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A continuity in the weak topologies of a vector integral operator

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Abstract

Let Y be a sequentially $\sigma(Y,X)$-complete Banach space satisfying the Radon-Nikodym property. Then, every vector integral operator $A:L_X \to M_U$ is continuous in the weak topologies $\sigma(L_X, L_Y^*)$ and $\sigma(M_U, M_V^*)$.

Let (S,Σ,μ), (T,Λ,ν) be spaces with σ-finite measures and $(S\times T, \Sigma\times\Lambda, \mu\times\nu)$ be their products; $L^0 = L^0(S)$ and $L^0 = L^0(T)$ be spaces of measurable real-valued functions on S and T.

A space $L \subset L^0$ is called a perfect space if $L = L^{**}$ and $S = \operatorname{supp} L \cap \operatorname{supp} L^*$, where

$$L^* = \left\{ \varphi^* \in L^0 : \int_S |\varphi(s)| \cdot |\varphi^*(s)| \, d\mu(s) < \infty, \quad \varphi \in L \right\}.$$

A perfect spaces $L, L^* \subset L^0$ form a dual pair $< L, L^* >$ with respect to a bilinear form

$$< \varphi, \varphi^* > = \int_S \varphi(s) \cdot \varphi^*(s) \, d\mu(s)$$

and there exists a sequence of the measurable sets $S_n \uparrow S$ such that functions $\chi_{S_n} \in L \cap L^*$. Spaces $L_\varphi^\infty, L_\varphi^1 \subset L^0(\operatorname{supp}\varphi), \varphi \in L_+$ form a dual pair $< L_\varphi^\infty, L_\varphi^1 >$ of a perfect spaces, where

$$L_\varphi^\infty = \{\psi \in L^0(\operatorname{supp}\varphi) : \exists \alpha : |\psi(s)| \le \alpha \cdot \varphi(s) \ \mu - a.e.\} = (L_\varphi^1)^*,$$

and

$$L_\varphi^1 = \left\{ \psi^* \in L_0(\operatorname{supp}\varphi) : \int_{\operatorname{supp}\varphi} \varphi(s) \cdot |\psi^*(s)| \, d\mu(s) < \infty \right\} = (L_\varphi^\infty)^*.$$

For perfect spaces the following equalities take place:

$$L = \cup L^{\infty}_{\varphi} : \varphi \in L_+, L^* = \cap L^1_{\varphi} : \varphi \in L_+.$$

Banach spaces X and Y form a dual pair $< X, Y >$ if they form a dual pair as a linear spaces with respect to a bilinear form $< x, y >$ and

$$\|x\| = \sup \left\{ | < x, y > | : \|y\| \leq 1 \right\}$$

and

$$\|y\| = \sup \left\{ | < x, y > | : \|x\| \leq 1 \right\}.$$

Let $L^0_X = L^0_X(S)$ and $L^0_Y = L^0_Y(S)$ be spaces of Bochner's measurable X- and Y-valued functions on S, $L^0_U = L^0_U(T)$ and $L^0_V = L^0_V(T)$ be spaces of Bochner's measurable U- and V-valued functions on T, where $< U, V >$ is a pair of Banach spaces in duality.

Let $B(X, U)$ is a Banach space of a linear mapping from X into U. A normed space $B_\sigma(X, U)$ consists of all mappings from $B(X, U)$ which are continuous in the weak topologies $\sigma(X, Y)$ and $\sigma(U, V)$.

A space $L_X = \{f \in L^0_X : \|f\| \in L\} \subset L^0_X$ is called a perfect space of the measurable Banach-valued functions if $L \subset L^0$ is a perfect space of a measurable real-valued functions.

The perfect spaces $L_X \subset L^0_X$ and $L^*_Y \subset L^0_Y$ form a dual pair $\ll L_X, L^*_Y \gg$ with respect to a bilinear form

$$\ll f, f^* \gg = \int_S < f(s), f^*(s) > \; d\mu(s).$$

In this case for perfect spaces the following equalities are valid:

$$L_X = \cup L^{\infty}_{\varphi X} : \varphi \in L_+, L^*_Y = \cap L^1_{\varphi Y} : \varphi \in L_+,$$

$$L_X = \left\{ f \in L^0_X : \int_S | < f(s), f^*(s) > | \, d\mu(s) < \infty, f^* \in L^*_Y \right\}$$

and

$$L^*_Y = \left\{ f^* \in L^0_Y : \int_S | < f(s), f^*(s) > | \, d\mu(s) < \infty, f \in L_X \right\}.$$

Let $\ll L_X, L^*_Y \gg$ and $\ll M_U, M^*_V \gg$ be two dual pairs of a perfect spaces of a measurable Banach-valued functions generated by a dual pairs $< L, L^* >$ and

$< M, M^* >$ of a perfect spaces of the measurable real-valued functions $L, L^* \subset L^0(S)$ and $M, M^* \subset L^0(T)$.

A linear mapping $A : L_X \to M_U$ is called a vector integral operator, if there exists a Bochner's measurable operator-valued function $K : S \times T \to B_\sigma(X, U)$ so that

$$(Af)(t) = \int_S K(s,t)f(s)\, d\mu(s)\ \nu - a.e., \quad f \in L_X$$

and

$$\int_S \|K(s,t)\| \cdot \|f(s)\|\, d\mu(s) < \infty\ \nu - a.e., \quad f \in L_X.$$

In [1] and [2] the authors showed the continuity of a real integral operators $A : L \to M$ in the weak topologies $\sigma(L, L^*)$ and $\sigma(M, M^*)$. For the vector integral operators we prove the following.

Theorem

Let a Banach space Y has a Radon-Nikodym property and is sequentially $\sigma(Y, X)$-complete. Then every vector integral operator $A : L_X \to M_U$ is continuous in the weak topologies $\sigma(L_X, L_Y^*)$ and $\sigma(M_U, M_V^*)$.

Proof. Let a function $g^* \in M_V^*$. We'll show that for every function $\varphi \in L_+$ there exists a function $f_\varphi^* \in L^1_{\varphi Y}$ so that

$$\ll Af, g^* \gg = \ll f, f_\varphi^* \gg, \quad f \in L^\infty_{\varphi X}.$$

Let a function $\varphi \in L_+$ and $\varphi(s) > 0\ \mu - a.e.$ A measurable function

$$\psi(t) = \int_S \|K(s,t)\| \cdot \varphi(s)\, d\mu(s) \cdot \|g^*(t)\| < \infty\ \nu - a.e.$$

There exists a sequence of a measurable sets $T_n \uparrow T$ such that

$$\chi_{T_n} \in M \cap M^*, \ \int_{T_n} \psi(t)\, d\nu(t) < \infty, \ n \geq 1.$$

For every $n \geq 1$ by Fubini's theorem [3] we have

$$\int_S \varphi(s) \cdot \int_{T_n} \|K(s,t)\| \cdot \|g^*(t)\|\, d\nu(t)\, d\nu(s) = \int_{T_n} \psi(t)\, d\nu(t) < \infty$$

and a measurable function

$$f_n^*(s) = \int_{T_n} K^*(s,t) g^*(t)\, d\nu(t) \in L^1_{\varphi Y}.$$

For every function $f \in L^\infty_{\varphi X}$ by Fubini's theorem we have

$$\begin{aligned}
&\int_{T_n} < (Af)(t), g^*(t) > \ d\nu(t) \\
&= \int_{T_n} < \int_S K(s,t)f(s)\ d\mu(s), g^*(t) > \ d\nu(t) \\
&= \int_{T_n} \int_S < K(s,t)f(s), g^*(t) > \ d\mu(s)\ d\nu(t) \\
&= \int_S \int_{T_n} < f(s), K^*(s,t)g^*(t) > \ d\nu(t)\ d\mu(s) \\
&= \int_S < f(s), \int_{T_n} K^*(s,t)g^*(t)\ d\nu(t) > \ d\mu(s) \\
&= \ll f, f_n^* \gg
\end{aligned}$$

and by Lebesgue's theorem we have

$$\begin{aligned}
\ll Af, g^* \gg = & \int_T < (Af)(t), g^*(t) > \ d\nu(t) \\
= & \lim_{n\to\infty} \int_{T_n} < (Af)(t), g^*(t) > \ d\nu(t) = \lim_{n\to\infty} \ll f, f_n^* \gg .
\end{aligned}$$

Hence a sequence $(f_n^*) \subset L^1_{\varphi Y}$ is a fundamental in the weak topology $\sigma(L^1_{\varphi Y}, L^\infty_{\varphi X})$. A space $L^1_{\varphi Y}$ is a sequentially complete in the weak topology $\sigma(L^1_{\varphi Y}, L^\infty_{\varphi X})$ if and only if Y is a sequentially $\sigma(Y,X)$-complete Banach space and has the Radon-Nikodym property [4]. Hence, there exists a function $f_\varphi^* \in L^1_{\varphi Y}$ such that

$$\ll Af, g^* \gg = \lim_{n\to\infty} \ll f, f_n^* \gg = \ll f, f_\varphi^* \gg,\ f \in L^\infty_{\varphi X}.$$

For a perfect space L there exists a sequence of a measurable mutually disjoint sets $S_m, m \geq 1$ such that

$$S = \sum_{m=1}^{\infty} S_m,\ \chi_{S_n} \in L \cap L^*, m \geq 1.$$

For every $m \geq 1$ there exists a function $f_m^* \in L^1_{\chi_{S_m} Y}(S_m)$ such that

$$\ll Af, g^* \gg = \ll f, f_m^* \gg, \quad f \in L^\infty_{\chi_{S_m} X}(S_m).$$

Let a measurable function $f_0^*(s) = \sum_{m=1}^\infty \chi_{S_m}(s) \cdot f_m^*(s)$. We'll show that a function $f_0^* \in L_X^*$ and

$$\ll Af, g^* \gg = \ll f, f_0^* \gg, \quad f \in L_X.$$

Let a function $f \in L_X$. Then a function $\varphi(s) = \|f(s)\| \in L_+$ and there exists a function $f^*_\varphi \in L^1_{\varphi Y}$ such that

$$\ll Af_0, g^* \gg = \ll f_0, g^*_\varphi \gg, \quad f_0 \in L^\infty_{\varphi X}.$$

For a measurable function f there exists a sequence of $\Sigma-$ step function $f_n, n \geq 1$ such that

$$f(s) = \mu - a.e. \lim_{n\to\infty} f_n(s), \ \|f_n(s)\| \leq \|f(s)\| \ \mu - a.e., \quad n \geq 1$$

([5], 2.6. Theorem 2). For every $n, m \geq 1$ $\Sigma-$ step functions $\chi_{S_m} \cdot f_n \in L^\infty_{\chi_{S_m} X} \cap L^\infty_{\varphi X}$. For every $n \geq 1$ we have

$$\begin{aligned} \int_A < f_n(s), f^*_\varphi(s) > \ d\mu(s) &= \ll \chi_A \cdot f_n, f^*_\varphi \gg \\ &= \ll A(\chi_A \cdot f_n), g^* \gg = \ll \chi_A \cdot f_n, f^*_m \gg \\ &= \int_A < f_n(s), f^*_m(s) > \ d\mu(s), \ A \subset S_m, m \geq 1 \end{aligned}$$

and then measurable functions

$$< f_n(s), f^*_\varphi(s) > \cdot \chi_{S_m}(s) = < f_n(s), f^*_m(s) > \cdot \chi_{S_m}(s) \ \mu - a.e., \quad m \geq 1$$

and then measurable functions

$$\begin{aligned} < f_n(s), f^*_\varphi(s) > &= \sum_{m=1}^\infty < f_n(s), f^*_\varphi(s) > \cdot \chi_{S_m}(s) \\ &= \sum_{m=1}^\infty < f_n(s), f^*_m(s) > \cdot \chi_{S_m}(s) = < f_n(s), f^*_0(s) > \ \mu - a.e. \end{aligned}$$

A measurable function

$$\begin{aligned} < f(s), f^*_0(s) > &= \ \mu - a.e. \lim_{n\to\infty} < f_n(s), f^*_0(s) > \\ &= \ \mu - a.e. \lim_{n\to\infty} < f_n(s), f^*_\varphi(s) > = < f(s), f^*_\varphi(s) > \in L^1(S) \end{aligned}$$

and

$$\begin{aligned} \ll Af, g^* \gg &= \ll f, f^*_\varphi \gg \\ &= \int_S < f(s), f^*_\varphi(s) > \ d\mu(s) = \int_S < f(s), f^*_0(s) > \ d\mu(s) \\ &= \ll f, f^*_0 \gg . \end{aligned}$$

Then a measurable function

$$f_0^* \in \left\{ f^* \in L_Y^0 : \int_S | < f(s), f^*(s) > | \, d\mu(s) < \infty, f \in L_X \right\} = L_Y^*.$$

Thus for every function $g^* \in M_V^*$ there exists a function $f^* \in L_Y^*$ such that

$$\ll Af, g^* \gg = \ll f, f^* \gg, \quad f \in L_X.$$

This proves the continuity of the vector integral operator $A : L_X \to M_U$ in the weak topologies $\sigma(L_X, L_Y^*)$ and $\sigma(M_U, M_V^*)$. $\square$

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