \cap M(t_0, \delta) \), then \( V(t_0 + 0, x) < a(\epsilon) \).

Let \( x_0 \in S_\alpha \cap M(t_0, \delta) \). Using successively (i), (ii) and (2) we obtain

\[
a(d(x(t; t_0, x_0), M(t))) \leq V(t, x(t; t_0, x_0))
\]

\[
\leq V(t_0 + 0, x_0)
\]

\[
< a(\epsilon)
\]
for $t \in J^+(t_0, x_0)$. From condition (D) it follows that $J^+(t_0, x_0) = (t_0, \infty)$, he $x(t; t_0, x_0) \in M(t, \epsilon)$ for all $t > t_0$, i.e. the set $M$ is stable with respect to system.

Let $\epsilon > 0$, $\alpha > 0$, $\eta > 0$ and $t_0 \in I$. From (ii) and (2) there follows the inequa

$$V(t, x(t; t_0, x_0)) \leq V(t_0 + 0, x_0) \exp[-c(t - t_0)].$$

Let

$$N(t_0, \eta, \alpha) = \sup\{V(t_0 + 0, x_0) : x_0 \in S_\alpha \cap D_{\eta, t_0}\}$$

and

$$\sigma = \sigma(t_0, \alpha, \eta, \epsilon) > \frac{1}{c} \ln \frac{N(t_0, \eta, \alpha)}{a(\epsilon)}.$$

Then from (i) and (5) it follows that for $t \geq t_0 + \sigma$ the following inequalities h

$$a(\delta(x(t; t_0, x_0), M(t))) \leq V(t, x(t; t_0, x_0))$$

$$\leq V(t_0 + 0, x_0) \exp[-c(t - t_0)]$$

$$\leq a(\epsilon),$$

which means that the set $M$ is globally equi-attractive with respect to system (1)

Finally we shall prove that the solutions of system (1) are equi-$M$-bounded.

In fact, let $t_0 \in I$, $\alpha > 0$ and $\eta > 0$. From lemma 1 it follows that there exis number $K(t_0, \alpha, \eta) > 0$ such that if $x \in S_\alpha \cap D_{\eta, t_0}$, then $V(t_0 + 0, x) \leq K(t_0, \alpha)$

From the condition $a(r) \to \infty$ for $r \to \infty$ it follows that there exists a numb $\beta = \beta(t_0, \alpha, \eta) > 0$ such that $a(\beta) > K(t_0, \alpha, \eta)$.

Let $x_0 \in S_\alpha \cap D_{\eta, t_0}$ and let $x(t) = x(t; t_0, x_0)$ be a solution of system (1). T from (i), (ii) and (2) we get

$$a(\delta(x(t), M(t))) \leq V(t, x(t))$$

$$\leq V(t_0 + 0, x_0)$$

$$\leq K(t_0, \alpha, \eta)$$

$$< a(\beta)$$

for each $t \in J^+(t_0, x_0)$. From condition (D) it follows that $J^+(t_0, x_0) = (t_0, \infty)$. It shows that

$$d(x(t; t_0, x_0), M(t)) < \beta \quad \text{for } t > t_0,$$

hence the solutions of system (1) are equi-$M$-bounded.

This completes the proof of theorem 1.
Theorem 2. Let conditions (A), (B), (C₁), (C₂), and (D) hold and functions \( V \in V₀ \) and \( a, b, c \in K \) exist such that

(i) \( a(d(x, M(t))) \leq V(t + 0, x) \leq b(d(x, M(t))) \), for \( (t, x) \in I \times \mathbb{R}^n \)
where \( a(r) \to \infty \) for \( r \to \infty \).

(ii) \( \dot{V}_1(t, x) \leq -c(d(x, M(t))) \), for \( (t, x) \in G \).

Then the set \( M \) is uniformly globally asymptotically stable with respect to system (1).

Proof. Let \( \epsilon > 0 \) be given. Choose the number \( \delta = \delta(\epsilon) > 0 \) so that \( b(\delta) < a(\epsilon) \). Then \( \alpha \), \( v_0 \in I \) and \( x_0 \in S \cap M(t_0, \delta) \) using successively (i), (ii), (2) we obtain

\[
a(d(x(t; t_0, x_0), M(t))) \leq V(t, x(t; t_0, x_0)) \\
\leq V(t_0, x_0) \\
\leq b(d(x_0, M(t_0))) \\
\leq b(\delta) \\
< a(\epsilon)
\]

\( t \in J^+(t_0, x_0) \). Hence \( J^+(t_0, x_0) = (t_0, \infty) \) and \( x(t; t_0, x_0) \in M(t, \epsilon) \) for any \( t > t_0 \) which means that the set \( M \) is uniformly stable with respect to system (1).

Let \( \epsilon > 0 \) be given. Choose the number \( \delta = \delta(\epsilon) > 0 \) so that \( b(\delta) < a(\epsilon) \).

Let \( \eta > 0 \),

\[
\sigma = \sigma(\eta, \epsilon) > \frac{b(\eta)}{c(\delta(\epsilon))},
\]

> 0, \( t_0 \in I, x_0 \in S \cap D_{\eta, t_0} \) and \( x(t) = x(t; t_0, x_0) \) be a solution of system (1). Assume that for any \( t \in [t_0, t_0 + \sigma] \) the inequality \( d(x(t), M(t)) \geq \delta(\epsilon) \) holds. Then from (ii) we obtain

\[
\int_{t_0}^t \dot{V}_1(s, x(s)) \, ds \leq -c(\delta(\epsilon)) [t - t_0]. \quad (6)
\]

On the other hand, if \( \{t_i\}_{i=1}^k \) are the moments at which the integral curve of the solution \( x(t) \) meets respectively the hypersurfaces \( \{x_i\} \), then from (2) we obtain,

\[
\int_{t_0}^t \dot{V}_1(s, x(s)) \, ds = \sum_{i=1}^k \int_{t_{i-1} + 0}^{t_i - 0} \dot{V}_1(s, x(s)) \, ds + \int_{t_k + 0}^{t_{i-0}} \dot{V}_1(s, x(s)) \, ds \\
= \sum_{i=1}^k [V(t_i, x(t_i)) - V(t_{i-1} + 0, x(t_i + 0))] \\
+ V(t, x(t)) - V(t_k + 0, x(t_k + 0)) \\
\geq V(t, x(t)) - V(t_0 + 0, x_0).
\]
From this inequality and (6) we deduce
\[ V(t, x(t)) \leq V(t_0 + 0, x_0) - c(\delta) [t - t_0] \quad \text{for } t \in [t_0, t_0 + \sigma), \]
whence for \( t = t_0 + \sigma \) we have
\[ V(t, x(t)) \leq V(t_0 + 0, x_0) - c(\delta) \sigma \leq b(\eta) - c(\delta) \sigma < 0, \]
which contradicts (i).

Hence there exists a number \( t' \in [t_0, t_0 + \sigma] \) such that
\[ d(x(t'), M(t')) < \delta(\epsilon). \]

Then for \( t \geq t' \), hence for any \( t \geq t_0 + \sigma \) as well the following inequalities hold
\[ \begin{align*}
a(d(x(t), M(t))) &\leq V(t, x(t)) \\
&\leq V(t', x(t')) \\
&\leq b(d(x(t'), M(t'))) \\
&\leq b(\delta) \\
&< a(\epsilon),
\end{align*} \]
i.e. \( x(t; t_0, x_0) \in M(t, \epsilon) \) which shows that the set \( M \) is uniformly globally attract with respect to system (1).

Finally we shall prove that the solutions of system (1) are uniformly \( M \)-bound

Let \( \eta > 0 \) be given. Choose \( \beta = \beta(\eta) > 0 \) so that \( a(\beta) > b(\eta) \). This is possi in view of the condition \( a(r) \to \infty \) for \( r \to \infty \).

Let \( t_0 \in I, \alpha > 0 \) and \( x_0 \in S_\alpha \cap D_{t_0, \infty} \). Applying successively (i), (ii) and (2) obtain
\[ \begin{align*}
a(d(x(t; t_0, x_0), M(t))) &\leq V(t, x(t; t_0, x_0)) \\
&\leq V(t_0 + 0, x_0) \\
&\leq b(d(x_0, M(t_0))) \\
&\leq b(\eta) \\
&< a(\beta)
\end{align*} \]
for \( t \in J^+(t_0, x_0) \). Hence \( J^+(t_0, \infty) = (t_0, \infty) \) and \( x(t; t_0, x_0) \in M(t, \epsilon) \) for \( t > t_0 \).

This completes the proof of theorem 2.

Corollary 1. Let the conditions of theorem 2 hold, condition (ii) being replaced the condition
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(iii) \( \dot{V}_{(1)}(t, x) \leq -c V(t, x) \), for \((t, x) \in G\), where \(c\) is a constant.

Then the set \(M\) is uniformly globally asymptotically stable with respect to system (1).

The assertion is an immediate consequence of theorem 2. But if we make use of
\[
V(t, x(t; t_0, x_0)) \leq V(t_0 + 0, x_0) \exp[-c(t - t_0)] \quad \text{for } t > t_0,
\]
the inequality following from (iii) and (2), then the proof can be carried out as the proof of theorem 1.

Analogous to the proof of the theorem 2 is the proof of the following two theorems:

**Theorem 3.** Let the conditions of theorem 2 hold, condition (i) being replaced by

(iv) \( a(d(x, M(t))) \leq V(t + 0, x) \leq b(d(x, M(t))), \) for \((t, x) \in I \times \mathbb{R}^n\),
where \(a \in \mathcal{K}, \ a(r) \rightarrow \infty \text{ for } r \rightarrow \infty\) and the function \(b : r \rightarrow b(r, s) \in \mathcal{K}\) for any \(s \geq 0\) fixed.

Then the set \(M\) is \(t\)-uniformly globally asymptotically stable with respect to system (1).

**Theorem 4.** Let the conditions of theorem 2 hold, condition (i) being replaced by

(v) \( a(d(x, M(t))) \leq V(t + 0, x) \leq b(d(x, M(t))), \) for \((t, x) \in I \times \mathbb{R}^n\),
where \(a \in \mathcal{K}, \ a(r) \rightarrow \infty \text{ for } r \rightarrow \infty\) and the function \(b : r \rightarrow b(t, r) \in \mathcal{K}\) for any \(t \in I\) fixed.

Then the set \(M\) is \(\alpha\)-uniformly globally asymptotically stable with respect to system (1).

**Theorem 5.** Let conditions (A), (B), (C_1), (C_3) and (D) hold and a function \(V \in \mathcal{V}_0\) exist such that

(i) \( d(x, M(t)) \leq V(t, x) \leq K(\eta, \alpha) d(x, M(t)) \), for \(\alpha > 0, \ \eta > 0, \ t \in I, \ x \in S_\alpha \cap D_{\eta, t}\).

(ii) \( \dot{V}_{(1)}(t, x) \leq -c V(t, x) \) for \((t, x) \in G\) where \(c > 0\) is a constant.

Then the set \(M\) is exponentially globally asymptotically stable with respect to system (1).

**Proof.** As in the proof of theorem 2 it can be proved that the solutions of system (1) are uniformly \(M\)-bounded. Hence each solution \(x(t; t_0, x_0)\) is defined in the interval \([t_0, \infty), \ t_0 \in I\).
Let $\alpha > 0$, $\eta > 0$, $t_0 \in I$, $x_0 \in S_\alpha \cap D_{\eta,t_0}$. From (ii) and (2) we obtain

$$V(t, x(t; t_0, x_0)) \leq V(t_0 + 0, x_0) \exp[-c(t - t_0)] \quad \text{for } t > t_0.$$ 

Then from (i) it follows that, for $t > t_0$:

$$d(x(t; t_0, x_0), M(t)) \leq V(t, x(t; t_0, x_0))$$
$$\leq V(t_0 + 0, x_0) \exp[-c(t - t_0)]$$
$$\leq K(\eta, \alpha) d(x_0, M(t_0)) \exp[-c(t - t_0)].$$

Theorem 5 is proved.

4. Examples

Example 1. Consider the system of differential equations with impulse effect at fix moments

$$\frac{dx}{dt} = \begin{cases} [B(t) + A(t)] x & \text{for } x > 0, t \neq t_i \\ 0 & \text{for } x \leq 0, t \neq t_i \end{cases}$$
$$\Delta x|_{t=t_i} = \begin{cases} I_i(x) & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

where $x \in \mathbb{R}^n$, $B(t)$ and $A(t)$ are $(n \times n)$-matrices of continuous functions on $I$ a $B(t)$ is diagonal and $A(t)$ is skew-symmetric, $I_i(x)$, $i = 1, 2, \ldots$, are continuous a such that $x + I_i(x) > 0$ and $\|x + I_i(x)\| \leq 0$ for $x > 0$ ($x > 0$ ($x \leq 0$) means $x_i > 0$ ($x_i \leq 0$) for $i = 1, 2, \ldots, n$, where $x_i$ is the $i$-th component of the vect $x \in \mathbb{R}^n$).

The moments $\{t_i\}$ of impulse effect form a strictly increasing sequence, i.e.

$$0 < t_1 < t_2 < \cdots < t_i < \cdots$$

and

$$\lim_{i \to \infty} t_i = \infty.$$ 

Let $M = I \times \{x \in \mathbb{R}^n : x \leq 0\}$. Consider the function

$$V(t, x) = \begin{cases} (x, x) & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$
\[ \dot{V}(t, x) = \begin{cases} \dot{2}(x, B(t)x) & \text{for } z > 0, \ t \neq t_i \\ 0 & \text{for } z \leq 0, \ t \neq t_i \end{cases} \]

\[ V(t_i + 0, x + I_i(x)) = \begin{cases} \|x + I_i(x)\|^2 & \text{for } z > 0 \\ 0 & \text{for } z \leq 0 \end{cases} \]

\[ V(t_i + 0, x + I_i(x)) \leq V(t_i, x). \]

Let \( B(t) = \text{diag}(b_1(t), b_2(t), \ldots, b_n(t)) \) and \( b_i(t) \leq -\gamma_i < 0 \) for \( i = 1, 2, \ldots, n \) and \( z \in I \).

Since \( d(x, M(t)) = \|x\| \) for \( t \in I \) and \( z > 0 \), then the conditions of theorem 2 are satisfied.

Hence the set \( M \) is uniformly globally asymptotically stable with respect to theorem (7).

**Example 2.** Consider the system

\[ \frac{dx}{dt} = a(t) y + b(t) x (x^2 + y^2) \]

\[ \frac{dy}{dt} = -a(t) x + b(t) y (x^2 + y^2), \quad t \neq t_i \]

\[ \Delta x|_{t=t_i} = c_i x(t_i), \]

\[ \Delta y|_{t=t_i} = y(t_i), \]

here the functions \( a(t) \) and \( b(t) \) are continuous on \( I \), \( b(t) \leq -\gamma < 0 \), \( -1 > c_i \leq 0 \), \( 1 < d_i \leq 0 \), for \( i = 1, 2, \ldots, 0 < t_1 < t_2 < \ldots, \) and \( \lim_{t \to \infty} t_i = \infty \). Let \( M = \{t, 0, 0 \} : t \in I \} \).

The function \( V(t, x, y) = x^2 + y^2 \) satisfies the conditions of theorem 2. In fact,

\[ \dot{V}(t, x, y) = 2 b(t) (x^2 + y^2)^2 \leq -\gamma (x^2 + y^2)^2 \quad t \neq t_i \]

\[ V(t_i + 0, x + c_i x, y + d_i y) \leq (1 + c_i)^2 x^2 + (1 + d_i)^2 y^2 \leq V(t_i, x, y). \]

Hence the set \( M \) is uniformly globally asymptotically stable with respect to theorem (8).
References


GLOBAL STABILITY OF SETS FOR SYSTEMS WITH IMPULSES


Received 25/FEB/88
Revised version received 28/APR/88

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