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# A dual property to uniform monotonicity in Banach lattices

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Abstract

For Banach lattices $X$ with strictly or uniformly monotone lattice norm dual, properties (o)-smoothness and (o)-uniform smoothness are introduced. Lindenstrauss type duality formulas are proved and duality theorems are derived. It is observed that (o)-uniformly smooth Banach lattices $X$ are order dense in $X^{**}$. An application to an optimization problem is given.

## 1. Introduction

Let $X$ be a Banach lattice with the dual $X^*$ and let $\|\cdot\|$ stands for the corresponding dual (monotone) norms. $X$ is said to be *strictly monotone* (STM) (we will often write $X \in$ STM etc.) if $\| x - y \| < \| x \|$ whenever $0 < y \leq x$. The strongest property in this direction is the *uniform monotonicity* (UM) of $X$ which means that $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0,1]$ where

$$\delta_X(\epsilon) = \inf\left\{1 - \| x - y \| : 0 \leq y \leq x, \| x \| = 1, \| y \| \geq \epsilon\right\} \quad (\epsilon \in [0,1]).$$

In [2] (Chap. XV) this is called a "UMB" space. It is worth noticing the following fact (cf. [5] and [4], p. 124).

**Lemma**

*For $\epsilon \in [0,1)$ the following formula holds true:*

$$\delta_X(\epsilon) = \frac{\sigma_X(\epsilon)}{1 + \sigma_X(\epsilon)}$$

*where $\sigma_X(\epsilon)$ is a modulus of the uniform monotonicity defined by*

$$\sigma_X(\epsilon) = \inf\left\{ \parallel x+y \parallel -1 : x,y \geq 0, \parallel x \parallel = 1, \parallel y \parallel \geq \epsilon \right\} \quad (\epsilon \in [0,1]).$$

*Proof.* It suffices to apply the following identity, with $\parallel u \parallel = 1$,

$$\frac{\parallel u+z \parallel -1}{\parallel u+z \parallel} = 1 - \left\| \frac{u+z}{\|u+z\|} - \frac{z}{\|u+z\|} \right\|$$

and pass to the infimum over the set $U_\epsilon = \{(u,z) : u,z \geq 0, \parallel u \parallel = 1, \parallel z \parallel \geq \epsilon\}$.□

Let us point out that the indicated correspondence of the different definitions of UM spaces is not longer true for local properties (eg. LUM, cf. [4]).

The UM and STM can be viewed as restrictions of the *uniform rotundity* (UR) and the *strict convexity* (R) to the positive cone $X_+$, respectively ([4], Proposition 1.2 and 1.3). Thus UR $\Rightarrow$ UM and R $\Rightarrow$ STM.

We will call $X$ *order smooth*, in abbreviation (o)-Sm, if for each $x \in S(X_+)$ (the positive part of the unit sphere in $X$) and each order interval $[u^*, v^*] \subset \partial_+ \parallel x \parallel$ there holds $u^* = v^*$, where $\partial_+ \parallel x \parallel = \{x^* \in S(X_+^*) : < x, x^* > = \parallel x \parallel\}$.

The strongest notion of smoothness of $X$ is the *order uniform smoothness*, in abbreviation (o)-USm. We say $X$ to be (o)-USm if $\rho_X(\tau)/\tau \to 0$, whenever $\tau \searrow 0$, where the *modulus of smoothness* $\rho_X(\tau)$ is defined as follows:

$$\rho_X(\tau) = \sup\left\{ \parallel x \vee \tau y \parallel -1 : 0 \leq x, y, \ \parallel x \parallel = 1, \parallel y \parallel = 1 \right\} \quad (\tau \in [0,1]) \ .$$

**Lemma**

*For all $\epsilon,\ \tau \in [0,1]$ the following inequalities hold true*

(i) $0 \leq \alpha_X(\epsilon) \leq \delta_X(\epsilon) \leq \epsilon,$

(ii) $0 \leq \rho_X(\tau)/\tau \leq \beta_X(\tau)/\tau \leq 1.$

By $\alpha_X(\epsilon)$ and $\beta_X(\tau)$ we mean the modulus of the uniform rotundity (cf. [4], Proposition 1.2) and the modulus of smoothness ($0 \leq \epsilon,\ \tau$):

$$\alpha_X(\epsilon) = \inf\{1 - \parallel x \pm y \parallel : \parallel x \parallel = 1, \parallel y \parallel \geq \epsilon\}$$

$$\beta_X(\tau) = \sup\left\{ \frac{\parallel x+\tau y \parallel + \parallel x - \tau y \parallel}{2} - 1 : \parallel x \parallel = 1, \parallel y \parallel = 1 \right\}.$$

**Corollary**

(a) *If $X$ is UR (resp. USm) then $X$ is UM (resp. (o)-USm).* (b) *If $X$ is an R (i.e. rotund) space (resp. Sm space , i.e. smooth) then $X$ is a STM space (resp. (o)-Sm space).*

Recall (cf. [4]) that any UM Banach lattice $X$ is a KB space (i.e. the norm is order continuous and $X$ is monotonically complete).

EXAMPLE: It follows easily from the definitions that $\delta_{L_1}(\epsilon) \equiv \epsilon$, $\rho_{L_\infty}(\tau) \equiv 0$. However $\delta_{L_\infty}(\epsilon) \equiv 0$ but $\rho_{L_1}(\tau) \equiv \tau$. Roughly speaking the space $L_1$ is the best (worst) UM (resp. (o)-USm ) space and the space $L_\infty$ is the best (worst) (o)-USm (resp. UM ) space since the respective modules attain their bounds.

## 2. (o)-Smoothness and strict monotonicity

The following theorem is true also for normed lattices.

**Theorem 1**

*Let $X$ be a Banach lattice with the dual $X^*$. Then*

(a) *if $X^*$ is a STM space then $X$ is (o)-Sm space,*
(b) *if $X^*$ is (o)-Sm space then $X$ is a STM space,*

*If moreover $X$ is reflexive then the converse implications are also true.*

*Proof.* (a) If X is not (o)-Sm then there exists a proper (order) interval $[u^*, v^*] \subset \partial_+ \parallel x \parallel$. Hence in particular $0 < u^* < v^*$ and $[u^*, v^*] \subset S(X_+)$ i.e. $X^*$ is not STM space which proves (a).

(b). Let $X^*$ be (o)-Sm space but $X$ is not STM, i.e. $\parallel x \parallel = \parallel x - y \parallel$ for some $y$ and $x \in S(X_+)$ such that $0 < y < x$. There exists a positive functional $x^* \in X^*$ satisfying $< x - y, x^* > = \parallel x - y \parallel$. Hence we conclude that also $< x, x^* > = \parallel x \parallel$. Let $u = x - y$. Denoting the canonical injections of x and u into $X^{**}$ by $\hat{x}$ and $\hat{u}$, respectively, we obtain finally that the proper interval $[\hat{u}, \hat{x}] \subset \partial_+ \parallel x^* \parallel$, a contradiction with the (o)-Sm of $X^*$.

The converse implications for $X$ reflexive are now clear. □

In the following we will try to explain the meaning of the (o)-Sm by means of the behavior of the function $t \to \eta(t)$ $(t > 0)$, where

$$\eta(t) = \frac{\parallel x \vee ty \parallel - \parallel x \parallel}{t} \quad (x, y \geq 0,\ t > 0).$$

**Lemma**

*The function $t \to \parallel x \vee ty \parallel$ is convex and the function $\eta(t)$ is nonnegative and nondecreasing for $t > 0$.*

*Proof.* Applying the formula $x \vee ty = \frac{1}{2}(x + ty + \mid x - ty \mid)$ the convexity of the function $\| x \vee ty \|$ easily follows. Now the standard reasoning yields the second assertion. □

As a corollary it follows that $\eta = \lim_{t \searrow 0} \eta(t) = \inf_{t>0} \eta(t)$ exists and the limit $\eta$ is finite and nonnegative.

Now we will prove the basic duality formula relating the notion of the (o)-smoothness with the behavior of divided difference of special kind.

**Theorem 2**

*Let $x, y$ be arbitrary in $S(X_+)$. The following duality formula holds true:*

$$\inf_{t>0} \frac{\| x \vee ty \| - \| x \|}{t} = \sup_{x^*, y^* \in \partial \|x\|, 0 \le y^* \le x^*} (< y, x^* - y^* >) \tag{1}$$

*where the "sup" on the right side is attained.*

*Proof.* First we will prove the inequality "$\le$". Let $x, y \in S(X_+)$ be arbitrary. In virtue of Lemma above the function $t \to \eta(t)$ is nondecreasing and nonnegative. Next, for the function $\eta(t)$ we have (cf. [1] pp. 55 and 175):

$$\eta(t) = \sup_{x^* \in S(X_+^*)} \sup_{(x^* \ge y^* \ge 0)} \left\{ < y, x^* - y^* > + \frac{1}{t}(< x, y^* > -1) \right\},$$

and $\eta = \lim_{t \searrow 0} \eta(t)$. Hence there exist nets $(t_\alpha)$, $(x_\alpha^*)$, $(y_\alpha^*)$ such that $t_\alpha \searrow 0$, $x_\alpha^* \in S(X_+^*)$, $0 \le y_\alpha^* \le x_\alpha^*$ and

$$< y, x_\alpha^* - y_\alpha^* > + \frac{1}{t_\alpha}(< x, y_\alpha^* > -1) \longrightarrow \eta.$$

Since the first term is bounded and $t_\alpha \searrow 0$ we conclude that $< y, y_\alpha^* > \to 1$ and therefore $< x, y_\alpha^* > \to 1$. Since $S(X_+^*)$ is weakly* compact there exist $x^* \in S(X_+^*)$ and $y^*$ with $0 \le y^* \le x^*$ such that $x_\beta^* \to x^*$ and $y_\beta^* \to y^*$ weakly* for a subnet $(\beta)$. Hence $< x, x^* > = < x, y^* > = \| x \|$ and consequently $x^*, y^* \in \partial \| x \|$, $0 \le y^* \le x^*$. Passing now to the limit above we see that with these $x^*$ and $y^*$ there holds

$$\inf_{t>0} \frac{\| x \vee ty \| - \| x \|}{t} = < y, x^* - y^* >,$$

and the inequality "$\le$" follows.

To prove the inequality "$\ge$" let us confine with the supremum in the formula for $\eta(t)$ to $x^*, y^* \in \partial \| x \|$ such that $x^* \ge y^* \ge 0$. Then $< x, y^* > = 1$ and the desired inequality follows which concludes the proof. □

We will relate the smoothness with the (o)-smoothness. Let $f(x) = \| x \|$ and $f_+(x, y)$ be the directional derivative of $f$ at $x$ in the direction $y$. It is a well known fact in convex analysis that $f_+(x, y) = \max \{< y, x^* >: x^* \in \partial \| x \|\}$. For the left directional derivative we have $-f_+(x, -y) = \min \{< y, x^* >: x^* \in \partial \| x \|\}$.

**Corollary**

*If $x$ in $S(X_+)$ is a smooth point then it is an (o)-smooth point. More precisely if $x \in S(X_+)$ and $y \geq 0$ then*

$$f_+(x,y) + f_+(x,-y) \geq \max_{x^*,y^* \in \partial\|x\|, 0 \leq y^* \leq x^*} (< y, x^* - y^* >) \geq 0.$$

*Moreover $x \in S(X_+)$ is an (o)-smooth point if and only if $X_+ \perp (\partial_+ \parallel x \parallel - \partial_+ \parallel x \parallel)$, where $y \perp x^*$ means that $< y, x^* >= 0$.*

Example: Any point $x \in S(l^2_\infty)$, $x \geq 0$, is an (o)-smooth point (in fact this space is (o)-USm). Indeed, it suffices to consider the extreme point $x = (1,1)$ only. In this case $\partial_+ \parallel x \parallel$ can be identified with the positive part of the unit sphere in $l^2_1$ which does not contain any order interval (the coordinatewise ordering is considered).

On the other hand the space $l^2_1$ is not (o)-smooth. Indeed, a point $x = (0,1)$ has $\partial_+ \parallel x \parallel$ containing an order interval $[y^*, x^*]$ ($x^* = (1,1)$, $y^* = (0,1)$) which is the largest possible.

## 3. Uniform properties and duality

In this paragraph Lindenstrauss type duality formulas relating the modulus of uniform monotonicity $\delta_X(\epsilon)$ and the modulus of (o)-uniform smoothness ((o)-USm) $\rho_X(\tau)$ are proved and the main duality theorem is derived.

Let us first observe that $\rho_X(\tau) \leq \rho_{X^{**}}(\tau)$ and $\delta_X(\epsilon) \geq \delta_{X^{**}}(\epsilon)$ ($\epsilon, \tau \in [0,1]$).

**Theorem 3**

*Let $x, y$ be arbitrary in $S(X_+)$. The following duality formulas hold true:*

(a) $\rho_X(\tau) = \rho_{X^{**}}(\tau)$,

(b) $\delta_X(\epsilon) = \delta_{X^{**}}(\epsilon)$ *and*

(c) $\rho_{X^*}(\tau) = \sup_{0 \leq \epsilon \leq 1} \big(\epsilon\tau - \delta_X(\epsilon)\big)$,

(d) $\delta_X(\epsilon) = \sup_{0 \leq \tau \leq 1} (\tau\epsilon - \rho_{X^*}(\tau))$

*where $\epsilon, \tau \in [0,1]$.*

*Proof.* (c). Let $x^*, y^* \in S(X_+^*)$ be arbitrary but fixed, $\tau \in [0,1]$ and $x \in S(X_+)$. Then

$$\begin{aligned} < x, x^* \vee \tau y^* > -1 &= \sup_{x \geq u \geq 0} (< x-u, x^* > + \tau < u, y^* >) - 1 \\ &\leq \sup_{x \geq u \geq 0} (\| x-u \| + \tau \| y^* \| \| u \|) - 1 \\ &\leq \sup_{(0 \leq \epsilon \leq 1)} \sup_{(0 \leq u \leq x, \|x\|=1, \|u\| \geq \epsilon)} (\| x-u \| - 1 + \tau\epsilon) \\ &= \sup_{0 \leq \epsilon \leq 1} \big(\tau\epsilon - \delta_X(\epsilon)\big). \end{aligned}$$

Now, passing to the "sup" over $x \in S(X_+)$ and then over $x^*, y^* \in S(X_+^*)$ we get

$$\rho_{X^*}(\tau) \leq \sup_{0 \leq \epsilon \leq 0} \big(\tau\epsilon - \delta_X(\epsilon)\big). \tag{2}$$

Now, let $\epsilon \in [0,1]$, and fix $x \in S(X_+)$ and $u$ such that $0 \leq u \leq x$. Then there exist $x^*, y^* \in S(X_+^*)$ such that $< x-u, x^* > = \| x-u \|$ and $< u, y^* > = \| u \|$. Hence for $\tau \in [0,1]$

$$\begin{aligned} \rho_{X^*}(\tau) &\geq \| x^* \vee \tau y^* \| - 1 \\ &\geq < x, x^* \vee \tau y^* > -1 \\ &= \sup_{0 \leq y \leq x} \big(< x-y, x^* > + \tau < y^*, y^* >\big) - 1 \\ &\geq \| x-u \| + \tau \| u \| - 1 \geq \tau\epsilon - (1 - \| x-u \|). \end{aligned}$$

Now, passing to the supremum over $x$ and $u$ indicated and then over $\epsilon \in [0,1]$, we obtain

$$\rho_{X^*}(\tau) \geq \sup_{0 \leq \epsilon \leq 1} \big(\tau\epsilon - \delta_X(\epsilon)\big). \tag{3}$$

Collecting (2) and (3) the property (c) follows.

To prove (b) we will estimate $\delta_X(\epsilon)$ from below. First in virtue of (c)

$$\delta_X(\epsilon) \geq \sup_{0 \leq \tau \leq 1} \big(\epsilon\tau - \sup_{(x^*, y^*) \in S(\tau)} (\| x^* \vee y^* \| - 1)\big) \tag{4}$$

for all $\epsilon \in [0,1]$, where $S(\tau) = \{(x^*, y^*) : x^*, y^* \geq 0, \| x^* \| = 1, \| y^* \| \leq \tau\}$.

Let $\epsilon, \eta, \tau \in (0,1]$ be arbitrary. For each $(x^*, y^*) \in S(\tau)$ there exists $x \in S(X_+)$ such that

$$\| x^* \vee y^* \| \quad \leq \quad < x, x^* \vee y^* > + \eta. \tag{5}$$

Denote $A_\epsilon = \{(x,y) : 0 \le y \le x, \| y \| \le \epsilon\}$ and $B_\epsilon = \{(x,y) : 0 \le y \le x, \| y \| \le \epsilon\}$. Given $x^*, y^* \in S(\tau)$ we have

$$\begin{aligned} < x, x^* \vee y^* > -1 &= \sup_{0 \le y \le x} (< x-y, x^* > + < y, y^* >) - 1 \\ &\le \max\{\sup_{A_\epsilon} (\| x-y \| -1 + \tau \| y \|), \\ &\qquad \sup_{B_\epsilon} (\| x-y \| -1 + \tau \| y \|)\} \\ &\le \max\{\tau\epsilon, -\delta_X(\epsilon) + \tau\} \\ &\le \tau\epsilon - \delta_X(\epsilon) + \tau. \end{aligned}$$

Hence with $x^*, y^*$ and $x$ as above, from (4) and (5) it follows that

$$\begin{aligned} \delta_X(\epsilon) &\ge \sup_{0 \le \tau \le 1} \{\epsilon\tau - \rho_{X^*}(\tau)\} \\ &\ge \tau\epsilon - (< x, x^* \vee y^* > -1 + \eta) \\ &= \delta_X(\epsilon) - \tau - \eta. \end{aligned}$$

Since $\eta$ and $\tau$ were arbitrary in $(0,1]$ we get the equality in (4) for each $\epsilon \in (0,1]$ as desired. The case $\epsilon = 0$ is obvious in virtue of $\rho_(\tau)X^*) \le \tau$ and $\delta_X(0) = 0$.

To prove (a) it suffices to prove that $\rho_X(\tau) \ge \rho_{X^{**}}(\tau)$. For this let $x^*, y^* \in X_+$ be such that $\| x^* \| = 1$, $0 \le y^* \le x^*$, $0 \ne y^*$ and let $\eta \in (0,1]$. Then there exist $x, y \in S(X_+)$ such that

$$\| x^* - y^* \| \le < x, x^* - y^* > + \eta \quad \text{and} \quad \| y^* \| \le < y, y^* > + \eta. \tag{6}$$

With these $x, y, x^*, y^*$ and $\eta$ we have

$$\begin{aligned} \rho_X(\tau) &\ge \| x \vee \tau y \| - 1 \\ &\ge < x \vee \tau y, x^* > -1 \\ &= \sup_{0 \le y^* \le x^*} \big(< x, x^* - y^* > + \tau < y, y^* >\big) - 1 \\ &\ge < x, x^* - y^* > + \tau < y, y^* > -1 \\ &\ge \| x^* - y^* \| - \eta + \tau(\| y^* \| - \eta) - 1. \end{aligned}$$

Taking the supremum over $x^*, y^*$ indicated we get

$$\begin{aligned} \rho_X(\tau) &\ge \sup_{0 \le \epsilon \le 1} \sup_{(\|x^*\|=1, 0 \le y^* \le x^*, \|y^*\| = \epsilon)} (\tau\epsilon + \| x^* - y^* \| - 1) - 2\eta \\ &= \sup_{0 \le \epsilon \le 1} (\tau\epsilon - \delta_{X^*}(\epsilon)) - 2\eta = \rho_{X^{**}}(\tau) - 2\eta. \end{aligned}$$

Since $\eta \in (0,1]$ was arbitrary we get the desired inequality and hence (a) follows.

Finally to prove (b) it suffices to apply (a) and (d) respectively:

$$\begin{aligned}\delta_{X^{**}}(\epsilon) &= \sup_{0\le\tau\le 1} (\tau\epsilon - \rho_{X^{***}}(\tau)) \\ &= \sup_{0\le\tau\le 1} (\tau\epsilon - \rho_{X^{*}}(\tau)) \\ &= \delta_X(\epsilon). \ \square\end{aligned}$$

In the Proposition below we collect basic properties of the modules $\delta_X(\epsilon)$ and $\rho_{X^*}(\tau)$.

**Proposition 4**

*The following properties hold true.*

(a) $\delta_X(\epsilon) \equiv 0$ *(resp.* $\epsilon$ *) if and only if* $\rho_{X^*}(\tau) \equiv \tau$ *(resp. 0).*

(b) $\epsilon\tau \le \delta_X(\epsilon) + \rho_{X^*}(\tau) \le \epsilon + \tau$. *Moreover, given* $\epsilon, \tau \in [0,1]$ *the equality on the right is attained if and only if* $\delta_X(\epsilon) = \epsilon$ *and* $\rho_{X^*}(\tau) = \tau$.

(c) *The functions* $\delta_X(\epsilon)$, $\rho_{X^*}(\tau)$ *are convex (nonnegative) and continuous on the interval* $[0,1]$ *with* $\delta_X(0) = \rho^*_X(0) = 0$ *and therefore nondecreasing.*

*Proof.* (a) In virtue of Theorem 3(d), $\delta_X(\epsilon) = 0$ for all $\epsilon \in [0,1]$ implies that $\tau\epsilon \le \rho_{X^*}(\tau) \le \tau$ for all $\epsilon \in [0,1]$. Hence $\rho_{X^*}(\tau) \equiv 0$. To prove the converse implication we put in Theorem 3(d) $\rho_{X^*}(\tau) \equiv \tau$. Hence $\delta_X(\epsilon) \equiv 0$. The remaining cases follow in the same way so we omit their proofs.

(b) It was already stated that $0 \le \delta_X(\epsilon) \le \epsilon$ and $0 \le \rho_{X^*}(\tau) \le \tau$. Hence and from (d) in Theorem 3, (b) follows.

(c) From (c) and (d) in Theorem 3 it follows that the functions $\delta_X(\epsilon)$, $\rho_{X^*}(\tau)$ are pointwise suprema of families of affine functions on the interval $(0,1)$. Therefore they are lsc and convex on $(0,1)$ and hence continuous and nondecreasing. From the definitions it follows that $\delta_X(0) = \rho_{X^*}(0) = 0$ and consequently they are continuous at zero from the right. Since $\delta_X(1) \ge \delta_X(\epsilon) = \sup_{0\le\tau\le 1}(\tau - \rho_{X^*}(\tau) - \tau(1-\epsilon)) \ge \delta_X(1) + (1-\epsilon)$, we conclude that $\delta_X(\epsilon)$ is left continuous at 1. The same reasoning applies to $\rho_{X^*}(\tau)$ so the proof is finished. $\square$

As a consequence of Theorem 3 we get the following duality theorem.

**Theorem 5**

*Let* $X$ *be a Banach lattice. Then*

(a) $X$ *is* UM (resp. (o)-USm) *if and only if* $X^{**}$ *is* UM (resp. (o)-USm).

(b) $X$ *is* UM *if and only if* $X^*$ *is* (o)-USm.

(c) $X^*$ *is* UM *if and only if* $X$ *is* (o)-USm.

*Proof.* (a) This follows immediately from Theorem 3 ((a),(b)).

(b) If $X^*$ is not an (o)-USm space then $\inf_{\tau>0} \rho_{X^*}(\tau)/\tau > \alpha$ for some $\alpha > 0$, because the function $\tau \to \rho_{X^*}(\tau)/\tau$ is nondecreasing and continuous on $(0,1]$ (apply the property of the function $\eta(t)$ from Sec. 2). Therefore

$$\delta_X(\epsilon) = \sup_{0\le\tau\le 1} \tau\big(\epsilon - \frac{\rho_{X^*}(\tau)}{\tau}\big) \le \sup_{0\le\tau\le 1} \tau(\epsilon - \alpha) = 0$$

whenever $0 < \epsilon \le \alpha$, i.e. $X$ is not UM.

To prove the converse implication let $X$ be not UM, i.e. $\delta_X(\epsilon_0) = 0$ for some $\epsilon_0 \in (0,1)$. Then

$$\frac{\rho_{X^*}(\tau)}{\tau} = \sup_{0\le\epsilon\le 1} \big(\epsilon - \frac{\delta_X(\epsilon)}{\tau}\big) \ge \epsilon_0 \ \text{ for } \ \tau \in (0,1],$$

i.e. $X^*$ is not (o)-USm. Collecting these all (b) follows.

Now let $X^*$ be UM. From (b) $X^{**}$ is then (o)-USm. Since $X$ embeds as a closed sublattice (isometrically) into $X^{**}$ we conclude that $X$ is (o)-USm. To prove the converse let $X$ be (o)-USm. Then in virtue of (a) $X^{**}$ is (o)-USm and hence (using (b)) $X^*$ is UM as desired. Thus (c) holds true and the proof is finished. □

As a corollary we get the following applications of the notion of UM and (o)-USm spaces.

**Theorem 6**

*Let $X$ be a Banach lattice. Then*

(a) *If $X$ is* UM *then $X$ is a* KB *-space.*

(b) *If $X$ is* (o)-USm *then $X^{**}$ is the band generated by $X$ in $X^{**}$ .*

*Proof.* (a) This is a known fact (cf. [2], Chap. XV, Theorem 21) so we omit the proof. (b) In virtue of Theorem 5 if $X$ is (o)-USm then $X^*$ is a UM-space. Now applying Theorem 2.4.14 from [8] we conclude that $X^{**}$ is the band generated by $X$ in $X^{**}$. □

Applying results from this section and characterizations of STM and UM Orlicz spaces for Luxemburg and Orlicz (in the Amemiya form, cf. [3], [7]) norm (see [4], [5], [3]), we derive in [7] characterizations of (o)-Sm and (o)-USm Orlicz spaces as well we obtain estimations for the modules $\delta_X(\epsilon)$, $\rho_{X^*}(\tau)$.

## 4. An application to optimization

Let $x$ be arbitrary but fixed on $S(X_+)$. Define a functional

$$f_x(y) = \| x \vee y \|, \quad \text{where } y \in X \text{ and } y \geq 0.$$

Clearly $f_x(y) \geq f_x(0)$ and $f_x(y) \geq f_x(x)$. Therefore the (order) interval $[0, x] \subset P_{f_x} = \{y \in X : y \geq 0, f_x(0) = \| x \vee y \|\}$.

DEFINITION. Let $x$ be fixed as above. We say that $P_{f_x}$ is a set of solutions of the optimization problem:

$$\begin{cases} f_x(y) \longrightarrow \min \\ y \geq 0. \end{cases}$$

As a corollary from Theorem 2 we get a criterion for potential members of $P_{f_x}$.

**Theorem 7**

*A necessary condition for $u \in P_{f_x}$ is*

$$\max_{x^*, y^* \in \partial \|x\|, 0 \leq y^* \leq x^*} < u, x^* - y^* > = 0.$$

*This condition is trivially satisfied if $x$ is an (o)-Sm point, i.e. $\partial \| x \| \cap X_+^*$ contains no proper order interval.*

EXAMPLE: Let us consider the space $l_1$ and let $x = (x_n)$ be in $S(l_1)$ with $x_n \geq 0$. Let $x^* \geq y^* \geq 0$ where $x^* = (\alpha_n), y^* = (\beta_n)$ are from $\partial \| x \|$. Thus $\bigvee_n \alpha_n = 1$ and $\bigvee_n \beta_n = 1$ with $\alpha_n \geq \beta_n \geq 0$. Hence, in particular, $x_n(\alpha_n - \beta_n) = 0$ for all n. Let $y = (y_n)$ be nonnegative such that $< y, x^* - y^* >= 0$. Hence if follows $y_n(\alpha_n - \beta_n) = 0$ for all n. Consequently the supports of $x^*$ and $y^*$ are the same: $\text{supp}_n(x_n) = \text{supp}_n(y_n)$. In fact we have a little more. Namely $y = (y_n)$ is in $P_{f_x}$ if $y_n \in [0, x_n]$ for all n, since $x \vee y = (x_n \vee y_n)$ and $\| x \vee y \| = \| x \|$.

In the theorem below full characterization of elements $y \in P_{f_x}$ is given.

**Theorem 8**

*Let $x \in S(X_+)$ be fixed and let $y \geq 0$. The following statements are equivalent.*

(a) $y \in P_{f_x}$.

(b) *There exists $x^* \in X_+^*$ such that*

  (i) $\| x^* \| = 1$ *and* $< x, x^* > = \| x \|$,

  (ii) $< x \vee y, x^* > = \| x \vee y \|$,

  (iii) $\forall_{(0 \leq y^* \leq x^*)} < y - x, y^* > \leq 0$.

*Proof.* (a)⇒(b). There exists $x^* \in S(X_+^*)$ such that $< x, x^* >=\| x \|$. Now, applying (a), we obtain

$$\begin{aligned}\| x \|=\| x \vee y \| &\geq < x \vee y, x^* > \\ &= \sup_{0 \leq y^* \leq x} (< x, x^* - y^* > + < y, y^* >) \\ &=< x, x^* > + \sup_{0 \leq y^* \leq x} < y - x, y^* > \\ &=\| x \| + \sup_{0 \leq y^* \leq x} < y - x, y^* >.\end{aligned}$$

Hence (b)(ii)-(iii) follow.

(a)⇐(b). We have to prove that for $y \geq 0$ satisfying (b) there holds $\| x \vee y \|= \| x \|$. In virtue of (b)

$$\begin{aligned}\| x \vee y \| &=< x \vee y, x^* > \\ &= \sup_{0 \leq y^* \leq x} (< x, x^* - y^* > + < y, y^* >) \\ &=< x, x^* > + \sup_{0 \leq y^* \leq x} < y - x, y^* > =\| x \|\end{aligned}$$

which finishes the proof. □

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